SPAs for performance modelling: Lecture 1 — Introduction

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Outline

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Performance Modelling

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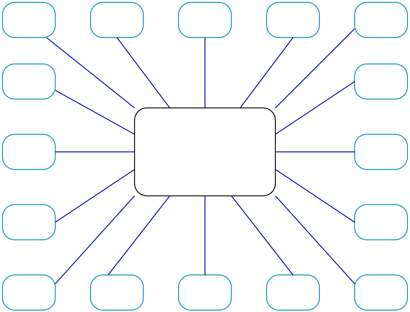
- Users typically want to optimise external measurements of the dynamics such as response time (as small as possible), throughput (as high as possible) or blocking probability (preferably zero);
- In contrast, system managers may seek to optimize internal measurements of the dynamics such as utilisation (reasonably high, but not too high), idle time (as small as possible) or failure rates (as low as possible).

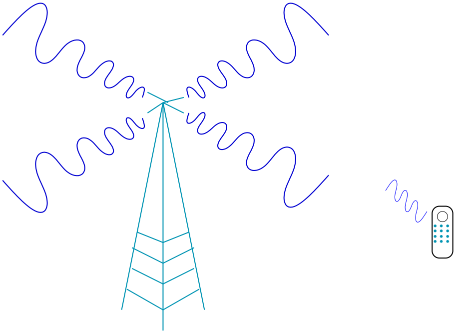
Performance Modelling: Motivation



Capacity planning

- How many clients can the existing server support and maintain reasonable response times?

Performance Modelling: Motivation

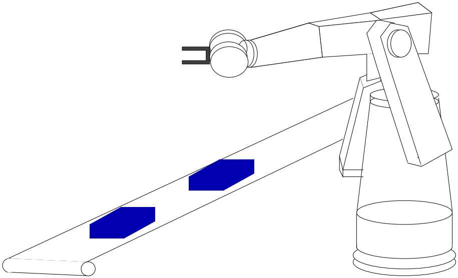


Mobile Telephone Antenna

System Configuration

- How many frequencies do you need to keep blocking probabilities low?

Performance Modelling: Motivation



System Tuning

- What speed of conveyor belt will minimize robot idle time and maximize throughput whilst avoiding lost widgets?

Key notions

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Once constructed, such a model becomes a tool with which we can investigate the behaviour of the system.

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Modelling computer systems: the challenges

Time What representation of time will we use?

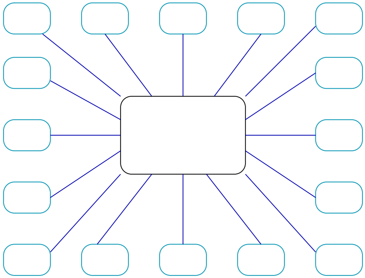
Randomness What kind of random number distributions will we use?

Probability How can we have probabilities in the model without uncertainty in the results?

Scale How can we escape the state-space explosion problem?

Percentages What can it mean to have a fraction of a process?

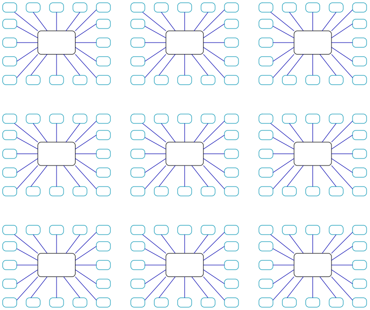
Quantitative Modelling: Motivation



Quality of Service issues

- Can the server maintain reasonable response times?

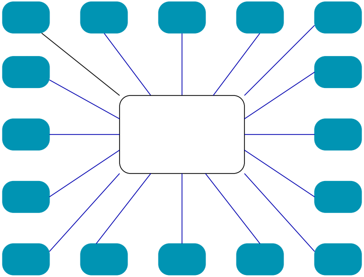
Quantitative Modelling: Motivation



Scalability issues

- How many times do we have to replicate this service to support all of the subscribers?

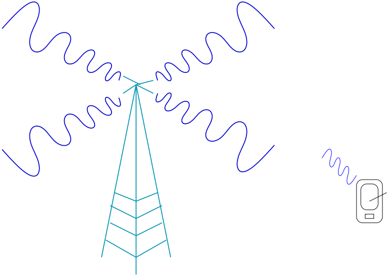
Quantitative Modelling: Motivation



Scalability issues

- Will the server withstand a distributed denial of service attack?

Quantitative Modelling: Motivation



Service-level agreements

- What percentage of downloads do complete within the time we advertised?

Outline

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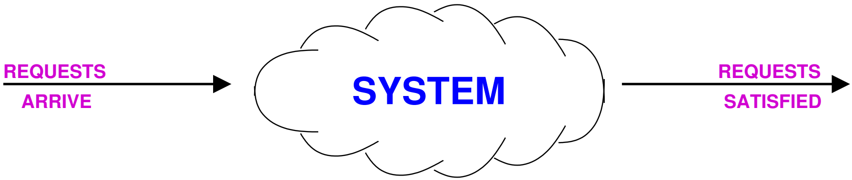
- Operational laws are simple equations may be regarded as a very abstract model of the average behaviour of almost any system, based on the operational view of the system.
- The laws are very general and make almost no assumptions about the behaviour of the random variables characterising the system.
- Another advantage of the laws is their simplicity: this means that they can be applied quickly without detailed knowledge. We will use them sometimes to derive further data from the output observed from models.

Observable variables



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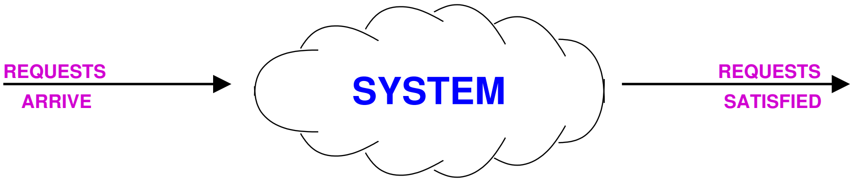

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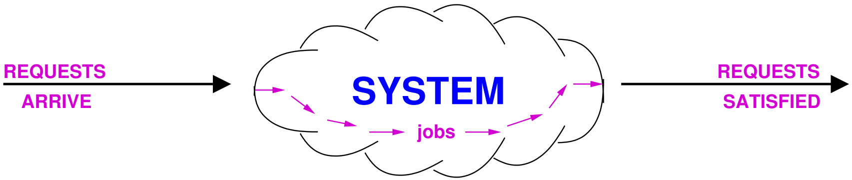

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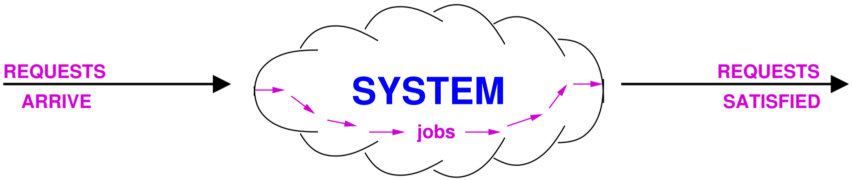

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- Each request generates a job or customer within the system.
- When the job has been processed the system responds to the environment with the completion of the corresponding request.

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- N, the average number of jobs in the system.

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$$S = B/C, \text{ the mean service time per completed job.}$$

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- We will assume that the system is job flow balanced. This means that the number of arrivals is equal to the number of completions during an observation period, i.e. $A = C$.

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- Note that if the system is job flow balanced the arrival rate will be the same as the completion rate, that is, $\lambda = X$.

Little's Law

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$$N = XW$$

The average number of jobs in a system is equal to the product of the throughput of the system and the average time spent in that system by a job.

Example

Consider a disk that serves 40 requests/second ($X = 40$) and suppose that on average there are 4 requests present in the disk system (waiting to be served or in service) ($N = 4$).

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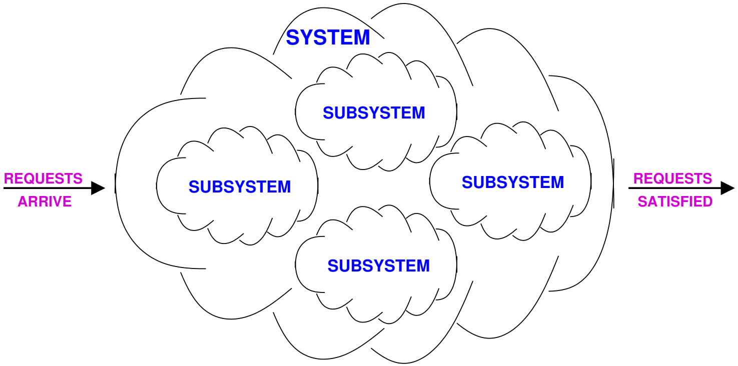
Little's law tells us that the average time spent at the disk by a request must be $4/40 = 0.1$ seconds.

If we know that each request requires 0.0225 seconds of disk service we can then deduce that the average queueing time is 0.0775 seconds.

Subsystems within Systems



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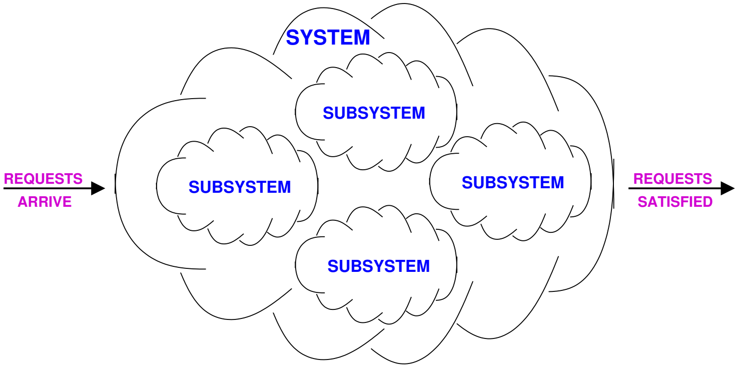

- A system may be regarded as being made up of a number of devices or resources.
- Each of these may be treated as a system in its own right from the perspective of operational laws.

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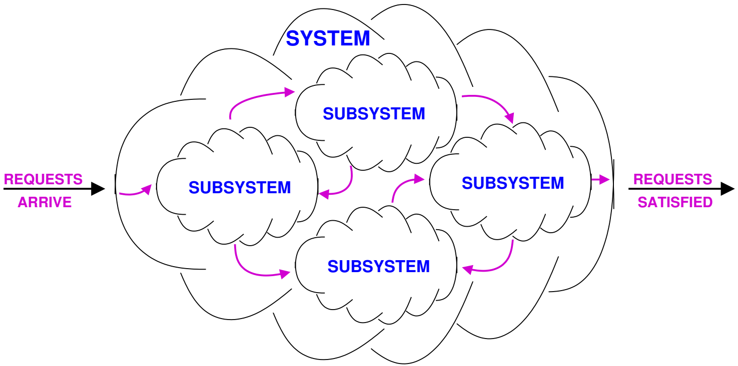

An external request generates a job within the system; this job may then circulate between the resources until all necessary processing has been done; as it arrives at each resource it is treated as a request, generating a job internal to that resource.

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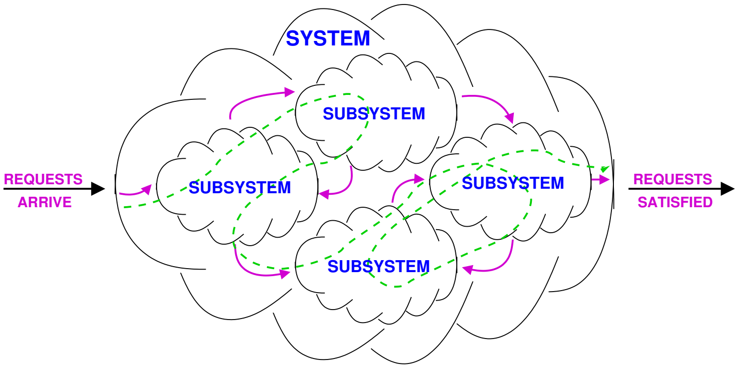

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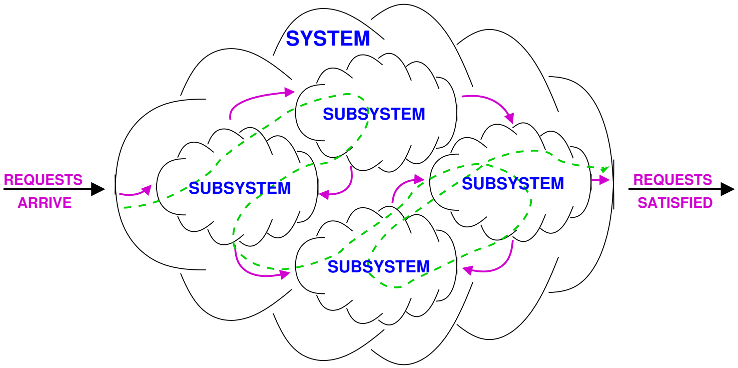
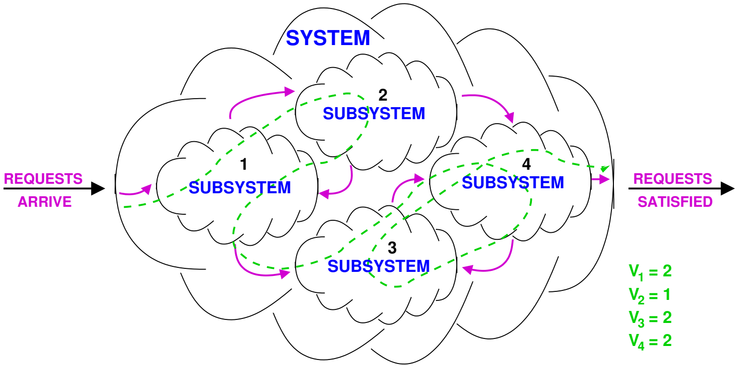


In an observation interval we can count not only completions external to the system, but also the number of completions at each resource within the system.

Visit count



We define the visit count, V_i, of the ith resource to be the ratio of the number of completions at that resource to the number of system completions $V_i \equiv C_i/C$.

Visit count: example

For example, if, during an observation interval, we measure 10 system completions and 150 completions at a specific disk, then on the average each system-level request requires 15 disk operations.

Forced Flow Law

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Forced Flow Law

$$X_i = XV_i$$

The throughput at the ith resource is equal to the product of the throughput of the system and the visit count at that resource.

Example

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- Thus the press throughput is 4 widgets per minute.

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- (Note that service time is not necessarily the same as the residence time of the job at that resource: in general a job might have to wait for some time before processing begins.)
- The total amount of service that a system job generates at the ith resource is called the service demand, D_i:

$$D_i = S_i V_i$$

Utilisation Law

The utilisation of a resource, the percentage of time that the ith resource is in use processing to a job, is denoted U_i.

Utilisation Law

$$U_i = X_i S_i = X D_i$$

The utilisation of a resource is equal to the product of the throughput of that resource and the average service requirement at that resource.

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- The utilisation law tells us that the utilisation of the disk must be $40 \times 0.0225 = 90\%$.

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- More generally in interactive systems, jobs spend time in the system not engaged in processing, or waiting for processing: this may be because of interaction with a human user, or may be for some other reason.
- The key feature of such a system is that the residence time can no longer be taken as a true reflection of the response time of the system.

Example

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- At the end of this non-processing period the job generates a fresh request.

Think time, residence time, response time

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- The think time represents the time between processing being completed and the job becoming available as a request again.
- Thus the residence time of the job, as calculated by Little's Law as the time from arrival to completion, is greater than the system's response time.

Interactive Response Time Law

The interactive response time law reflects this: it calculates the response time, R as follows:

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Note that if the think time is zero, $Z = 0$ and $R = W$, then the interactive response time law simply becomes Little's Law.

Interactive Response Time Law: Example

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- Suppose that the library catalogue system has 64 interactive users connected via Browsers, that the average think time is 30 seconds, and that system throughput is 2 interactions/second.
- Then the interactive response time law tells us that the response time must be $64/2 - 30 = 2$ seconds.

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The discrete event view

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The state of the system is characterised by variables which take distinct values and which change by discrete events, i.e. at a distinct time something happens within the system which results in a change in one or more of the state variables.

The discrete event view: example

We might be interested in the number of nodes in a communication network which are currently waiting to send a message N.

- If a node, which was not previously waiting, generates a message and is now waiting to send then $N \to N + 1$, or
- If a node, which was previously waiting, successfully transmits its message then $N \to N - 1$.

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At levels of abstraction above the hardware clock continuous time models are generally appropriate for computer and communication systems.

Quantitative modelling

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In contrast, performance modelling is quantitative modelling as we must take into account explicit values for time (latency, service time etc.) and probability (choices, alternative outcomes, mixed workload).

Probability will be used to represent randomness (e.g. from human users) but also as an abstraction over unknown values (e.g. service times).

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- The sample space of a random experiment is the set of all individual outcomes of the experiment.
- These individual outcomes are also called sample points or elementary events.
- An event is a subset of a sample space.

Random variables

We are interested in the dynamics of a system as events happen over time.

A function which associates a (real-valued) number with the outcome of an experiment is known as a random variable.

Formally, a random variable X is a real-valued function defined on a sample space Ω.

Measurable functions

If X is a random variable, and x is a real number, we write $X \leq x$ for the event

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We require that for a random variable X, the set $X \leq x$ is an event for each real x. This is necessary so that probability calculations can be made. A function having this property is said to be a measurable function or measurable in the Borel sense.

Distribution function

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We associate another function $p(\cdot)$, called the probability mass function, with X (pmf), for each x:

$$p(x) = \Pr[X = x]$$

Distribution function

For each random variable X we define its distribution function F for each real x by

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(If X is a continuous random variable, then X can assume infinitely many values, and so it is reasonable that the probability of its assuming any specific value we choose beforehand is zero.)

Exponential random variables, distribution function

The random variable X is said to be an exponential random variable with parameter λ ($\lambda > 0$) or to have an exponential distribution with parameter λ if it has the distribution function

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Some authors call this distribution the negative exponential distribution.

Exponential random variables, density function

The density function $f = dF/dx$ is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Mean, or expected value

If X is a continuous random variable with density function $f(\cdot)$, we define the mean or expected value of X, $\mu = E[X]$ by

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If X is a discrete random variable with probability mass function $p(\cdot)$, we define the mean or expected value of $X \in S$, $\mu = E[X]$ by

$$E(X) = \sum_{x \in S} x p(x)$$

Mean, or expected value, of the exponential distribution

Suppose X has an exponential distribution with parameter $\lambda > 0$.

Then

$$\mu = E[X] = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

Exponential inter-event time distribution

The time interval between successive events can also be deduced.

Let $F(t)$ be the distribution function of T, the time between events. Consider $\Pr(T > t) = 1 - F(t)$:

$$\begin{aligned}\Pr(T > t) &= \Pr(\text{No events in an interval of length } t) \\ &= 1 - F(t) \\ &= 1 - (1 - e^{-\lambda t}) \\ &= e^{-\lambda t}\end{aligned}$$

Memoryless property of the exponential distribution

The memoryless property of the exponential distribution is so called because the time to the next event is independent of when the last event occurred.

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Suppose that the last event was at time 0. What is the probability that the next event will be after $t + s$, given that time t has elapsed since the last event, and no events have occurred?

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\begin{aligned}
\Pr(T > t + s \mid T > t) &= \frac{\Pr(T > t + s \text{ and } T > t)}{\Pr(T > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\
&= e^{-\lambda s}
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This value is independent of t (and so the time already spent has not been remembered).

Outline

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- A stochastic process is a set of random variables $\{X(t), t \in T\}$.
- T is called the index set usually taken to represent time.
- Since we consider continuous time models $T = \mathbb{R}^{\geq 0}$, the set of non-negative real numbers.

State Space

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These paths are called sample paths or realisations of the stochastic process.

Properties of Stochastic Processes

In this course we will focus on stochastic processes with the following properties:

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$\{X(t)\}$ **is a Markov process**.

This implies that $\{X(t)\}$ has the Markov or memoryless property: given the value of $X(t)$ at some time $t \in T$, the future path $X(s)$ for $s > t$ does not depend on knowledge of the past history $X(u)$ for $u < t$, i.e. for $t_1 < \cdots < t_n < t_{n+1}$,

$$\Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \ldots, X(t_1) = x_1) = \Pr(X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n)$$

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$\{X(t)\}$ **is irreducible**.

This implies that all states in S can be reached from all other states, by following the transitions of the process. If we draw a directed graph of the state space with a node for each state and an arc for each event, or transition, then for any pair of nodes there is a path connecting them, i.e. the graph is strongly connected.

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$\{X(t)\}$ **is stationary**:

for any $t_1, \ldots t_n \in T$ and $t_1 + \tau, \ldots, t_n + \tau \in T$ $(n \geq 1)$, then the process's joint distributions are unaffected by the change in the time axis and so,

$$F_{X(t_1+\tau)\ldots X(t_n+\tau)} = F_{X(t_1)\ldots X(t_n)}$$

Properties of Stochastic Processes

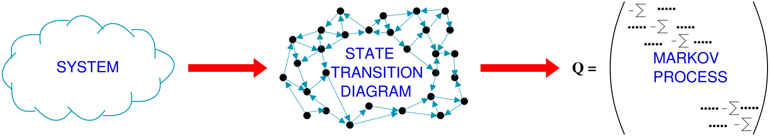
In this course we will focus on stochastic processes with the following properties:

$\{X(t)\}$ **is time homogeneous**:

the behaviour of the system does not depend on when it is observed. In particular, the transition rates between states are independent of the time at which the transitions occur. Thus, for all t and s, it follows that

$$\Pr(X(t+\tau) = x_k \mid X(t) = x_j) = \Pr(X(s+\tau) = x_k \mid X(s) = x_j).$$

Performance Modelling using CTMC



A stochastic process $X(t)$ is a Markov process iff for all $t_0 < t_1 < \ldots < t_n < t_{n+1}$, the joint probability distribution of $(X(t_0), X(t_1), \ldots, X(t_n), X(t_{n+1}))$ is such that

$$\Pr(X(t_{n+1}) = s_{i_{n+1}} \mid X(t_0) = s_{i_0}, \ldots, X(t_n) = s_{i_n}) = \Pr(X(t_{n+1}) = s_{i_{n+1}} \mid X(t_n) = s_{i_n})$$

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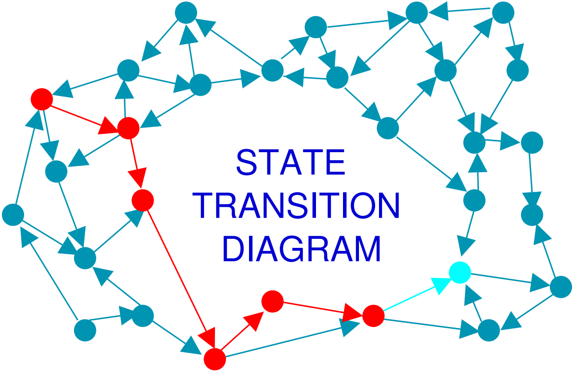

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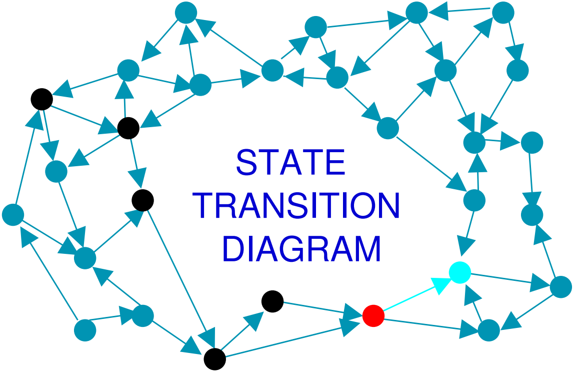

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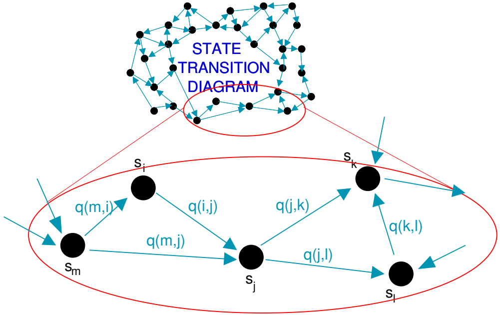
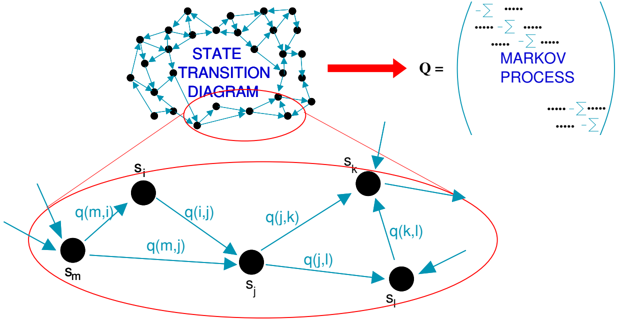


A negative exponentially distributed duration is associated with each transition.

Performance Modelling using CTMC



these parameters form the entries of the infinitesimal generator matrix Q

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Note that it follows from the Markov property that sojourn times are memoryless.

Transition rates and transition probabilities

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Thus, for $i \neq j, i, j \in S$,
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The transition probability p_{ij} is the probability, given that a transition out of state i occurs, that it is the transition to state j. By the definition of conditional probability, this is $p_{ij} = q_{ij}/q_i$.

Infinitesimal Generator Matrix

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For a state space of size N, this is a $N \times N$ matrix, where entry $q(i,j)$ or $q_{i,j}$, records the transition rate of moving from state x_i to state x_j.

By convention, the diagonal entries $q_{i,i}$ are the negative row sum for row i, i.e.

$$q_{i,i} = \sum_{j=1, j \neq i}^{N} q_{i,j}$$

Steady state probability distribution

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From this probability distribution we will derive performance measures based on subsets of states where some condition holds.

Existence of a steady state probability distribution

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This distribution is reached when the initial state no longer has any influence.

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This is called the probability flux from state x_i to state x_j.

Global balance equations

In steady state, equilibrium is maintained so for any state the total probability flux out is equal to the total probability flux into the state.

$$\boxed{\underbrace{\boldsymbol{\pi}_{\mathbf{i}} \times \sum_{x_j \in S, j \neq i} q_{ij}}_{\text{flux \textbf{out} of } x_i} = \underbrace{\sum_{x_j \in S, j \neq i} (\boldsymbol{\pi}_{\mathbf{j}} \times q_{ji})}_{\text{flux \textbf{into} } x_i}}$$

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(If this were not true the distribution over states would change.)

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Expressing the unknown values $\boldsymbol{\pi}_\mathbf{i}$ as a row vector $\boldsymbol{\pi}$, we can write this as a matrix equation:

$$\boxed{\boldsymbol{\pi}\ \mathbf{Q} = 0}$$

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With these $n+1$ equations we can use standard linear algebra techniques to solve the equations and find the n unknowns, $\{\boldsymbol{\pi_i}\}$.

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- The CPUs execute in private memory for a random time before issuing a common memory access request. Assume that this random time is exponentially distributed with parameter λ (the average time a CPU spends executing in private memory between two common memory access requests is $1/\lambda$).
- The common memory access duration is also assumed to be exponentially distributed, with parameter μ (the average duration of a common memory access is $1/\mu$).

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If the system has only one processor, it has only two states:

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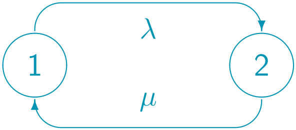
The system behaviour can be modelled by a 2-state Markov process whose state transition diagram and generator matrix are as shown below:



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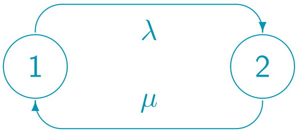

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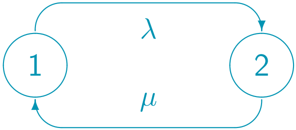
If we consider the probability flux in and out of state 1 we obtain: $\boldsymbol{\pi_1}\ \lambda = \boldsymbol{\pi_2}\mu$. Similarly, for state 2: $\boldsymbol{\pi_2}\ \mu = \boldsymbol{\pi_1}\lambda$.

Example



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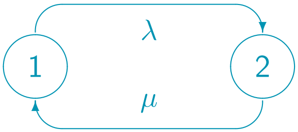

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Thus the steady state probability distribution is
$\boldsymbol{\pi} = \left(\frac{\mu}{\mu+\lambda}, \frac{\lambda}{\mu+\lambda}\right)$.

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$$\begin{tikzcd} 1 \arrow[rr, bend left, "\lambda"] & & 2 \arrow[ll, bend left, "\mu"] \end{tikzcd} \qquad \mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

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From this we can deduce, for example, that the probability that the processor is executing in private memory is $\mu/(\mu+\lambda)$.

Solving the global balance equations

In general our systems of equations will be too large to contemplate solving them by hand, so we want to be able to take advantage of linear algebra packages which can solve matrix equations of the form $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $N \times N$ matrix, $\mathbf{x}$ is a column vector of N unknowns, and $\mathbf{b}$ is a column vector of N values.

Solving the global balance equations

First we must resolve two problems:

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This problem is resolved by transposing the equation, i.e. $\mathbf{Q}^T\boldsymbol{\pi} = 0$, where the right hand side is now a column vector of zeros, rather than a row vector.

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Now we can use any linear algebra solution package, such as `maple` or `xmaple` to solve the resulting equation:

$$\mathbf{Q}_N^T \boldsymbol{\pi} = \mathbf{e}_N$$

Example

Consider the two-processor version of the multiprocessor with processors A and B.

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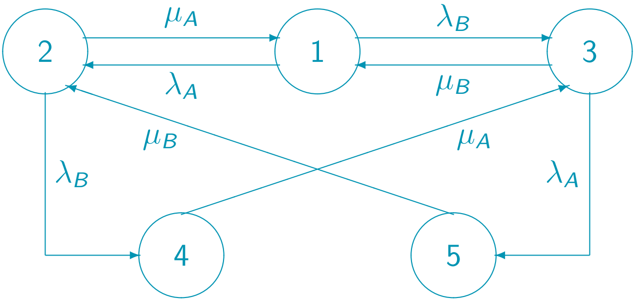
We assume that the processors have different timing characteristics, the private memory access of A being governed by an exponential distribution with parameter λ_A, the common memory access of B being governed by an exponential distribution with parameter μ_B, etc.

Example: state space

Now the state space becomes:

1. A and B both executing in their private memories;
2. B executing in private memory, and A accessing common memory;
3. A executing in private memory, and B accessing common memory;
4. A accessing common memory, B waiting for common memory;
5. B accessing common memory, A waiting for common memory;

Example: state space



Example: generator matrix

$$\mathbf{Q} = \begin{pmatrix} -(\lambda_A + \lambda_B) & \lambda_A & \lambda_B & 0 & 0 \\ \mu_A & -(\mu_A + \lambda_B) & 0 & \lambda_B & 0 \\ \mu_B & 0 & -(\mu_B + \lambda_A) & 0 & \lambda_A \\ 0 & 0 & \mu_A & -\mu_A & 0 \\ 0 & \mu_B & 0 & 0 & -\mu_B \end{pmatrix}$$

Example: modified generator matrix

$$\mathbf{Q}_N^T = \begin{pmatrix} -(\lambda_A + \lambda_B) & \mu_A & \mu_B & 0 & 0 \\ \lambda_A & -(\mu_A + \lambda_B) & 0 & 0 & \mu_B \\ \lambda_B & 0 & -(\mu_B + \lambda_A) & \mu_A & 0 \\ 0 & \lambda_B & 0 & -\mu_A & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Example: steady state probability distribution

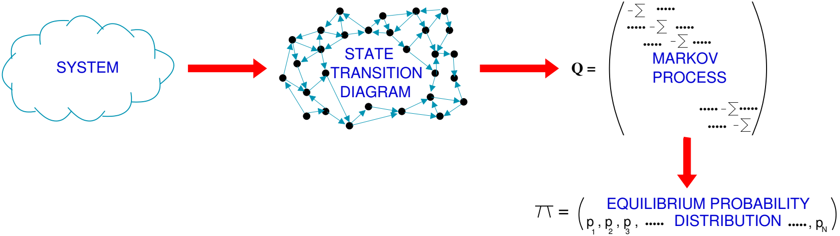
If we choose the following values for the parameters:

$$\lambda_A = 0.05 \qquad \lambda_B := 0.1 \qquad \mu_A = 0.02 \qquad \mu_B = 0.05$$

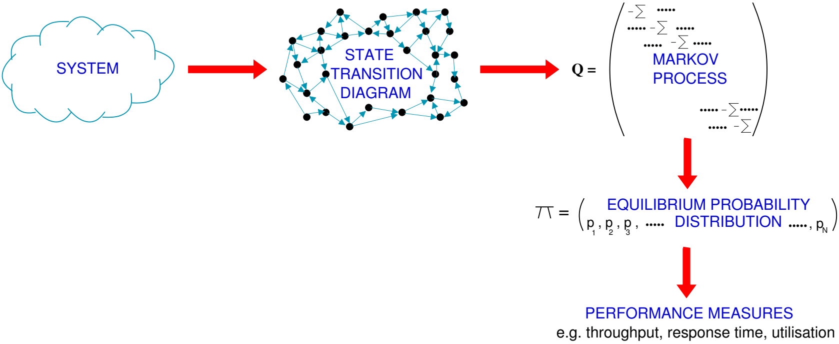
solving the matrix equation, and rounding figures to 4 significant figures, we obtain:

$$\boldsymbol{\pi} = (0.0693,\ 0.0990,\ 0.1683,\ 0.4951,\ 0.1683)$$

Deriving Performance Measures



Deriving Performance Measures



Deriving Performance Measures

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These different methods can be thought of as corresponding to different types of measure:

- state-based measures, e.g. utilisation;
- rate-based measures, e.g. throughput;
- other measures which fall outside the above categories, e.g. response time.

State-based measures

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If we consider the multiprocessor example, the utilisation of the common memory, U_{mem}, is the total probability that the model is in one of the states in which the common memory is in use:

$$U_{mem} = \boldsymbol{\pi_2} + \boldsymbol{\pi_3} + \boldsymbol{\pi_4} + \boldsymbol{\pi_5} = 93.07\%$$

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For example, if we consider jobs waiting for the common memory to be queued in that subsystem, then the average number of jobs in the common memory, N_{mem}, is:

$$N_{mem} = (1 \times \boldsymbol{\pi_2}) + (1 \times \boldsymbol{\pi_3}) + (2 \times \boldsymbol{\pi_4}) + (2 \times \boldsymbol{\pi_5}) = 1.594$$

Rate-based measures

Rate-based measures are those which correspond to the predicted rate at which some event occurs.

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This will be the product of the rate of the event, and the probability that the event is enabled, i.e. the probability of being in one of the states from which the event can occur.

Example: rate-based measures

In order to calculate the throughput of the common memory, we need the average number of accesses from either processor which it satisfies in unit time.

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X_{mem} is thus calculated as:

$$X_{mem} = (\mu_A \times (\boldsymbol{\pi_2} + \boldsymbol{\pi_4})) + (\mu_B \times (\boldsymbol{\pi_3} + \boldsymbol{\pi_5})) = 0.0287$$

or, approximately one access every 35 milliseconds.

Other measures

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For example, applying Little's Law to the common memory we see that

$$W_{mem} = N_{mem}/X_{mem} = 1.594/0.0287 = 55.54 \text{ milliseconds}$$

Assumptions

Stochastic Hypothesis

"The behaviour of a real system during a given period of time is characterised by the probability distributions of a stochastic process."

Assumptions

- All delays and inter-event times are exponentially distributed.

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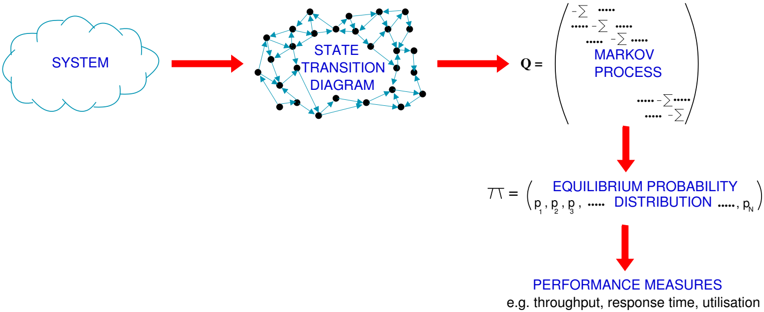
- The Markov/memoryless assumption that future behaviour is only dependent on the current state, not on the past history is a reasonable assumption for computer and communication systems, if we choose our states carefully.
- We generally assume that the Markov process is finite, time homogeneous and irreducible.

Difficulties of working with Markov processes



Whilst Markov process-based modelling has many advantages, working directly in terms of the state transition diagram or infinitesimal generator matrix is at best time-consuming and error prone, and often simply infeasible.

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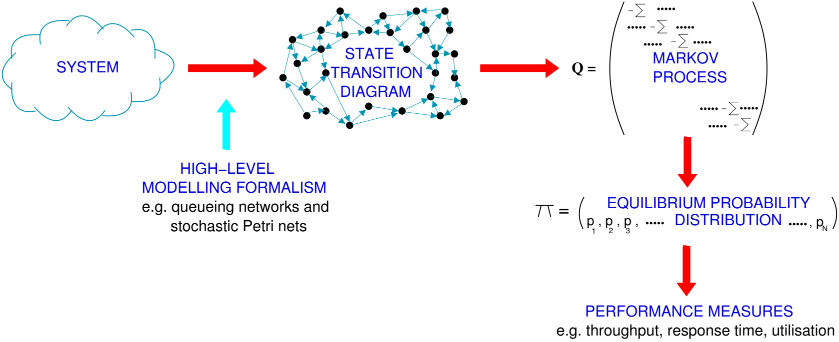

For this reason various high level modelling formalisms have been introduced to make the job of constructing the state transition diagram and/or infinitesimal generator matrix easier.

Outline

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- Performance Evaluation Process Algebra (PEPA) sought to address these problems by the introduction of a suitable process algebra.
- We have sought to investigate and exploit the interplay between the process algebra and the continuous time Markov chain (CTMC).

Performance Evaluation Process Algebra

- PEPA (Performance Evaluation Process Algebra) is a high-level modelling language for distributed systems. It can be used to develop models of existing systems (abstraction) or designs for proposed ones (specification).
- PEPA can capture performance information in a process algebra setting. It is a stochastic process algebra.
- For technical details the definitive reference is *A Compositional Approach to Performance Modelling*, Hillston, Cambridge University Press, 1996.

Strengths of stochastic process algebras

SPAs have strengths in the areas of semantic definition, inherent compositionality and the existence of important equivalence relations (including bisimulation). This relation provides the basis for aggregation of PEPA models.

Terminology

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To represent unimportant or unknown actions there is a distinguished action type, τ.

Quantitative aspects

Every activity in PEPA has an associated activity rate which may be any positive real number, or the distinguished symbol "$\top$", meaning unspecified, read as 'top'.

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Components and activities are primitives. PEPA also provides a small set of combinators.

PEPA syntax

$$
\begin{array}{lcll}
S & ::= & (\alpha, r).S & \text{(prefix)} \\
  & | & S_1 + S_2 & \text{(choice)} \\
  & | & X & \text{(variable)} \\
C & ::= & C_1 \underset{L}{\bowtie} C_2 & \text{(cooperation)} \\
  & | & C \,/\, L & \text{(hiding)} \\
  & | & S & \text{(sequential)}
\end{array}
$$

PEPA: informal semantics (sequential sublanguage)

$(\alpha, r).S$

The activity (α, r) takes time Δt (drawn from the exponential distribution with parameter r).

$S_1 + S_2$

In this choice either S_1 or S_2 will complete an activity first. The other is discarded.

PEPA: informal semantics (combinators)

$C_1 \underset{L}{\bowtie} C_2$

All activities of C_1 and C_2 with types in L are shared: others remain individual.
NOTATION: write $C_1 \parallel C_2$ if L is empty.

$C \,/\, L$

Activities of C with types in L are hidden (τ type activities) to be thought of as internal delays.

Example: M/M/1/N/N queue

$$
\begin{aligned}
Arrival_0 &\stackrel{def}{=} (accept, \lambda).Arrival_1 \\
Arrival_i &\stackrel{def}{=} (accept, \lambda).Arrival_{i+1} + (serve, \top).Arrival_{i-1} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 < i < N) \\
Arrival_N &\stackrel{def}{=} (serve, \top).Arrival_{N-1} \\
Server &\stackrel{def}{=} (serve, \mu).Server
\end{aligned}
$$

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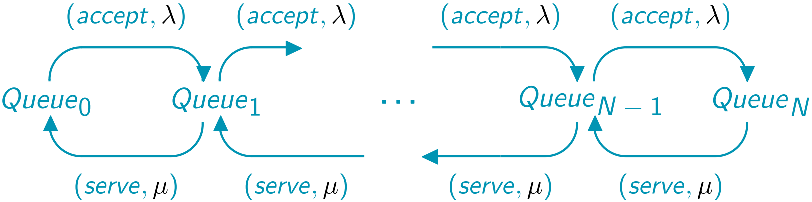

$$Queue_i \equiv Arrival_i \underset{\{serve\}}{\bowtie} Server$$