SPAs for performance modelling: Lecture 6 — Collective Dynamics

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of EDINBURGH

Outline

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Introduction

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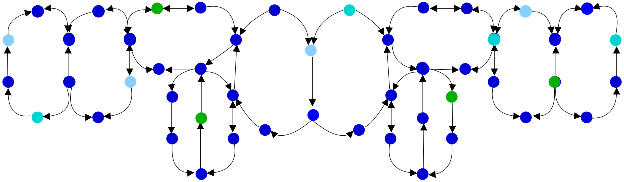
- Here we reconsider the approach with a particular focus on the types of systems that it is well-suited for: systems with collective dynamics.
- Moreover we give a more formal derivation of the system of ordinary differential equations that are used to approximate the discrete event system we are interested in.

Collective Dynamics

The behaviour of many systems can be interpreted as the result of the collective behaviour of a large number of interacting entities.



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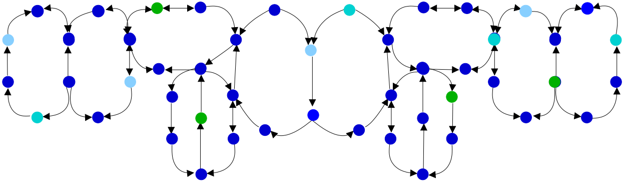

For such systems we are often as interested in the population level behaviour as we are in the behaviour of the individual entities.

Collective Behaviour

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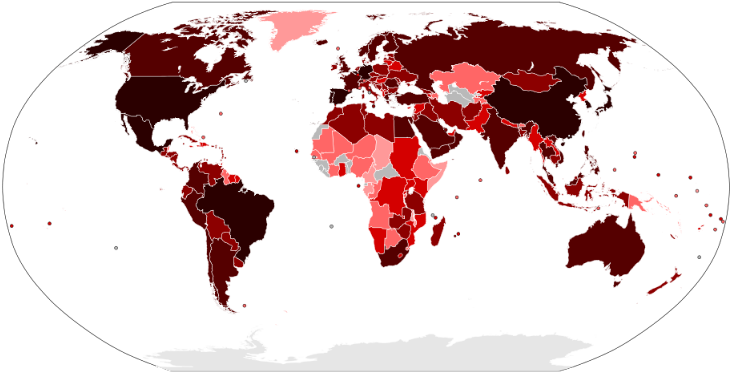
This is also true in the man-made and engineered world:



Spread of H1N1 virus in 2009

Collective Behaviour

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Love Parade, Germany 2006

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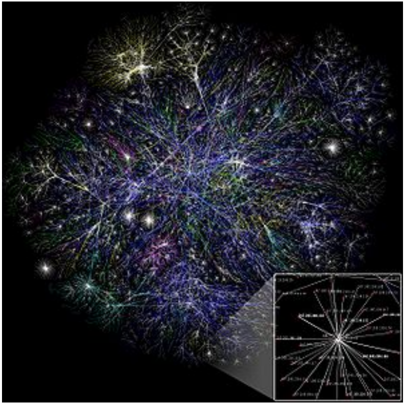

Map of the Internet 2009

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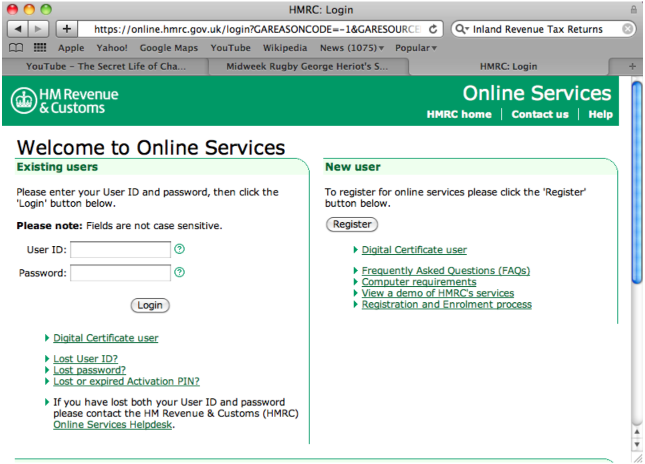

Self assessment tax returns 31st January each year

Process Algebra and Collective Dynamics

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- Incorporate formal apparatus for reasoning about the behaviour of systems;
- Stochastic extensions, such as PEPA, enable quantified behaviour of the dynamics of systems.

In the CODA project we are developing stochastic process algebras and associated theory, tailored to the construction and evaluation of the collective dynamics of large systems of interacting entities.

Performance as an emergent behaviour

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This allows us to model much larger systems than previously possible but in making the shift we are no longer able to collect any information about individuals in the system.

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Example Service Level Agreement

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Qualitative Service Level Agreement

Less than 1% of the responses received within 3 seconds will read "System is overloaded, try again later".

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Large scale software systems
Issues of scalability are important for user satisfaction and resource efficiency but such issues are difficult to investigate using discrete state models.

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Biochemical signalling pathways
Understanding these pathways has the potential to improve the quality of life through enhanced drug treatment and better drug design.

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Epidemiological systems
Improved modelling of these systems could lead to improved disease prevention and treatment in nature and better security in computer systems.

Outline

Solving discrete state models

Under the SOS semantics a PEPA model is mapped to a Continuous Time Markov Chain (CTMC) with global states determined by the local states of all the participating components.

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Alternatively they may be studied using stochastic simulation. Each run generates a single trajectory through the state space. Many runs are needed in order to obtain average behaviours.

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

Continuous Approximation

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Use ordinary differential equations to represent the evolution of those variables over time.

New mathematical structures: differential equations

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Appropriate for models in which there are large numbers of components of the same type, i.e. models of populations and situations of collective dynamics.

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We seek to take advantage of Kurtz's Theorem from the 1970's which gives conditions under which a sequence of population Markov chains converges to a deterministic behaviour (within a given time horizon), i.e. $\forall t < T$ as $N \longrightarrow \infty$.

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For example, a model $P[X_P] \underset{L}{\bowtie} Q[X_Q]$, is scaled as $P[n \times X_P] \underset{L}{\bowtie} Q[n \times X_Q]$ for increasing n.

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In the literature this is sometimes called the occupancy measure.

Scaling Conditions

- We have a sequence $\mathbf{X}^{(N)}$ of population CTMCs, for increasing total population N.

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- We assume that each transition in the Markov chain is characterised by an update vector $\mathbf{v}$ (cf. the columns of the activity matrix)
- For each such transition τ, the normalized update is $\bar{\mathbf{v}} = \mathbf{v}/N$ and the rate function is $\bar{r}_\tau(\overline{\mathbf{X}}^{(N)}) = Nf_\tau(\overline{\mathbf{X}}^{(N)})$ (density dependence).

Fluid ODE

For a sequence of population CTMCs that satisfy these conditions we can define the Fluid ODE:

Fluid ODE

The fluid ODE is $\dot{\mathbf{x}} = F(\mathbf{x})$, where

$$F(\mathbf{x}) = \sum_{\tau \in \mathcal{T}} \mathbf{v}_\tau f_\tau(\mathbf{x})$$

Fluid approximation theorem

Hypothesis

- $\overline{\mathbf{X}}^{(N)}(t)$: a sequence of normalized population CTMC, residing in $E \subset \mathbb{R}^n$
- $\exists \mathbf{x_0} \in S$ such that $\overline{\mathbf{X}}^{(N)}(0) \to \mathbf{x_0}$ in probability (initial conditions)
- $\mathbf{x}(t)$: solution of $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = F(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x_0}$, residing in E.

Theorem

For any finite time horizon $T < \infty$, it holds that:

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} ||\overline{\mathbf{X}}^{(N)}(t) - \mathbf{x}(t)|| > \varepsilon \right) \to 0.$$

T.G.Kurtz. *Solutions of ordinary differential equations as limits of pure jump Markov processes.* Journal of Applied Probability, 1970.

Simple example revisited

$$
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Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{def}{=} (task2, r_2).Proc_0 \\
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CTMC interpretation

Processors (N_P)	Resources (N_R)	States ($2^{N_P+N_R}$)
1	1	4
2	1	8
2	2	16
3	2	32
3	3	64
4	3	128
4	4	256
5	4	512
5	5	1024
6	5	2048
6	6	4096
7	6	8192
7	7	16384
8	7	32768
8	8	65536
9	8	131072
9	9	262144
10	9	524288
10	10	1048576

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ODE interpretation

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\begin{aligned}
\frac{dx_1}{dt} &= -r_1 \min(x_1, x_3) + r_2\, x_1 \\
&\quad x_1 = \text{no. of } Proc_1 \\
\frac{dx_2}{dt} &= r_1 \min(x_1, x_3) - r_2\, x_1 \\
&\quad x_2 = \text{no. of } Proc_2 \\
\frac{dx_3}{dt} &= -r_1 \min(x_1, x_3) + r_4\, x_4 \\
&\quad x_3 = \text{no. of } Res_0 \\
\frac{dx_4}{dt} &= r_1 \min(x_1, x_3) - r_4\, x_4 \\
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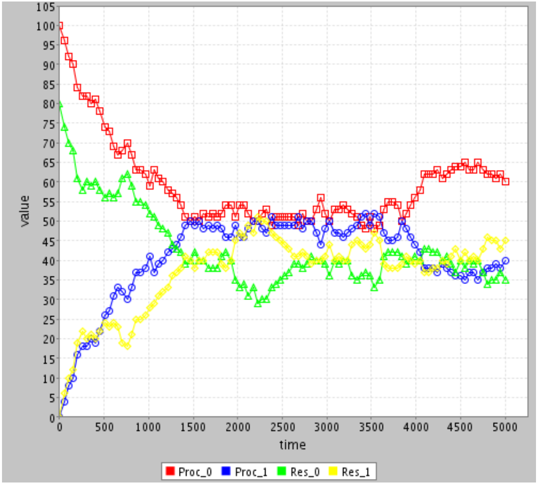
100 processors and 80 resources (simulation run A)



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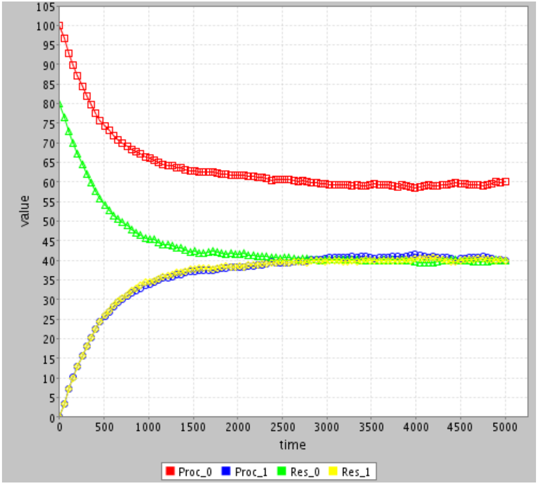

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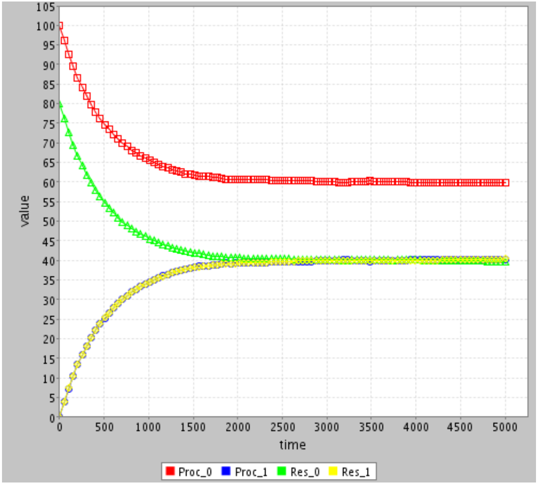
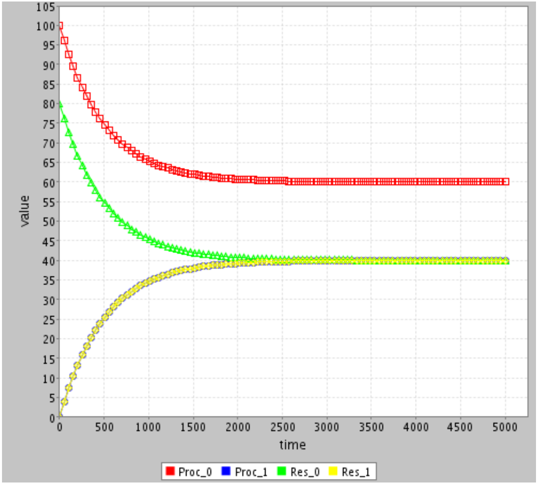


100 processors and 80 resources (ODE solution)



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Deriving a Fluid Approximation of a PEPA model

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SPA MODEL $\xrightarrow{\text{SOS rules}}$ LABELLED TRANSITION SYSTEM $\xrightarrow[\text{diagram}]{\text{state transition}}$ CTMC $\mathbf{Q}$

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Generating functions

- The population-based semantics gives rise to generating functions, denoted by $f_\alpha(\xi, l)$, giving the rate at which an activity of type α is executed, and the state change due to its execution as a vector l.

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- Thus in the previous example the shared action `task1` is captured by the function

$$f_{task1}\big(\xi, (-1, 1, -1, 1)\big) = \min(r\,\xi_{Proc_0}, r\,\xi_{Res_0}),$$

`task1` decreases the population counts of $Proc_0$ and Res_0 and, correspondingly, increases the population counts of $Proc_1$ and Res_1 at a rate which is dependent upon the current state.

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- This is just as we saw with the activity matrix construction but now obtained through SOS rules.

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2. Collect the transitions of the reduced context (Jump Multiset)
3. Calculate the rate of the transitions in terms of an arbitrary state of the CTMC.

Once this is done we can extract the vector field $F_{\mathcal{M}}(x)$ from the jump multiset.

Semantics by example

In these slides I will illustrate this approach to the scalable semantics using the previous Processor-Resource example.

But the full SOS rules can be found in the paper:

Tribastone M, Gilmore S, Hillston J,
Scalable Differential Analysis of Process Algebra Models
Transactions on Software Engineering 38(1), 2012

Context Reduction

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System &\stackrel{\text{def}}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
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$$\Downarrow$$

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Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

Location Dependency

$$System \stackrel{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

Location Dependency

$$System \stackrel{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

$$\Downarrow$$

$$\{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\} \parallel \{Proc_0, Proc_1\}$$

Location Dependency

$$System \stackrel{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

$$\Downarrow$$

$$\{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\} \parallel \{Proc_0, Proc_1\}$$

Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
\mathit{Proc_0} &\stackrel{\mathit{def}}{=} (\mathit{task1}, r_1).\mathit{Proc_1} \\
\mathit{Proc_1} &\stackrel{\mathit{def}}{=} (\mathit{task2}, r_2).\mathit{Proc_0} \\
\mathit{Res_0} &\stackrel{\mathit{def}}{=} (\mathit{task1}, r_3).\mathit{Res_1} \\
\mathit{Res_1} &\stackrel{\mathit{def}}{=} (\mathit{reset}, r_4).\mathit{Res_0} \\
\mathit{System} &\stackrel{\mathit{def}}{=} \mathit{Proc_0}[N_P] \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0}[N_R]
\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\stackrel{def}{=} (task1, r_3).Res_1 \\
Res_1 &\stackrel{def}{=} (reset, r_4).Res_0 \\
System &\stackrel{def}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\frac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1}$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\stackrel{def}{=} (task1, r_3).Res_1 \\
Res_1 &\stackrel{def}{=} (reset, r_4).Res_0 \\
System &\stackrel{def}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$
\frac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{}
$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
\mathit{Proc_0} &\stackrel{\mathit{def}}{=} (\mathit{task1}, r_1).\mathit{Proc_1} \\
\mathit{Proc_1} &\stackrel{\mathit{def}}{=} (\mathit{task2}, r_2).\mathit{Proc_0} \\
\mathit{Res_0} &\stackrel{\mathit{def}}{=} (\mathit{task1}, r_3).\mathit{Res_1} \\
\mathit{Res_1} &\stackrel{\mathit{def}}{=} (\mathit{reset}, r_4).\mathit{Res_0} \\
\mathit{System} &\stackrel{\mathit{def}}{=} \mathit{Proc_0}[N_P] \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0}[N_R]
\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$
\dfrac{\dfrac{\mathit{Proc_0} \xrightarrow{\mathit{task1}, r_1} \mathit{Proc_1}}{\mathit{Proc_0} \xrightarrow{\mathit{task1}, r_1 \xi_1}_{*} \mathit{Proc_1}} \qquad \dfrac{\mathit{Res_0} \xrightarrow{\mathit{task1}, r_3} \mathit{Res_1}}{\mathit{Res_0} \xrightarrow{\mathit{task1}, r_3 \xi_3}_{*} \mathit{Res_1}}}{\mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0} \xrightarrow{\mathit{task1}, r(\xi)}_{*} \mathit{Proc_1} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_1}}
$$

Apparent Rate Calculation

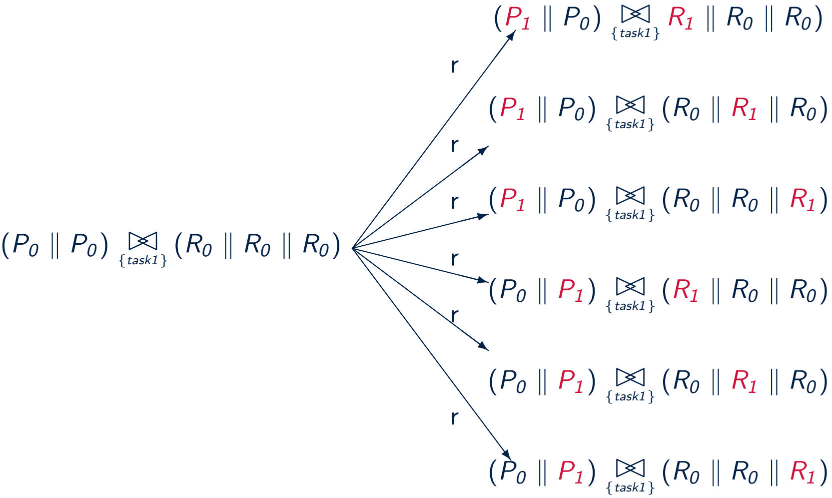
$$\dfrac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

Apparent Rate Calculation

$$\dfrac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

$$\begin{aligned} r(\xi) &= \frac{r_1 \xi_1}{r^*_{task1}\left(Proc_0, \xi\right)} \; \frac{r_3 \xi_4}{r^*_{task1}\left(Res_0, \xi\right)} \min\left(r^*_{task1}\left(Proc_0, \xi\right), r^*_{task1}\left(Res_0, \xi\right)\right) \\ &= \frac{r_1 \xi_1}{r_1 \xi_1} \; \frac{r_3 \xi_3}{r_3 \xi_3} \min\left(r_1 \xi_1, r_3 \xi_3\right) \\ &= \min\left(r_1 \xi_1, r_3 \xi_3\right) \end{aligned}$$

$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC



$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \xrightarrow{r} (P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} R_1 \parallel R_0 \parallel R_0)$$

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \xrightarrow{r} (P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_1 \parallel R_0)$$

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \xrightarrow{r} (P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_1)$$

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \xrightarrow{r} (P_0 \parallel P_1) \underset{\{task1\}}{\bowtie} (R_1 \parallel R_0 \parallel R_0)$$

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$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \xrightarrow{r} (P_0 \parallel P_1) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_1)$$

$$r = \frac{r_1}{2r_1}\frac{r_3}{3r_3}\min(2r_1, 3r_3) = \frac{1}{6}\min(2r_1, 3r_3)$$

$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0)$$

$\xrightarrow{r}$ $(P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} R_1 \parallel R_0 \parallel R_0)$

$\xrightarrow{r}$ $(P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_1 \parallel R_0)$

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$\xrightarrow{r}$ $(P_0 \parallel P_1) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_1)$

$$r = \frac{r_1}{2r_1} \frac{r_3}{3r_3} \min(2r_1, 3r_3) = \frac{1}{6} \min(2r_1, 3r_3)$$

$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(2,0,3,0) \xrightarrow{\min(2r_1, 3r_3)} (1,1,2,1)$$

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0) \longrightarrow$$

$$(P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_1 \parallel R_0 \parallel R_0)$$

$$(P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_1 \parallel R_0)$$

$$(P_1 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_1)$$

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$$(P_0 \parallel P_1) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_1)$$

$$r = \frac{r_1}{2r_1}\frac{r_3}{3r_3}\min(2r_1, 3r_3) = \frac{1}{6}\min(2r_1, 3r_3)$$

Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1 \xi_1, r_3 \xi_3\right)$$

Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Equivalent Transitions

Some transitions may give the same information:

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$
$$Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\xi_4 r_4}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_0$$

i.e., Res_1 may perform an action independently from the rest of the system.

This is captured by the procedure used for the construction of the generator function $f(\xi, l, \alpha)$

Construction of $f(\xi, I, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $I = (0, 0, 0, 0)$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$
- Add -1 to all elements of l corresponding to the indices of the components in the lhs of the transition

$$l = (-1, 0, 0, -1)$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$
- Add -1 to all elements of l corresponding to the indices of the components in the lhs of the transition

$$l = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of l corresponding to the indices of the components in the rhs of the transition

$$l = (-1 + 1, 0, +1, -1) = (0, 0, +1, -1)$$

Construction of $f(\xi, I, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $I = (0, 0, 0, 0)$
- Add -1 to all elements of I corresponding to the indices of the components in the lhs of the transition

$$I = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of I corresponding to the indices of the components in the rhs of the transition

$$I = (-1 + 1, 0, +1, -1) = (0, 0, +1, -1)$$

$$f\left(\xi, (0, 0, +1, -1), reset\right) = \xi_4 r_4$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2'}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

$$f(\xi, (+1, -1, 0, 0), task2) = \xi_2 r_2$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2'}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

$$f(\xi, (+1, -1, 0, 0), task2) = \xi_2 r_2$$

$$f(\xi, (0, 0, +1, -1), reset) = \xi_4 r_4$$

Capturing behaviour in the Generator Function

$$
\begin{aligned}
Proc_0 &\stackrel{\text{def}}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{\text{def}}{=} (task2, r_2).Proc_0 \\
Res_0 &\stackrel{\text{def}}{=} (task1, r_3).Res_1 \\
Res_1 &\stackrel{\text{def}}{=} (reset, r_4).Res_0 \\
System &\stackrel{\text{def}}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

Capturing behaviour in the Generator Function

$$
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Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
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Res_1 &\stackrel{def}{=} (reset, r_4).Res_0 \\
System &\stackrel{def}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \text{ and } \xi_3 + \xi_4 = N_R$$

Capturing behaviour in the Generator Function

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Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
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Res_1 &\stackrel{def}{=} (reset, r_4).Res_0 \\
System &\stackrel{def}{=} Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]
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Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \text{ and } \xi_3 + \xi_4 = N_R$$

Generator Function

$$f(\xi, l, \alpha): \quad \begin{aligned}
f(\xi, (-1, 1, -1, 1), task1) &= \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1, -1, 0, 0), task2) &= r_2\xi_2 \\
f(\xi, (0, 0, 1, -1), reset) &= r_4\xi_4
\end{aligned}$$

Extraction of the ODE from f

Generator Function

$$
\begin{aligned}
f(\xi, (-1,1,-1,1), \mathit{task1}) &= \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1,-1,0,0), \mathit{task2}) &= r_2\xi_2 \\
f(\xi, (0,0,1,-1), \mathit{reset}) &= r_4\xi_4
\end{aligned}
$$

Differential Equations

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} = F_{\mathcal{M}}(x) &= \sum_{l\in\mathbb{Z}^d} l \sum_{\alpha\in\mathcal{A}} f(x, l, \alpha) \\
&= (-1,1,-1,1)\min(r_1x_1, r_3x_3) + (1,-1,0,0)r_2x_2 \\
&\quad + (0,0,1,-1)r_4x_4
\end{aligned}
$$

Extraction of the ODE from f

Generator Function

$$
\begin{aligned}
f(\xi, (-1,1,-1,1), \mathit{task1}) &= \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1,-1,0,0), \mathit{task2}) &= r_2\xi_2 \\
f(\xi, (0,0,1,-1), \mathit{reset}) &= r_4\xi_4
\end{aligned}
$$

Differential Equations

$$
\begin{aligned}
\frac{dx_1}{dt} &= -\min(r_1x_1, r_3x_3) + r_2x_2 \\
\frac{dx_2}{dt} &= \min(r_1x_1, r_3x_3) - r_2x_2 \\
\frac{dx_3}{dt} &= -\min(r_1x_1, r_3x_3) + r_4x_4 \\
\frac{dx_4}{dt} &= \min(r_1x_1, r_3x_3) - r_4x_4
\end{aligned}
$$

Density Dependence

Density dependence of parametric apparent rates

Let $r^{*}_{\alpha}(P, \xi)$ be the parametric apparent rate of action type α in process P. For any $n \in \mathbb{N}$ and $\alpha \in \mathcal{A}$,

$$r^{*}_{\alpha}(P, \xi) = n \cdot r^{\star}_{\alpha}(P, \xi / n)$$

Density Dependence

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$$r^{*}_{\alpha}(P, \xi) = n \cdot r^{\star}_{\alpha}(P, \xi/n)$$

Density dependence of parametric transition rates

If $P \xrightarrow{(\alpha, r(\xi))}_{*} Q$ then, for any $n \in \mathbb{N}$, $r(\xi) = n \cdot r(\xi/n)$

Generating functions give rise to density dependent rates

Generating functions give rise to density dependent rates

Let $\mathcal{M}$ be a PEPA model with generating functions $f(\xi, l, \alpha)$ derived as demonstrated. Then the corresponding sequence of CTMCs will be density dependent.

Lipschitz continuity

Since Lipschitz continuity is preserved by summation, in order to verify that the vector field $F_{\mathcal{M}}(x)$ is Lipschitz it suffices to prove that any parametric rate generated by the semantics is Lipschitz.

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Lipschitz continuity of parametric apparent rates

Let $r^*_\alpha(P, \xi)$ be the parametric apparent rate of action type α in process P. There exists a constant $L \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d, x \neq y$,

$$\frac{\| r^\star_\alpha(P, x) - r^\star_\alpha(P, y) \|}{\| x - y \|} \leq L$$

Lipschitz continuity

Since Lipschitz continuity is preserved by summation, in order to verify that the vector field $F_{\mathcal{M}}(x)$ is Lipschitz it suffices to prove that any parametric rate generated by the semantics is Lipschitz.

Lipschitz continuity of parametric apparent rates

Let $r_{\alpha}^{*}(P, \xi)$ be the parametric apparent rate of action type α in process P. There exists a constant $L \in \mathbb{R}$ such that for all $x, y \in \mathbb{R}^d, x \neq y$,

$$\frac{\| r_{\alpha}^{\star}(P, x) - r_{\alpha}^{\star}(P, y) \|}{\| x - y \|} \leq L$$

Lipschitz continuity of rate functions

If $P \xrightarrow{(\alpha, r(x))}_{*} P'$ then $r(x) \leq r_{\alpha}^{*}(P, x)$ and thus it follows that $r(x)$ is Lipschitz continuous.

Kurtz's Theorem

Kurtz's Theorem for PEPA

Let $x(t), 0 \leq t \leq T$ satisfy the initial value problem $\frac{dx}{dt} = F(x(t)),\ x(0) = \delta$, specified from a PEPA model.

Let $\{X_n(t)\}$ be a family of CTMCs with parameter $n \in \mathbb{N}$ generated as explained and let $X_n(0) = n \cdot \delta$. Then,

$$\forall \varepsilon > 0 \lim_{n \to \infty} \mathbb{P}\left(\sup_{t \leq T} \| X_n(t)/n - x(t) \| > \varepsilon \right) = 0.$$

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Let $\{X_n(t)\}$ be a family of CTMCs with parameter $n \in \mathbb{N}$ generated as explained and let $X_n(0) = n \cdot \delta$. Then,

$$\forall \varepsilon > 0 \lim_{n \to \infty} \mathbb{P} \left(\sup_{t \leq T} \| X_n(t)/n - x(t) \| > \varepsilon \right) = 0.$$

Moreover Lipschitz continuity of the vector field guarantees existence and uniqueness of the solution to the initial value problem. This allows the time horizon to be extended to ∞.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

A note about passive actions

The scalable semantics which have been presented do not have rules for passive actions.

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The scalable semantics which have been presented do not have rules for passive actions.

The reason is that the passive partner within a model will act as a switch.

- When it is present in any number the rate of the action will proceed at the rate determined by the activity rate and the population of the other partner.
- When it is absent the rate will become zero.

A note about passive actions

The scalable semantics which have been presented do not have rules for passive actions.

The reason is that the passive partner within a model will act as a switch.

- When it is present in any number the rate of the action will proceed at the rate determined by the activity rate and the population of the other partner.
- When it is absent the rate will become zero.

This will cause a discontinuity in the rate of the activity meaning that the Lipschitz continuity condition required to apply Kurtz's theorem will no longer hold.

Outline

Internet worms: Background

- Internet worms are malicious programs that exploit operating system security weaknesses to propagate themselves.

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- Internet worms are malicious programs that exploit operating system security weaknesses to propagate themselves.
- While the security flaws go unpatched, the worm spreads epidemic-like and uses large amounts of available bandwidth.
- Far more destructive is the worms' effect on the Internet routing infrastructure, as the worms tend to overload the connecting routers with nonexistent IP lookups.
- Worms like Nimbda, Slammer, Code Red, Sasser and Code Red 2 have caused the Internet to become unusable for many hours at a time until security patches could be applied and routers fixed.

Internet worms: Background

- Internet worms are malicious programs that exploit operating system security weaknesses to propagate themselves.
- While the security flaws go unpatched, the worm spreads epidemic-like and uses large amounts of available bandwidth.
- Far more destructive is the worms' effect on the Internet routing infrastructure, as the worms tend to overload the connecting routers with nonexistent IP lookups.
- Worms like Nimbda, Slammer, Code Red, Sasser and Code Red 2 have caused the Internet to become unusable for many hours at a time until security patches could be applied and routers fixed.
- The estimated cost of computer worms and related activities is at least $50 billion a year.

An Internet-scale Problem

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- We wish to study the emergent behaviour of Internet worms as they spread to thousands and then hundreds-of-thousands of hosts.
- Explicit state-based methods for calculating steady-state, transient or passage-time measures are limited to state-spaces of the order of 10^9.
- By transforming our stochastic process algebra model into a set of ODEs, we can obtain a plot of model behaviour against time for models with global state spaces in excess of 10^{10000} states.

Susceptible-Infective-Removed (SIR) model

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Susceptible-Infective-Removed over a network

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- This state is termed removed and is an absorbing state for that component in the system.

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- Additionally, an attempted network connection can fail or timeout as indicated by the *fail* action.
- This might be due to network contention or the lack of availability of a susceptible machine to infect.
- As large scale worm infections tend not to waste time determining whether a given host is already infected or not, we assume that a certain number of infections will attempt to reinfect hosts; in this instance, the host is unaffected.

Susceptible-Infective-Removed over a network

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\begin{aligned}
S &\stackrel{def}{=} (infectS, \beta).I \\
I &\stackrel{def}{=} (infectI, \beta).I + (infectS, \beta).I + (patch, \gamma).R \\
R &\stackrel{def}{=} Stop
\end{aligned}
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Net &\stackrel{def}{=} (infectI, \beta).Net' \\
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Sys \stackrel{def}{=} (S[N] \parallel I) \underset{L}{\bowtie} Net[M] \quad \text{where } L = \{ infectI, infectS \}
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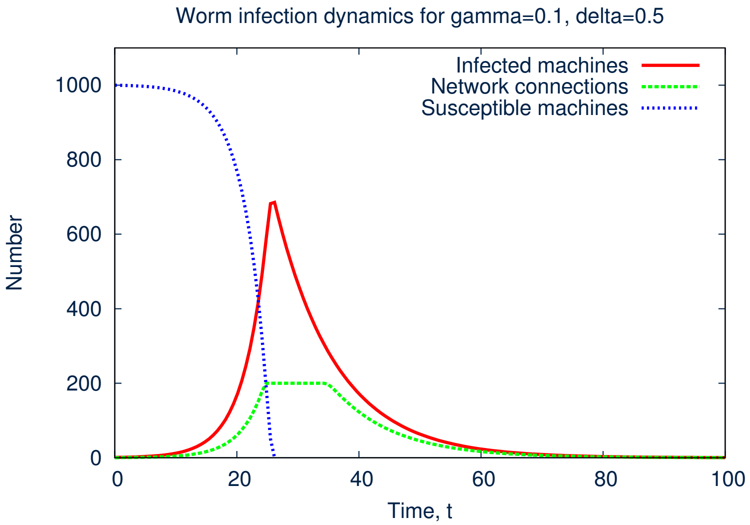
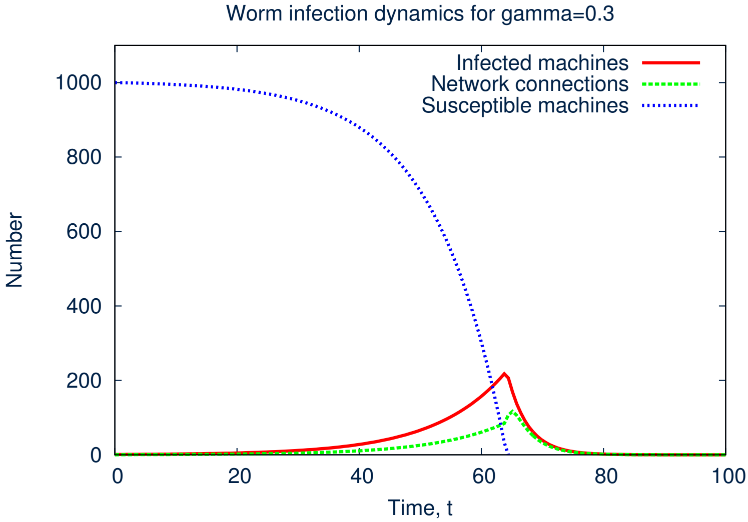
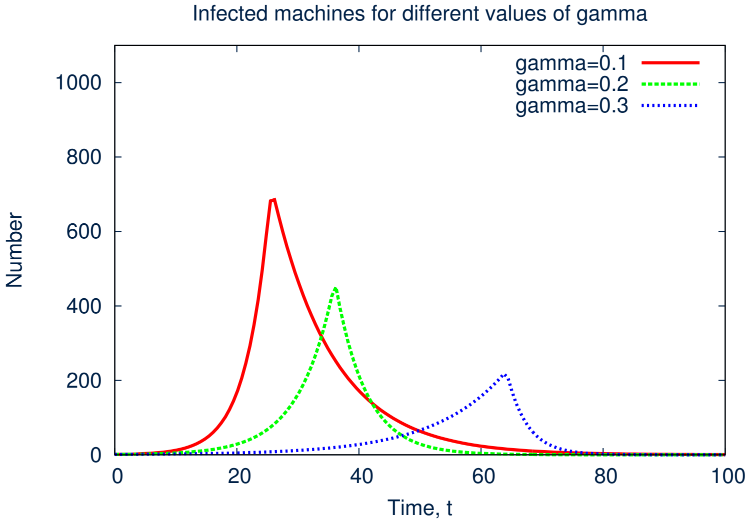
Patch rate $\gamma = 0.1$. Connection failure rate $\delta = 0.5$



Patch rate $\gamma = 0.3$. Connection failure rate $\delta = 0.5$



Increasing machine patch rate γ from 0.1 to 0.3



Susceptible-Infective-Removed-Reinfection (SIRR) model

- As with the SIR model, we constrain infection to occur over a limited network resource, constrained by the number of independent network connections in the system, M.

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- As with the SIR model, we constrain infection to occur over a limited network resource, constrained by the number of independent network connections in the system, M.
- A small modification in the process model of infection allows for removed computers to become susceptible again after a delay.
- We use this to model a faulty or incomplete security upgrade or the mistaken removal of security patches which had previously defended the machine against attack.

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Sys \stackrel{def}{=} (S[1000] \parallel I) \underset{L}{\bowtie} Net[M] \quad \text{where } L = \{infectI, infectS\}
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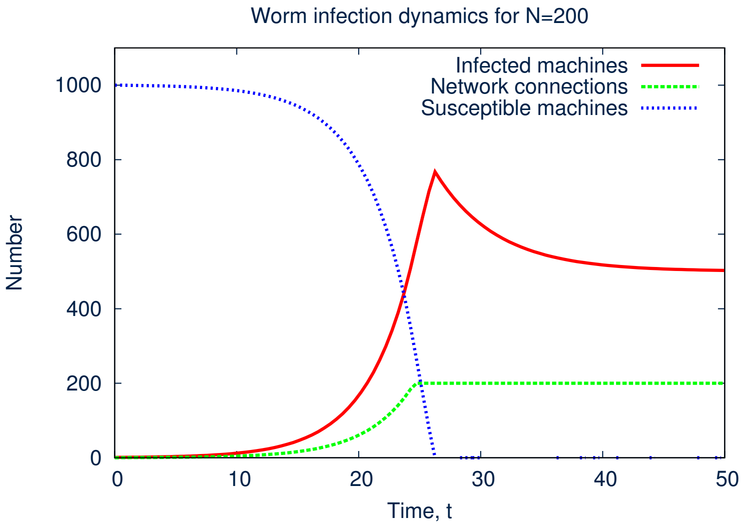
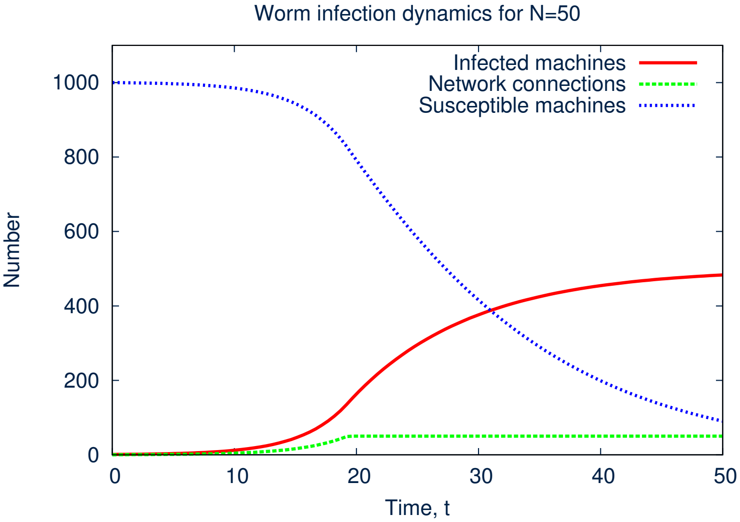
Unsecured SIR model (200 network channels)



Unsecured SIR model (50 network channels)



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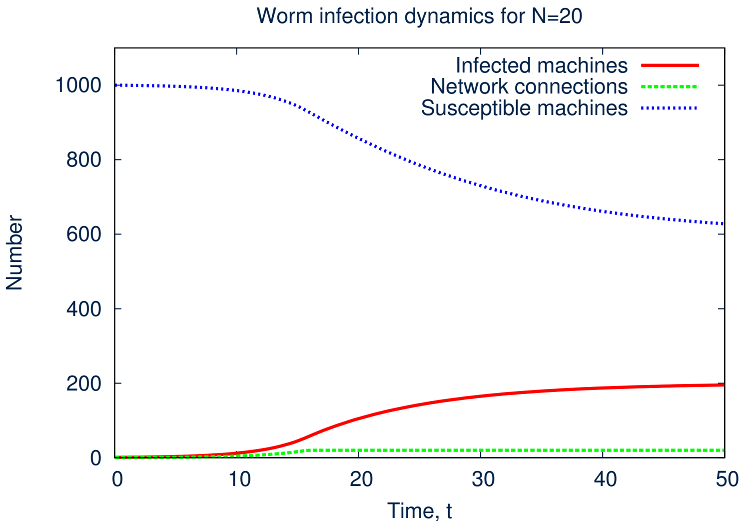

Conclusions

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- The scale of the effects of Internet worms defeats attempts to model their behaviour in very close detail.
- Process algebra modelling allows the details of interactions to be recorded on the individual level but then abstracted away into appropriate population-based representations.
- The scale of problems which can be modelled in this way vastly exceeds those which are founded on explicit state representations.