Higher-Order Computability: Errata, Omissions and Updates

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> *'One should not want to be so like a god as not to have to correct something here and there in one's created works.'*
>
> — Ludwig van Beethoven, Letter 199.

This document contains an up-to-date list of known errata in the book *Higher-Order Computability* by John Longley and Dag Normann (published in 2015 by Springer-Verlag as part of the *Computability in Europe* series). It also lists a few omissions and other updates relating to the book's content.

Thanks are due to Alex Kavvos for spotting many of the listed errors in Chapters 1 and 3.

If you discover any suspected errors not listed here, or have made interesting progress in the subject area covered by the book, please contact John Longley at: `jrl@staffmail.ed.ac.uk` .

Errata

- Page 14, line 13: $\mathbf{T}_1 \rightarrow \mathbf{T}_2$ should read $\mathbf{T}_1 \longrightarrow\!\!\!\rhd\, \mathbf{T}_2$.
- Page 15, line 20: $\mathbf{T}_3 \rightarrow \mathsf{E}$ should read $\mathbf{T}_3 \longrightarrow\!\!\!\rhd\, \mathsf{E}$.
- Page 53, line 2 from bottom: 'Section 3.4' should read 'Example 3.5.1'.
- Page 55, bottom of page. The list of closure properties should include one further item:

  For any $X \in {<}B{>}$, the identity function $X \rightarrow X$ is computable,
- Page 56, line 12: $\mathsf{K}(B, C)$ should read $\mathsf{K}(B; C)$.
- Page 57, lines 15–16: 'a Turing machine that accepts' should read 'a Turing machine U that accepts'.
- Page 60: Some amendments to the proof of Theorem 3.1.15 are necessary:
  - Lines 2–3: the phrase 'with a weak terminal (I, i)' should be applied to $\mathbf{C}$ rather than $\mathbf{D}$.
  - Lines 7–9: this sentence (beginning 'Conversely') should be replaced by the following text:

Conversely, given $f : A \rightharpoonup B$ and $\widehat{f} \in \mathbf{C}[I, A \Rightarrow B]$ with $\widehat{f}(i)\downarrow$ and $\widehat{f}(i) \cdot a \simeq f(a)$ for all a, take $\widehat{g} = f \circ (\Lambda x.i) \in \mathbf{C}[I, A \Rightarrow B] = \mathbf{D}[I, A \Rightarrow B]$ so that $\widehat{g}$ is total and represents $g = \Lambda(x,a).f(a) : I \times A \rightharpoonup B$. Now let $\overline{g} \in \mathbf{D}[I \bowtie A, B]$ also represent g as in Definition 3.1.8. Then $\overline{g} \circ \langle \Lambda a.i, id_A \rangle \in \mathbf{D}[A, B] = \mathbf{C}[A, B]$, and it is routine to check that $\overline{g} \circ \langle \Lambda a.i, id_A \rangle = f$.

  - Lines 11-12: $k' \circ \langle \Lambda a.i, id_A \rangle$ should read:

    $k' \circ \pi_A$ (where $\pi_A \in \mathbf{D}[I \bowtie A, A]$)

  Also, $\widehat{k}$ would be a better notation than $\widehat{k'}$ in these lines.

- Page 62, line 8 from end: $\check{f}' \in \mathbf{C}[A \Rightarrow B]$ should read $\check{f}' \in \mathbf{C}[A, B]$.

- Page 63, lines 14–15. This should read '…condition 1 of Definition 3.1.13 is satisfied, and conditions 2 and 3 are immediate …'.

- Page 67, line 2: the phrase 'the models $\mathsf{K}(B; C)$ are examples' is spurious and should be deleted.

- Page 83, bottom of page. The particular definition of λ^* given here is not suitable for use in the partial setting: if $a \cdot b \uparrow$ then we will not have $[\![\lambda x.ab]\!]\downarrow$. The second clause in the definition should instead read: $\lambda^* x^\sigma.a = k_{\tau\sigma} a$ for each $a \in \mathbf{A}^\sharp(\tau)$. This change entails a reworking of Proposition 3.3.5 (see next erratum).

- Page 84, bottom of page. In view of the necessary change to the definition of λ^* on page 83, Proposition 3.3.5 must be weakened considerably. In the sentence preceding the proposition, one should insert 'certain' before 'β-reductions', and the statement of the proposition should now read:

  (i) If M is a meta-expression, x is a variable and a is a constant or variable, then $[\![(\lambda x.M)a)^\dagger]\!]_\nu \succeq [\![M[x \mapsto a]^\dagger]\!]_\nu$.
  (ii) If M, N are meta-expressions, $x \notin \mathrm{FV}(N)$, no free occurrence of x in M occurs under a λ, and $[\![N^\dagger]\!]_\nu\downarrow$, then $[\![(\lambda x.M)N)^\dagger]\!]_\nu \succeq [\![M[x \mapsto N]^\dagger]\!]_\nu$.

  Fortunately, the above restricted kinds of β-reduction are sufficient to cover all β-reductions on meta-expressions elsewhere in the book.

  The proof of the proposition (top of page 85) obviously also needs to be replaced. We here refer the reader to Longley's PhD thesis (Theorem 1.1.9) for the easy proof.

- Page 85, line 4: the last sentence of the proof may be improved by adding 'along with the compositionality of $[\![-]\!]$' at the end.

- Page 88, statement of Proposition 3.3.14: the reference to $0, suc, rec_{\mathtt{N}}$ as 'constants' is somewhat loose and informal: technically these are *variables* in the sense of Definition 3.3.1.

- Page 101, line 2: Example 1.1.3 should be Example 1.1.10.

- Page 102, line 2 from bottom: 'finitary data' should read 'countable data'.

- Page 103: lines 3–1 from bottom: the text between ‘as follows:’ and ‘interpreting the constants’ should be replaced by:

  > regarding M as a meta-expression, expand it to an applicative expression $M^\dagger$ as indicated in Section 3.3.1, and evaluate $M^\dagger$ in $\mathbf{A}$,

- Page 106, first line of Section 3.6: ‘conclude this section’ should read ‘conclude this chapter’.
- Page 117, middle of page. The definition of λ^* should be replaced as described in the erratum for page 83, bottom of page.
- Page 119, lines 2–3. With the revised definition of λ^* as per the erratum for page 83, we can no longer appeal to the equation $(M[x \mapsto N])^\dagger = M^\dagger[x \mapsto N^\dagger]$ to justify the β-rule. Nonetheless, the validity of the β-rule can in this setting be shown by an easy induction on the structure of M.
- Page 135, line 12 from bottom: $Bool_A$ should read $Bool(A)$.
- Page 164, statement of Lemma 4.4.11, last line: ‘regular term’ should read ‘β-normal regular term’.
- Page 164, Corollary 4.4.12: C should be $\mathcal{Z}$.
- Page 197, statement of Theorem 5.2.18. There is a $\times$ missing before $\mathsf{S}(l_{r-1})$.
- Page 201, line 2. Here ‘condition 3’ should be ‘condition 2’.
- Page 202, line 6. Within list item 2, Ψ should be Φ.
- Page 205, line 3. One occurrence of ‘or not’ should be deleted.
- Page 225, line 3 from bottom: there are some brackets missing (they are present at the corresponding places in the preceding line).
- Page 232, line 9 from bottom: ‘non-trival’ should read ‘non-trivial’.
- Pages 234–236. The clauses listed in Definition 6.2.1 do indeed hold for the intended translation $[-]^S$, as do the claimed properties of this translation (in particular Theorem 6.2.6). However, there are two problems.

  The first problem[1] is that these clauses do not constitute a valid *coinductive definition* of the translation as claimed, as they do not provide the ‘guardedness’ necessary to ensure that we make progress in the construction of the tree. For instance, suppose there were a number m such that $m = \lceil Eval_{\mathtt{N}}(m) \rceil$. Then Definition 6.2.1 would tell us simply that $[Eval_{\mathtt{N}}(m)]^S$ is equal to itself, and so would not determine its value. (One could construe the given clauses as a *least fixed point* definition of $[-]^S$, but this is not convenient for establishing the required properties.) Similar issues stemming from the inherent circularity associated with *Eval* are discussed elsewhere in Chapters 5 and 6.

[1]Drawn to our attention by a conversation with Paul Levy.

The second problem is that even aside from this, the outlined proof of the right-to-left implications in Theorem 6.2.6 does not go through as claimed. For instance, if M is some complex Kleene expression that evaluates to 0 so that $[M]^S = 0$, then the (trivial) generation of the statement $[\![[M]^S]\!]() = 0$ via Definition 6.2.5 gives no direct information about $[\![M]\!]()$.

Both problems can be addressed by first defining a related translation $[-]^R$ from Kleene expressions to NSP *meta-terms*, then defining $[-]^S = \ll [-]^R \gg$ and showing that $[-]^S$ satisfies the clauses of the current definition. The claimed proof of Theorem 6.2.6 then goes through with $[-]^R$ in place of $[-]^S$, and one completes the picture with an extra lemma showing that $[\![[M]^S]\!]$ and $[\![[M]^R]\!]$ agree for any β-normal M.

Specifically, the definition of $[-]^R$ adapts the current definition of $[-]^S$ by omitting all evaluations to normal form. The key clauses are:

- $[MN]^R = \lambda\vec{z}.\,\mathtt{case}\ [M]^R[N]^R\vec{z}^{\eta}\ \mathtt{of}\ (i \Rightarrow i)$.
- $[Suc(M)]^R = \mathtt{case}\ [M]^R\ \mathtt{of}\ (i \Rightarrow i+1)$.
- $[Primrec(M,N,P)]^R =$

$$\mathtt{case}\ [M]^R\ \mathtt{of}\ (j \Rightarrow \mathtt{case}\ [N]^R\ \mathtt{of}\ (i_0 \Rightarrow \mathtt{case}\ [P]^R 0 i_0\ \mathtt{of}\ (i_1 \Rightarrow \cdots \\ \cdots \Rightarrow \mathtt{case}\ [P]^R(j-1)i_{j-1}\ \mathtt{of}\ (i_j \Rightarrow i_j)))) \;.$$

- $[Eval_\sigma(M, N_0, \ldots, N_{n-1})]^R =$

$$\mathtt{case}\ [M]^R\ \mathtt{of}\ (\lceil P \rceil \Rightarrow [\beta nf(PN_0 \ldots N_{n-1})]^R) \;.$$

The clauses for x, $\lambda x.M$, $\widehat{0}$ and C_f follow those of the original definition. The definition of $[-]^R$ is now a valid coinductive definition because all invocations of $[-]^R$ in the right hand sides are guarded.

We then define $[M]^S = [\![[M]^R]\!]$. Using the evaluation theorem, it is then easy to show that $[M]^S$ satisfies the clauses originally given. Note that the first three of these, along with the fact that NSPs form a λ-algebra, suffice to show that if $M \simeq M'$ then $[M]^S = [M']^S$, so that $[PN_0 \ldots N_{n-1}]^S$ and $[\beta nf(PN_0 \ldots N_{n-1})]^S$ are interchangeable in the clause for *Eval*.

The outlined proof of Theorem 6.2.6 now goes through smoothly if $[-]^S$ is replaced by $[-]^R$.

Finally, we need to check that for β-normal M we have $[\![[M]^S]\!]_\nu(\vec{\Phi}) \simeq [\![[M]^R]\!]_\nu(\vec{\Phi})$ (and likewise $[\![[M]^S]\!]^*_\nu \simeq [\![[M]^R]\!]^*_\nu$). The '$\succeq$' direction here follows easily from Theorem 6.2.9(i), but for the '$\preceq$' direction, we require a partial converse to this theorem. Call an NSP meta-term T *benign* if all β-redexes within T have the form $(\lambda\vec{x}.M)\vec{q}$ where the q_i are number literals $\lambda.n_i$. (The point here is that for such arguments q_i the definedness of $[\![q_i]\!]^*$ is unproblematic.) Notice that if M is β-normal then $[M]^R$ is benign. The desired conclusion follows once we know that $[\![\ll G \gg]\!]_\nu \preceq [\![G]\!]_\nu$ for all benign ground meta-terms G and all suitable ν. The proof of this lemma is included under 'Omissions' below.

- Page 239, first bullet point. 'Section 6.1.1' should read 'Subsection 6.1.1'.

- Page 247, definition of ld, case for $pQ_0 \ldots Q_{r-1}$: both occurrences of p in this line should be P.

- Pages 254–258. There is a small, fixable bug in the proof of Theorem 6.3.27. In the penultimate sentence of the proof (on page 258), we claim that $F_1(g^{F_1}_{()}) = K$. For this, we first use the established fact that $g^{F_1}_{()}(0) = \langle 0, \ldots, 0\rangle$; it is natural denote this by y_0. However, to proceed further, one wants that y_0 is distinct from all of $y_1, \ldots, y_d$, which we do not know to be the case.

  The problem is readily fixed by simply adding, at the point at which each y_w is selected (where $1 \leq w \leq d$), the further requirement that $y_w \neq y_0$. For this, we need to adjust the definitions of the moduli m^w from earlier in the proof: we should now take $m^0 > n^0 + 2$, $m^1 > n^0 + n^1 + 3$ and so on. Then, when picking the path through the tree fpr Ψ_d at the bottom of page 256, we should start by defining $y_0 = g^{F_0}_{()}(0)$ (so that actually $y_0 = c$, then insert the requirements that y_1, y_2 and indeed each y_w differs from y_0. (There is also a typo here: on the last line of page 256, y_0 was intended to read y_1, but this should now be modified to y_0, y_1.)

  In the third-to-last line of page 257, the claim that $g^{F_1}_{\vec{z},0)(0)=y_w}$ will now make sense even when $w = 0$.

- Page 255. In the displayed formula near the middle of the page, $\lambda n F$ should be just λF.

- Page 256, line 12 from bottom. the superscript F_1 on g here should really be just F.

- Page 257, line 8 from bottom. Not an error, but as a further hint for the confused reader, the reason why $\langle \vec{z}, 0, \vec{x}, 0\rangle$ differs from y_d is that $w < d$ and z_w was chosen to be non-zero at the bottom of page 256.

- Page 258, line 2. Here $g^{F_1}_{(z_0,\ldots,z_{d-1})}$ is missing its argument (0).

- Page 258, line 4. The second g is missing its superscript F_1.

- Page 258, line 12 from bottom. 'Section 7.3.3' should read 'Subsection 7.3.3'.

- Page 260, statement of Corollary 6.3.32: The two equations are reproduced from the top of page 255, but here the notation Φ_n serves no purpose: a single equation defining $\Phi(n)(F)$ would suffice.

- Page 275, line 14 from bottom. The phrase 'as well as making Π^1_1 nature of termination (when $\mathbf{A} = \mathsf{S}$) somewhat more intelligible' is spurious. It is totality, not termination, that is Π^1_1, as explained in Subsection 5.2.3.

- Page 303. Theorem 7.1.40 is not correct as stated—both it and the preceding discussion require some additional hygiene conditions. Fortunately, these conditions hold in all the models we are interested in, so the main point of Theorem-7.1.40 is not affected.

  Firstly, in lines 2–3 of the page, we require that $\mathbf{P}$ is a model of PCF + *byval* and is normalizable at type $\overline{1}$, but we should also require that the interpretation of *byval* is itself a normalizer (cf. the discussion on page 286). This also applies to the statement of Theorem 7.1.40.

Secondly, for Theorem 7.1.40 we require the additional hypothesis that the following equation is valid in $\mathbf{P}$ (uniformly in f, g):

$$(byval\ g) \circ (byval\ f) \;=\; byval\,((byval\ g) \circ f)\ .$$

This is needed at the very end of the proof, in order to show that if $M \rightsquigarrow M'$ via the case-commuting rule ((b4) on page 214) then $[\![M]\!]^f = [\![M']\!]^f$.

Whilst these modifications render our treatment technically correct, it turns out that the essential content of this theorem may be presented in a much improved way. This both clarifies the role of the equational theory induced by SP^0 (some of which is currently concealed by our conditions $[\![c]\!]^{\mathbf{P}} = [\![[\![c]\!]^{\mathsf{SP}^0}]\!]^f$) and dispenses with the hypothesis that $\mathbf{P}$ is simple (thus removing the anomaly mentioned at the very end of the subsection, page 304). This improved treatment will be outlined under 'Updates' below.

- Page 327, just before bullet points: the parenthesized sentence should be deleted. It is more confusing than helpful, as the relevant notion of 'finite' is rather subtle and has not been spelt out here.
- Page 327, first bullet point: G_i does indeed model PCF + *catch*, but to obtain definability of all finite strategies, one should also include the *byval* operator from Section 7.1 (or something similar).
- Page 327, last bullet point: G_{ib} should be G.
- Page 328, third bullet point in middle of page: the '$.d$' in the definition of e is spurious, as is the phrase 'd with ξ,' two lines later on. The phrase 'of `N`' should also be deleted.
- Page 328, lines 8–7 from bottom (the sentence defining $S(p)$): should add 'along with ϵ' at the end.
- Page 328, line 3 from end: 'the preceding λ' should read 'the λ that binds it'.
- Page 337, Exercise 7.5.15. The full stop at the end should come before the right bracket.
- Page 355, line 17: the second $\mathsf{Ext}_n(\sigma)$ should be $\mathsf{Ext}_n(\tau)$.
- Page 445, line 12: in the statement of Corollary 9.2.6, F' should be declared to be a partial function $\mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$, not a total function. (This still gives all we need for the proof of Theorem 9.2.7).
- Page 555, reference 211 (Normann, D.: Closing the gap ...). The citation details should read: Archive for Mathematical Logic **36(4-5)**, 269–287 (1997).

Omissions

- Pages 236–238: As indicated in the Errata above, the following lemma is needed to support the corrected treatment of the translation $[-]^S$:

  Lemma. If G is a benign ground meta-term and ν is a suitable valuation, then $[\![\ll G \gg]\!]_\nu \preceq [\![G]\!]_\nu$.

  Proof. Note that if $[\![\ll G \gg]\!]_\nu = a$, then by inspection of Definition 6.2.5 there is a well-founded portion $g \sqsubseteq \ll G \gg$ such that $[\![g]\!]_\nu = a$. We now consider the 'reduction tree' $\mathcal{T}(G)$ with root G generated as follows:

  - If H is a node of $\mathcal{T}(G)$ and H head-reduces to H', then H' is the sole child of H.
  - If H is a node of form n or $\bot$ then H has no children.
  - If $H = \texttt{case}\ x(\lambda\vec{z}_0.D_0)\ldots(\lambda\vec{z}_{r-1}.D_{r-1})\ \texttt{of}\ (i \Rightarrow E_i)$, then each of the D_i and E_i are children of H.

  Now let $\mathcal{T}_0(G)$ be a well-founded subtree of $\mathcal{T}(G)$ that covers all of g, and for each node H, let $\mathcal{T}_0(H)$ be the full subtree of this rooted at H. Then each $\mathcal{T}_0(H)$ witnesses the reduction of H to some meta-term $\mid H \mid$ (allowing for parallel reductions of subterms D_i and E_i), where $g \sqsubseteq \mid G \mid$. We now show by induction on the height of nodes H within $\mathcal{T}_0(H)$ that

  $$[\![H]\!]_\mu \ \simeq\ [\![\mid H \mid]\!]_\mu \quad \text{for all } \mu \supseteq \nu \text{ covering the free variables of } H\ .$$

  In the case $H \rightsquigarrow_h H'$ this is easy to check, as the reduction is either a non-β reduction or a β-reduction with number literals as arguments. In the case $H = \texttt{case}\ x(\lambda\vec{z}_0.D_0)\ldots(\lambda\vec{z}_{r-1}.D_{r-1})\ \texttt{of}\ (i \Rightarrow E_i)$, we infer from the induction hypothesis that $[\![\lambda\vec{z}_i.D_i]\!]^*_\mu \simeq [\![\lambda\vec{z}_i.\mid D_i \mid]\!]^*_\mu$ for each i, whence $[\![H]\!]_\mu \simeq [\![\mid H \mid]\!]_\mu$ by compositionality. Finally, this yields that $[\![G]\!]_\nu = [\![\mid G \mid]\!]_\nu = a$ since $g \sqsubseteq \mid G \mid$. $\Box$

- Page 241, Theorem 6.3.3: One should mention here a very similar and earlier result of Clairambault and Harmer[2], who show in a fairly general setting that well-founded strategies (which they call *noetherian* strategies) are closed under application. When specialized to well-bracketed innocent strategies, this result is actually the same as ours, modulo the somewhat non-trivial isomorphism between the NSP model and the game model (see our Proposition 7.4.5 and the ensuing discussion). Clairambault and Harmer give a proof that works directly with game plays and strategies, and it is instructive to compare this with ours.

  These authors themselves trace the idea of their proof back to Coquand[3].

Updates

- A brief account of the improved treatment of the ideas around Theorem 7.1.40 (see the erratum for page 303) will be added here soon.

[2] Clairambault, P. and Harmer, R., *Totality in arena games*, Annals of Pure and Applied Logic 161(5), 673–689 (2009)

[3] Coquand, T., *A semantics of evidence for classical arithmetic*, Journal of Symbolic Logic 60(1), 325–337 (1995)