SPAs for performance modelling:
Lecture 1 — Introduction

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Outline

1. Performance Modelling
2. Operational laws
3. Discrete event modelling
4. CTMC-based performance modelling
   - Derivation of Performance Measures
   - Assumptions
5. Continuous Time Markov Chains
6. PEPA
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There are often conflicting interests at play:

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- **Users** typically want to optimise external measurements of the dynamics such as **response time** (as small as possible), **throughput** (as high as possible) or **blocking probability** (preferably zero);

- In contrast, **system managers** may seek to optimize internal measurements of the dynamics such as **utilisation** (reasonably high, but not too high), **idle time** (as small as possible) or **failure rates** (as low as possible).
Performance Modelling: Motivation

Capacity planning

- How many clients can the existing server support and maintain reasonable response times?
System Configuration

- How many frequencies do you need to keep blocking probabilities low?
System Tuning
- What speed of conveyor belt will minimize robot idle time and maximize throughput whilst avoiding lost widgets?
A model can be constructed to represent some aspect of the dynamic behaviour of a system.
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Once constructed, such a model becomes a tool with which we can investigate the behaviour of the system.
Modelling computer systems: the challenges

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  - Network latency
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  - Server may be down
  - Routers may be down
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Modelling computer systems: the challenges

**Time** What representation of time will we use?

**Randomness** What kind of random number distributions will we use?

**Probability** How can we have probabilities in the model without uncertainty in the results?

**Scale** How can we escape the state-space explosion problem?

**Percentages** What can it mean to have a fraction of a process?
Quantitative Modelling: Motivation

- Quality of Service issues
  - Can the server maintain reasonable response times?
Quantitative Modelling: Motivation

Scalability issues

- How many times do we have to replicate this service to support all of the subscribers?
Scalability issues

- Will the server withstand a distributed denial of service attack?
Quantitative Modelling: Motivation

Service-level agreements

- What percentage of downloads do complete within the time we advertised?
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- The laws are very general and make almost no assumptions about the behaviour of the random variables characterising the system.

- Another advantage of the laws is their simplicity: this means that they can be applied quickly without detailed knowledge. We will use them sometimes to derive further data from the output observed from models.
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Each request generates a job or customer within the system.

When the job has been processed the system responds to the environment with the completion of the corresponding request.
If we observed such an abstract system we might measure the following quantities:

- $T$, the length of time we observe the system;
- $A$, the number of request arrivals we observe;
- $C$, the number of request completions we observe;
- $B$, the total amount of time during which the system is busy ($B \leq T$);
- $N$, the average number of jobs in the system.
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\[ S = \frac{B}{C}, \] the mean service time per completed job.
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Note that if the system is job flow balanced the arrival rate will be the same as the completion rate, that is, $\lambda = \mu$. 
Little’s Law

\[ N = XW \]

*The average number of jobs in a system is equal to the product of the throughput of the system and the average time spent in that system by a job.*
Example

Consider a disk that serves 40 requests/second ($X = 40$) and suppose that on average there are 4 requests present in the disk system (waiting to be served or in service) ($N = 4$).
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Little’s law tells us that the average time spent at the disk by a request must be \(4/40 = 0.1\) seconds.

If we know that each request requires 0.0225 seconds of disk service we can then deduce that the average queueing time is 0.0775 seconds.
A system may be regarded as being made up of a number of devices or resources.
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Each of these may be treated as a system in its own right from the perspective of operational laws.
An external request generates a job within the system; this job may then circulate between the resources until all necessary processing has been done; as it arrives at each resource it is treated as a request, generating a job internal to that resource.
An external request generates a job within the system; this job may then circulate between the resources until all necessary processing has been done; as it arrives at each resource it is treated as a request, generating a job internal to that resource.
In an observation interval we can count not only completions external to the system, but also the number of completions at each resource within the system.
We define the visit count, $V_i$, of the $i$th resource to be the ratio of the number of completions at that resource to the number of system completions $V_i \equiv C_i / C$. 
For example, if, during an observation interval, we measure 10 system completions and 150 completions at a specific disk, then on average each system-level request requires 15 disk operations.
Forced Flow Law

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Forced Flow Law

\[ X_i = XV_i \]

*The throughput at the *ith* resource is equal to the product of the throughput of the system and the visit count at that resource.*
Example

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- We know that the lathe processes 8 widgets in a minute and we want to know the throughput of the press.

\[
\text{Throughput of the workcell} = \frac{\text{Throughput of the lathe}}{\text{Lathe throughput}} = \frac{8}{4} = 2.
\]

The throughput of the press will be

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The total amount of service that a system job generates at the \( i \)th resource is called the service demand, \( D_i \):

\[
D_i = S_i V_i
\]
Utilisation Law

The utilisation of a resource, the percentage of time that the $i$th resource is in use processing to a job, is denoted $U_i$.

\[
U_i = X_i S_i = XD_i
\]

*The utilisation of a resource is equal to the product of the throughput of that resource and the average service requirement at that resource.*
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- The utilisation law tells us that the utilisation of the disk must be $40 \times 0.0225 = 90\%$. 
Interactive Response Time Law

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- More generally in interactive systems, jobs spend time in the system not engaged in processing, or waiting for processing: this may be because of interaction with a human user, or may be for some other reason.

- The key feature of such a system is that the residence time can no longer be taken as a true reflection of the response time of the system.
Example

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- At the end of this non-processing period the job generates a fresh request.
The think time represents the time between processing being completed and the job becoming available as a request again.
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Thus the residence time of the job, as calculated by Little’s Law as the time from arrival to completion, is greater than the system’s response time.
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Note that if the think time is zero, $Z = 0$ and $R = W$, then the interactive response time law simply becomes Little’s Law.
Suppose that the library catalogue system has 64 interactive users connected via Browsers, that the average think time is 30 seconds, and that system throughput is 2 interactions/second.
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Then the interactive response time law tells us that the response time must be $64/2 - 30 = 2$ seconds.
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The discrete event view

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The state of the system is characterised by variables which take distinct values and which change by discrete events, i.e. at a distinct time something happens within the system which results in a change in one or more of the state variables.
We might be interested in the number of nodes in a communication network which are currently waiting to send a message \( N \).

- If a node, which was not previously waiting, generates a message and is now waiting to send then \( N \rightarrow N + 1 \), or
- If a node, which was previously waiting, successfully transmits its message then \( N \rightarrow N - 1 \).
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**Discrete time:** such models only consider the system at predetermined moments in time, which are typically evenly spaced, e.g. at each clock “tick”.

**Continuous time:** such models consider the system at the time of each event so the time parameter in such models is conceptually continuous.
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At levels of abstraction above the hardware clock continuous time models are generally appropriate for computer and communication systems.
Quantitative modelling

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Probability will be used to represent randomness (e.g. from human users) but also as an abstraction over unknown values (e.g. service times).
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An event is a subset of a sample space.
Random variables

We are interested in the dynamics of a system as events happen over time.

A function which associates a (real-valued) number with the outcome of an experiment is known as a random variable.

Formally, a random variable $X$ is a real-valued function defined on a sample space $\Omega$. 
If $X$ is a random variable, and $x$ is a real number, we write $X \leq x$ for the event

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Measurable functions

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We require that for a random variable $X$, the set $X \leq x$ is an event for each real $x$. This is necessary so that probability calculations can be made. A function having this property is said to be a measurable function or measurable in the Borel sense.
Distribution function

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(If $X$ is a continuous random variable, then $X$ can assume infinitely many values, and so it is reasonable that the probability of its assuming any specific value we choose beforehand is zero.)
The random variable $X$ is said to be an exponential random variable with parameter $\lambda$ ($\lambda > 0$) or to have an exponential distribution with parameter $\lambda$ if it has the distribution function

$$F(x) = \begin{cases} 
1 - e^{-\lambda x} & \text{for } x > 0 \\
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Some authors call this distribution the negative exponential distribution.
The density function \( f = dF/dx \) is given by

\[
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0 & \text{if } x \leq 0
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Mean, or expected value

If $X$ is a continuous random variable with density function $f(\cdot)$, we define the mean or expected value of $X$, $\mu = E[X]$ by

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)\,dx$$
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If $X$ is a discrete random variable with probability mass function $p(\cdot)$, we define the mean or expected value of $X \in S$, $\mu = E[X]$ by

$$E(X) = \sum_{x \in S} xp(x)$$
Suppose $X$ has an exponential distribution with parameter $\lambda > 0$. Then

$$\mu = E[X] = \int_{-\infty}^{\infty} x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$$
The time interval between successive events can also be deduced.

Let $F(t)$ be the distribution function of $T$, the time between events. Consider $\Pr(T > t) = 1 - F(t)$:

$$
\Pr(T > t) = \Pr(\text{No events in an interval of length } t) \\
= 1 - F(t) \\
= 1 - (1 - e^{-\lambda t}) \\
= e^{-\lambda t}
$$
The memoryless property of the exponential distribution is so called because the time to the next event is independent of when the last event occurred.
Suppose that the last event was at time 0. What is the probability that the next event will be after $t + s$, given that time $t$ has elapsed since the last event, and no events have occurred?

$$\Pr(T > t + s | T > t) = \Pr(T > t + s \text{ and } T > t) \Pr(T > t) = e^{-\lambda (t + s)} e^{-\lambda t} = e^{-\lambda s}$$

This value is independent of $t$ (and so the time already spent has not been remembered).
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\[
\Pr(T > t + s \mid T > t) = \frac{\Pr(T > t + s \text{ and } T > t)}{\Pr(T > t)}
\]

\[
= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}
\]

\[
= e^{-\lambda s}
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Memoryless property of the exponential distribution

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Pr(T > t + s \mid T > t) = \frac{Pr(T > t + s \text{ and } T > t)}{Pr(T > t)}
\]

\[= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}
\]

\[= e^{-\lambda s}
\]

This value is independent of \( t \) (and so the time already spent has not been remembered).
Outline

1. Performance Modelling
2. Operational laws
3. Discrete event modelling
4. CTMC-based performance modelling
5. Continuous Time Markov Chains
   - Derivation of Performance Measures
   - Assumptions
6. PEPA
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Stochastic Process

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- A stochastic process is a set of random variables \( \{ X(t), t \in T \} \).

- \( T \) is called the index set usually taken to represent time.

- Since we consider continuous time models \( T = \mathbb{R}^{\geq 0} \), the set of non-negative real numbers.
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Any set of instances of $\{X(t), t \in T\}$ can be regarded as a path of a particle moving randomly in the state space, $S$, its position at time $t$ being $X(t)$.

These paths are called **sample paths** or **realisations** of the stochastic process.
Properties of Stochastic Processes

In this course we will focus on stochastic processes with the following properties:
Properties of Stochastic Processes

In this course we will focus on stochastic processes with the following properties:

\( \{X(t)\} \) is a Markov process.

This implies that \( \{X(t)\} \) has the Markov or memoryless property: given the value of \( X(t) \) at some time \( t \in T \), the future path \( X(s) \) for \( s > t \) does not depend on knowledge of the past history \( X(u) \) for \( u < t \), i.e. for \( t_1 < \cdots < t_n < t_{n+1} \),

\[
\Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \ldots, X(t_1) = x_1) = \Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n)
\]
In this course we will focus on stochastic processes with the following properties:

\(\{X(t)\}\) is irreducible.

This implies that all states in \(S\) can be reached from all other states, by following the transitions of the process. If we draw a directed graph of the state space with a node for each state and an arc for each event, or transition, then for any pair of nodes there is a path connecting them, i.e. the graph is strongly connected.
Properties of Stochastic Processes

In this course we will focus on stochastic processes with the following properties:

\( \{X(t)\} \) is stationary:

for any \( t_1, \ldots, t_n \in T \) and \( t_1 + \tau, \ldots, t_n + \tau \in T \) \((n \geq 1)\), then the process's joint distributions are unaffected by the change in the time axis and so,

\[
F_{X(t_1+\tau) \ldots X(t_n+\tau)} = F_{X(t_1) \ldots X(t_n)}
\]
Properties of Stochastic Processes

In this course we will focus on stochastic processes with the following properties:

\( \{X(t)\} \) is time homogeneous:

the behaviour of the system does not depend on when it is observed. In particular, the transition rates between states are independent of the time at which the transitions occur. Thus, for all \( t \) and \( s \), it follows that

\[
\Pr(X(t + \tau) = x_k \mid X(t) = x_j) = \Pr(X(s + \tau) = x_k \mid X(s) = x_j).
\]
A stochastic process $X(t)$ is a Markov process iff for all $t_0 < t_1 < ... < t_n < t_{n+1}$, the joint probability distribution of $(X(t_0), X(t_1), ..., X(t_n), X(t_{n+1}))$ is such that

$$\Pr(X(t_{n+1}) = s_{i_{n+1}} | X(t_0) = s_{i_0}, ..., X(t_n) = s_{i_n}) = \Pr(X(t_{n+1}) = s_{i_{n+1}} | X(t_n) = s_{i_n})$$
Performance Modelling using CTMC

STATE TRANSITION DIAGRAM
Performance Modelling using CTMC
A negative exponentially distributed duration is associated with each transition.
Performance Modelling using CTMC

these parameters form the entries of the infinitesimal generator matrix $Q$
Exit rate and sojourn time

In any stochastic process the time spent in a state is called the sojourn time.
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In a Markov process the rate of leaving a state $x_i$, $q_i$ the exit rate, is exponentially distributed with the rate which is the sum of all the individual transitions that leave the state, i.e. $q_i = \sum_{j=1,j\neq i}^{N} q_{i,j}$. This follows from the superposition principle of exponential distributions. It follows that the sojourn time will be $1/q_i$. Note that it follows from the Markov property that sojourn times are memoryless.
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Transition rates and transition probabilities

At time $\tau$, the probability that there is a state transition in the interval $(\tau, \tau + dt)$ is $q_i dt + o(dt)$. 

The $q_{ij}$ are called the instantaneous transition rates. The transition probability $p_{ij}$ is the probability, given that a transition out of state $i$ occurs, that it is the transition to state $j$. By the definition of conditional probability, this is $p_{ij} = q_{ij} / q_i$. 

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Thus, for $i \neq j, i, j \in S$,
\[
Pr(X(\tau + dt) = j \mid X(\tau) = i) = q_{ij} dt + o(dt)
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For a state space of size $N$, this is a $N \times N$ matrix, where entry $q(i,j)$ or $q_{i,j}$, records the transition rate of moving from state $x_i$ to state $x_j$. 
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By convention, the diagonal entries \( q_{i,i} \) are the negative row sum for row \( i \), i.e.

\[
q_{i,i} = \sum_{j=1, j \neq i}^{N} q_{i,j}
\]
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Steady state probability distribution

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This is termed the **steady state probability distribution**.

From this probability distribution we will derive performance measures based on subsets of states where some condition holds.
Existence of a steady state probability distribution

For every time-homogeneous, finite, irreducible Markov process with state space $S$, there exists a steady state probability distribution

$$\{\pi_k, x_k \in S\}$$
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This distribution is the same as the limiting or long term probability distribution:

$$\pi_k = \lim_{t \to T} \Pr(X(t) = x_k \mid X(0) = x_0)$$
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This distribution is reached when the initial state no longer has any influence.
In steady state, $\pi_i$ is the proportion of time that the process spends in state $x_i$. 
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Thus, in an instant of time, the probability that a transition will occur from state $x_i$ to state $x_j$ is the probability that the model was in state $x_i$, $\pi_i$, multiplied by the transition rate $q_{ij}$. 
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This is called the probability flux from state $x_i$ to state $x_j$. 
Global balance equations

In steady state, equilibrium is maintained so for any state the total probability flux out is equal to the total probability flux into the state.

\[ \pi_i \times \sum_{x_j \in S, j \neq i} q_{ij} = \sum_{x_j \in S, j \neq i} (\pi_j \times q_{ji}) \]

flux out of \( x_i \)  
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(flux out of \( x_i \)) (flux into \( x_i \))

(If this were not true the distribution over states would change.)
Recall that the diagonal elements of the infinitesimal generator matrix $Q$ are the negative sum of the other elements in the row, i.e. $q_{ii} = - \sum_{x_j \in S, j \neq i} q_{ij}$.
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We can use this to rearrange the flux balance equation to be:

$$\sum_{x_j \in S} \pi_j q_{ji} = 0.$$
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Expressing the unknown values $\pi_i$ as a row vector $\pi$, we can write this as a matrix equation:

$$\pi \, Q = 0.$$
Normalising constant

The $\pi_i$ are unknown — they are the values we wish to find.
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Fortunately, since $\{\pi_i\}$ is a probability distribution we also know that the normalisation condition holds:

$$\sum_{x_i \in S} \pi_i = 1$$
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With these $n + 1$ equations we can use standard linear algebra techniques to solve the equations and find the $n$ unknowns, $\{\pi_i\}$. 
Example

Consider a system with multiple CPUs, each with its own private memory, and one common memory which can be accessed only by one processor at a time.
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- The CPUs execute in private memory for a random time before issuing a common memory access request. Assume that this random time is exponentially distributed with parameter $\lambda$ (the average time a CPU spends executing in private memory between two common memory access requests is $1/\lambda$).
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- The CPUs execute in private memory for a random time before issuing a common memory access request. Assume that this random time is exponentially distributed with parameter $\lambda$ (the average time a CPU spends executing in private memory between two common memory access requests is $1/\lambda$).

- The common memory access duration is also assumed to be exponentially distributed, with parameter $\mu$ (the average duration of a common memory access is $1/\mu$).
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If the system has only one processor, it has only two states:

1. The processor is executing in its private memory;
2. The processor is accessing common memory.
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1. The processor is executing in its private memory;
2. The processor is accessing common memory.

The system behaviour can be modelled by a 2-state Markov process whose state transition diagram and generator matrix are as shown below:

\[ Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \]
Example

\[ Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \]

If we consider the probability flux in and out of state 1 we obtain: \( \pi_1 \lambda = \pi_2 \mu \). Similarly, for state 2: \( \pi_2 \mu = \pi_1 \lambda \).
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We know from the normalisation condition that: 
\[ \pi_1 + \pi_2 = 1. \]
Example

\[
\begin{pmatrix}
\lambda & \\
\mu & \\
\end{pmatrix}
\]

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Thus the steady state probability distribution is
\[\pi = \left(\frac{\mu}{\mu + \lambda}, \frac{\lambda}{\mu + \lambda}\right).\]
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Thus the steady state probability distribution is
\[
\pi = \left( \frac{\mu}{\mu + \lambda}, \frac{\lambda}{\mu + \lambda} \right).
\]

From this we can deduce, for example, that the probability that the processor is executing in private memory is \[\frac{\mu}{\mu + \lambda}.\]
In general our systems of equations will be too large to contemplate solving them by hand, so we want to be able to take advantage of linear algebra packages which can solve matrix equations of the form $Ax = b$, where $A$ is an $N \times N$ matrix, $x$ is a column vector of $N$ unknowns, and $b$ is a column vector of $N$ values.
Solving the global balance equations

First we must resolve two problems:

1. Our global balance equation is expressed in terms of a row vector of unknowns $\pi$, $\pi Q = 0$: the unknowns.
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1. Our global balance equation is expressed in terms of a row vector of unknowns $\pi$, $\pi Q = 0$: the unknowns.

This problem is resolved by transposing the equation, i.e. $Q^T \pi = 0$, where the right hand side is now a column vector of zeros, rather than a row vector.
Solving the global balance equations

2 We must eliminate the redundancy in the global balance equations and add in the normalisation condition.
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We replace one of the global balance equations by the normalisation condition. In $Q^T$ this corresponds to replacing one row by a row of 1’s. We usually choose the last row and denote the modified matrix $Q^T_N$. 
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We must also make the corresponding change to the “solution” vector $\mathbf{0}$, to be a column vector with 1 in the last row, and zeros everywhere else. We denote this vector, $\mathbf{e}_N$. 
Solving the global balance equations

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We replace one of the global balance equations by the normalisation condition. In $Q^T$ this corresponds to replacing one row by a row of 1’s. We usually choose the last row and denote the modified matrix $Q_N^T$.

We must also make the corresponding change to the “solution” vector $\mathbf{0}$, to be a column vector with 1 in the last row, and zeros everywhere else. We denote this vector, $e_N$.

Now we can use any linear algebra solution package, such as maple or xmaple to solve the resulting equation:

$$Q_N^T \pi = e_N$$
Example

Consider the two-processor version of the multiprocessor with processors $A$ and $B$. 
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Consider the two-processor version of the multiprocessor with processors $A$ and $B$.

We assume that the processors have different timing characteristics, the private memory access of $A$ being governed by an exponential distribution with parameter $\lambda_A$, the common memory access of $B$ being governed by an exponential distribution with parameter $\mu_B$, etc.
Example: state space

Now the state space becomes:

1. $A$ and $B$ both executing in their private memories;
2. $B$ executing in private memory, and $A$ accessing common memory;
3. $A$ executing in private memory, and $B$ accessing common memory;
4. $A$ accessing common memory, $B$ waiting for common memory;
5. $B$ accessing common memory, $A$ waiting for common memory;
Example: state space

\[ \begin{aligned}
\lambda_B & \quad \mu_A \\
\lambda_A & \quad \mu_B \\
\lambda_B & \quad \mu_B \\
\lambda_A & \quad \mu_A
\end{aligned} \]
Example: generator matrix

\[ Q = \begin{pmatrix}
-(\lambda_A + \lambda_B) & \lambda_A & \lambda_B & 0 & 0 \\
\mu_A & -(\mu_A + \lambda_B) & 0 & \lambda_B & 0 \\
\mu_B & 0 & -(\mu_B + \lambda_A) & 0 & \lambda_A \\
0 & 0 & \mu_A & -\mu_A & 0 \\
0 & \mu_B & 0 & 0 & -\mu_B \\
\end{pmatrix} \]
Example: modified generator matrix

\[
Q_N^T = \begin{pmatrix}
-(\lambda_A + \lambda_B) & \mu_A & \mu_B & 0 & 0 \\
\lambda_A & -(\mu_A + \lambda_B) & 0 & 0 & \mu_B \\
\lambda_B & 0 & -(\mu_B + \lambda_A) & \mu_A & 0 \\
0 & \lambda_B & 0 & -\mu_A & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
Example: steady state probability distribution

If we choose the following values for the parameters:

\[ \lambda_A = 0.05 \quad \lambda_B := 0.1 \quad \mu_A = 0.02 \quad \mu_B = 0.05 \]

solving the matrix equation, and rounding figures to 4 significant figures, we obtain:

\[ \pi = (0.0693, 0.0990, 0.1683, 0.4951, 0.1683) \]
Deriving Performance Measures

SYSTEM

STATE

TRANSITION

DIAGRAM

\[ Q = \begin{pmatrix}
-\sum \\
-\sum \\
-\sum \\
-\sum \\
\end{pmatrix}
\]

\[ \pi = \begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
\cdots \\
\end{pmatrix} \]

MARKOV PROCESS

EQUILIBRIUM PROBABILITY DISTRIBUTION
Deriving Performance Measures

System Markov Chain

\[ Q = \begin{pmatrix}
\cdots & -\sum & \cdots \\
\cdots & -\sum & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & -\sum & \cdots \\
\cdots & -\sum & \cdots
\end{pmatrix} \]

\[ \pi = \begin{pmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\cdots \\
\pi_N
\end{pmatrix} \]

Performance Measures

e.g. throughput, response time, utilisation
Broadly speaking, there are three ways in which performance measures can be derived from the steady state distribution of a Markov process.
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Broadly speaking, there are three ways in which performance measures can be derived from the steady state distribution of a Markov process.

These different methods can be thought of as corresponding to different types of measure:

- **state-based measures**, e.g. utilisation;
- **rate-based measures**, e.g. throughput;
- **other measures** which fall outside the above categories, e.g. response time.
State-based measures correspond to the probability that the model is in a state, or a subset of states, which satisfy some condition.
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For example, utilisation will correspond to those states where a resource is in use.
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If we consider the multiprocessor example, the utilisation of the common memory, $U_{mem}$, is the total probability that the model is in one of the states in which the common memory is in use:

$$U_{mem} = \pi_2 + \pi_3 + \pi_4 + \pi_5 = 93.07\%$$
State-based measures

Other examples of state-based measures are idle time, or the number of jobs in a system.
State-based measures

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For example, if we consider jobs waiting for the common memory to be queued in that subsystem, then the average number of jobs in the common memory, $N_{mem}$, is:

$$N_{mem} = (1 \times \pi_2) + (1 \times \pi_3) + (2 \times \pi_4) + (2 \times \pi_5) = 1.594$$
Rate-based measures are those which correspond to the predicted rate at which some event occurs.
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This will be the product of the rate of the event, and the probability that the event is enabled, i.e. the probability of being in one of the states from which the event can occur.
Example: rate-based measures

In order to calculate the throughput of the common memory, we need the average number of accesses from either processor which it satisfies in unit time.

\[ X_{\text{mem}} = (\mu_A \times (\pi_2 + \pi_4)) + (\mu_B \times (\pi_3 + \pi_5)) = 0.0287 \]

or, approximately one access every 35 milliseconds.
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\[ X_{mem} \text{ is thus calculated as:} \]

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For example, applying Little’s Law to the common memory we see that

\[ W_{\text{mem}} = \frac{N_{\text{mem}}}{X_{\text{mem}}} = \frac{1.594}{0.0287} = 55.54 \text{ milliseconds} \]
Stochastic Hypothesis

“The behaviour of a real system during a given period of time is characterised by the probability distributions of a stochastic process.”
Assumptions

- All delays and inter-event times are exponentially distributed.
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■ The Markov/memoryless assumption that future behaviour is only dependent on the current state, not on the past history is a reasonable assumption for computer and communication systems, if we choose our states carefully.

■ We generally assume that the Markov process is finite, time homogeneous and irreducible.
Whilst Markov process-based modelling has many advantages, working directly in terms of the state transition diagram or infinitesimal generator matrix is at best time-consuming and error prone, and often simply infeasible.
Difficulties of working with Markov processes

For this reason various high level modelling formalisms have been introduced to make the job of constructing the state transition diagram and/or infinitesimal generator matrix easier.
Outline

1. Performance Modelling
2. Operational laws
3. Discrete event modelling
4. CTMC-based performance modelling
5. Continuous Time Markov Chains
   - Derivation of Performance Measures
   - Assumptions
6. PEPA
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- Performance Evaluation Process Algebra (PEPA) sought to address these problems by the introduction of a suitable process algebra.
- We have sought to investigate and exploit the interplay between the process algebra and the continuous time Markov chain (CTMC).
PEPA (Performance Evaluation Process Algebra) is a high-level modelling language for distributed systems. It can be used to develop models of existing systems (abstraction) or designs for proposed ones (specification).

PEPA can capture performance information in a process algebra setting. It is a stochastic process algebra.

Strengths of stochastic process algebras

SPAs have strengths in the areas of semantic definition, inherent compositionality and the existence of important equivalence relations (including bisimulation). This relation provides the basis for aggregation of PEPA models.
Terminology

The components in a PEPA model engage, cooperatively or individually, in activities.
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To represent unimportant or unknown actions there is a distinguished action type, $\tau$. 
Every activity in PEPA has an associated activity rate which may be any positive real number, or the distinguished symbol “⊤”, meaning unspecified, read as ‘top’.
Quantitative aspects

Every activity in PEPA has an associated activity rate which may be any positive real number, or the distinguished symbol "⊤", meaning unspecified, read as ‘top’.

Components and activities are primitives. PEPA also provides a small set of combinators.
PEPA syntax

\[
S ::= (\alpha, r).S \quad \text{(prefix)} \\
| \quad S_1 + S_2 \quad \text{(choice)} \\
| \quad X \quad \text{(variable)} \\
C ::= C_1 \bowtie L C_2 \quad \text{(cooperation)} \\
| \quad C / L \quad \text{(hiding)} \\
| \quad S \quad \text{(sequential)}
\]
(\(\alpha, r\)).S

The activity \((\alpha, r)\) takes time \(\Delta t\) (drawn from the exponential distribution with parameter \(r\)).

\(S_1 + S_2\)

In this choice either \(S_1\) or \(S_2\) will complete an activity first. The other is discarded.
$C_1 \parallel_L C_2$

All activities of $C_1$ and $C_2$ with types in $L$ are shared: others remain individual.

**NOTATION:** write $C_1 \parallel C_2$ if $L$ is empty.

$C / L$

Activities of $C$ with types in $L$ are hidden ($\tau$ type activities) to be thought of as internal delays.
Example: M/M/1/N/N queue

\[
\begin{align*}
\text{Arrival}_0 & \mathrel{\overset{\text{def}}{=}} (\text{accept}, \lambda).\text{Arrival}_1 \\
\text{Arrival}_i & \mathrel{\overset{\text{def}}{=}} (\text{accept}, \lambda).\text{Arrival}_{i+1} + (\text{serve}, \top).\text{Arrival}_{i-1} \\
\text{Arrival}_N & \mathrel{\overset{\text{def}}{=}} (\text{serve}, \top).\text{Arrival}_{N-1} \\
\text{Server} & \mathrel{\overset{\text{def}}{=}} (\text{serve}, \mu).\text{Server}
\end{align*}
\]
Example: M/M/1/N/N queue

Queue_0 \equiv \text{Arrival}_0 \ {serve} \ Server
Queue_1 

... 

Queue_{N-1} 
Queue_N

(accept, \lambda) \quad (accept, \lambda) \quad (accept, \lambda) \quad (accept, \lambda)

(serve, \mu) \quad (serve, \mu) \quad (serve, \mu) \quad (serve, \mu)