PReMo: an analyzer for Probabilistic Recursive Models

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Abstract. This paper describes PReMo, a tool for analyzing Recursive Markov Chains, and their controlled/game extensions: (1-exit) Recursive Markov Decision Processes and Recursive Simple Stochastic Games.

Introduction. Recursive Markov Chains (RMCs) [4, 5] are a natural abstract model of probabilistic procedural programs and other systems involving recursion and probability. They are formally equivalent to probabilistic Pushdown Systems (pPDSs) ([2, 3]), and they define a class of infinite-state Markov chains that generalize a number of well studied stochastic models such as Stochastic Context-Free Grammars (SCFGs) and Multi-Type Branching Processes. In a series of recent papers ([4–7]), the second author and M. Yannakakis have developed algorithms for analysis and model checking of RMCs and their controlled and game extensions: 1-exit Recursive Markov Decision Processes (1-RMDPs) and 1-exit Recursive Simple Stochastic Games (1-RSSGs). These extensions allow modelling of nondeterministic and interactive behavior.

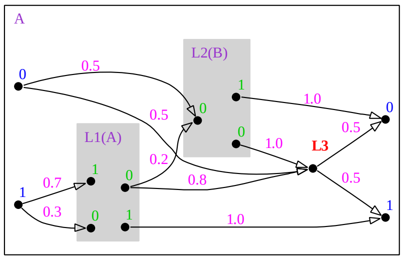
In this paper we describe PReMo, a software tool for analysing models based on RMCs, 1-RMDPs, and 1-RSSGs. PReMo allows these models to be specified in several different input formats, including a simple imperative-style language for specifying RMCs and RSSGs, and an input format for SCFGs. For RMCs/RSSGs, PReMo generates a graphical depiction of the model, useful for visualizing small models (see Figure 1). PReMo has implementations of numerical algorithms for a number of analyses of RMCs and 1-RSSGs. From an RMC, PReMo generates a corresponding system of nonlinear polynomial equations, whose Least Fixed Point (LFP) solution gives precisely the termination probabilities for vertex-exit pairs in the RMC. For 1-RSSGs, it generates a system of nonlinear min-max equations, whose LFP gives the *values* of the termination game starting at each vertex. Computation of termination probabilities is a key ingredient for model checking and other analyses for RMCs and pPDSs ([4, 5, 2]). PReMo provides a number of optimized numerical algorithms for computing termination probabilities. Methods provided include both dense and sparse versions of a decomposed Newton's method developed in [4], as well as versions of value iteration, optimized using nonlinear generalizations of Gauss-Seidel and SOR techniques. The latter methods also apply to analysis of 1-RSSGs.

In addition to computing termination probabilities, PReMo can compute the (maximum/minimum/game) *expected termination time* in 1-RMCs, 1-RMDPs, and 1-RSSGs. It does so by generating a different monotone system of linear

```
A(2,2);
B(1,2);

A {
    L1(A);
    L2(B);
    entry 0:
        0.5: goto L3; 0.5: call L2(0);
    entry 1:
        0.3: call L1(0); 0.7: call L1(1);
    L1 {
        exit 0:
            0.8: goto L3; 0.2: call L2(0);
        exit 1:
            1.0: return 1;
    }
    L2 {
        exit 0:
            1.0: goto L3;
        exit 1:
            1.0: return 0;
    }
    L3 {
        0.5: return 0; 0.5: return 1;
    }
}

B {
    L4(A);
    entry 0:
        0.3: call L4(0); 0.3: call L4(1); 0.4: goto L5;
    L4 {
        exit 0:
            0.5: return 0; 0.5: return 1;
        exit 1:
            0.5: return 0; 0.5: goto L5;
    }
    L5 {
        0.4: goto L5; 0.6: return 1;
    }
}
```



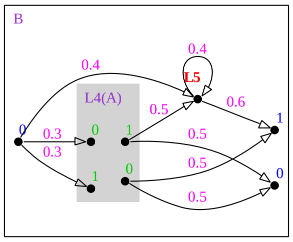


Fig. 1. Source code of an RMC, and its visualization generated by PReMo

(min-max) equations, whose LFP is the value of the game where the objectives of the two players are to maximize/minimize the expected termination time (these expected times can be infinity). (This analysis extends, to a game setting, the expected reward analysis for pPDSs (equivalently, RMCs) studied in [3]. The generalization works for 1-RMDPs and 1-RSSGs, which correspond to controlled/game versions of *stateless* pPDSs, also known as pBPAs. We do not explicate the theory behind these game analyses here. It is a modification of results in [6, 7], and will be explicated elsewhere.)

PReMo is implemented entirely in Java, and has the following main components: (1) A parsers for text descriptions of RMCs, RSSGs, and SCFGs, using one of several input formats; (2) A menu-driven GUI (using the Standard Widget Library(SWT)), with an editor for different input formats, and menu choices for running different analyses with different methods; (3) A graphical depiction generator for RMCs and RSSGs, which produces output using the `dot` format. (4) *Optimized solvers:* Several solvers are implemented for computation of termination probabilities/values for RMCs and 1-RSSGs, and also computation of expected termination times for 1-RMCs, 1-RMDPs, 1-RSSGs. We conducted a range of experiments. Our experiments indicate very promising potential for several methods. In particular, our decomposed Sparse Newton's method performed very well on most models we tried, up to quite large sizes. Although these numerical methods appear to work well in practice on most instances, there are no theoretical guarantees on their performance, and there are many open questions about the complexity of the underlying computational problems (see [4–7]).

We can see PReMo source code for an RMC, together with a visualization that PReMo generates for it, in Figure 1. Informally, an RMC consists of several component Markov Chains (in Fig. 1, these are named A and B) that can call each other recursively. Each component consists of nodes and boxes with possible probabilistic transitions between them. Each box is mapped to a specific component so that every time we reach an entry of this box, we jump to the corresponding entry of the component it is mapped to. When/if we finally reach an exit node of that component, we will jump back to a respective exit of the box that we have entered this component from. This process models, in an obvious way, function invocation in a probabilistic procedural program. Every potential function call is represented by a box. Entry nodes represent parameter values passed to the function, while exit nodes represent returned values. Nodes within a component represent control states inside the function. Documentation about the input languages is available on the PReMo web page.

The core numerical computation for all the analyses provided by PReMo involves solving a monotone systems of nonlinear min-max equations. Namely, we have a vector of variables $\mathbf{x} = (x_1, \ldots, x_n)$, and one equation per variable of the form $x_i = P_i(\mathbf{x})$, where $P_i(\mathbf{x})$ is a polynomial-min-max expression with rational coefficients. In vector notation, this system of equations can be denoted $\mathbf{x} = P(\mathbf{x})$. The goal is to find the Least Fixed Point solution, i.e., the least non-negative solution, $\mathbf{q}^* \in \mathbb{R}^n_{\geq 0}$, of these equations, which is $\lim_{k\to\infty} P^k(\mathbf{0})$. In brief, the solvers in PReMo work as follows (see [4, 6] for more background). First, we decompose the equations into SCCs and calculate the solution "bottom-up", solving the Bottom SCCs first and plug in the solution as constants in higher SCCs. To solve each SCC, PReMo provides several methods:

Value iteration: nonlinear Jacobi & Gauss-Seidel. Optimized forms of nonlinear *value iteration* have been implemented for computing the LFP of $\mathbf{x} = P(\mathbf{x})$. Jacobi, or basic iteration, just computes $\mathbf{x}^0 = \mathbf{0}, \mathbf{x}^1, \mathbf{x}^2, \ldots$, where $\mathbf{x}^i = P(\mathbf{x}^{i-1})$. Gauss-Seidel iteration optimizes this slightly: inductively, having computed x_j^{k+1} for $j < i$, let $x_i^{k+1} := P_i(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i^k, x_{i+1}^k, \ldots, x_n^k)$. Successive Overrelaxation (SOR) is an "optimistic" modification of Gauss-Seidel, which isn't guaranteed to converge in our case.

Dense and sparse decomposed Newton's method. Newton's method attempts to compute solutions to $F(\mathbf{x}) = \mathbf{0}$. In n-dimensions, it works by iterating $\mathbf{x}^{k+1} := \mathbf{x}^k - (F'(\mathbf{x}^k))^{-1}F(\mathbf{x}^k)$ where $F'(\mathbf{x})$ is the *Jacobian* matrix of partial derivatives of F. In our case we apply this method for $F(\mathbf{x}) = P(\mathbf{x}) - \mathbf{x}$. It was shown in [4] that if the system is decomposed into SCCs appropriately, convergence to the LFP is guaranteed, if we start with $\mathbf{x}^0 = \mathbf{0}$. The expensive task at each step of Newton is the matrix inversion $(F'(\mathbf{x}^k))^{-1}$. Explicit matrix inversion is too expensive for huge matrices. But this matrix is typically sparse for RMCs, and we can handle much larger matrices if instead of inverting $(F'(\mathbf{x}^k))$ we solve the following equivalent sparse linear system of equations: $(F'(\mathbf{x}^k))(\mathbf{x}^{k+1} - \mathbf{x}^k) = F(\mathbf{x}^k)$ to compute the value of $\mathbf{x}^{k+1} - \mathbf{x}^k$, and then add $\mathbf{x}^k$ to obtain $\mathbf{x}^{k+1}$. We used the solver library MTJ (Matrix Toolkit for Java) and tried various sparse linear solvers. Our Dense Newton's method uses LU decomposition to invert $(F'(\mathbf{x}^k))$.

Iterative numerical solvers can only converge to within some error to the actual solution. PReMo provides different mechanisms for users to choose when to stop the iteration: absolute tolerance, relative tolerance, and a specified number of iterations. In, e.g., the absolute tolerance mode, the algorithm stops after the first iteration when the absolute difference in the value for all variables changed less than a given $\epsilon > 0$. This does not in general guarantee closeness to the actual solution, but it behaves well in practice.

Experimental results. We ran a wide range of experiments on a Pentium 4 3GHz with 1GB RAM, running Linux Fedora 5, kernel 2.6.17, using Java 5.0. Please see the appendix more details about our experimental results.

SCFGs generated from the Penn Treebank NLP corpora. We checked the *consistency*[1] of a set of large SCFGs, with 10,000 to 50,000 productions, used by the Natural Language Processing (NLP) group at University of Edinburgh and derived by them from the Penn Treebank NLP corpora. These SCFGs were assumed to be consistent by construction. Our most efficient method (Sparse Newton) solved all these SCFGs in a few seconds (see Table 1). Two out of seven SCFGs were (very) inconsistent, namely those derived from the `brown` and `switchboard` corpora of Penn Treebank, with termination probabilities as low as 0.3 for many nonterminals. This inconsistency was a surprise to our NLP colleagues, and was subsequently identified by them to be caused by annotation errors in Penn Treebank itself ([1]). Note that both dense and sparse versions of decomposed Newton's method are by far the fastest. Since the largest SCCs are no bigger than 1000 vertices, dense Newton also worked on these examples. Most of the time for Newton's method was in fact taken up by the initialization phase, for computing all the partial derivatives in entries of the Jacobian $F'(\mathbf{x})$. We thus optimized the computation of the Jacobian in several ways.

name	#prod	max-scc	Jacobi	Gauss Seidel	SOR ω=1.05	DNewton	SNewton
brown	22866 ✘	448	312.084(9277)	275.624(7866)	diverge	2.106(8)	2.115(9)
lemonde	32885 ✓	527	234.715(5995)	30.420(767)	diverge	1.556(7)	2.037(7)
negra	29297 ✓	518	16.995(610)	4.724(174)	4.201(152)	1.017(6)	0.499(6)
swbd	47578 ✘	1123	445.120(4778)	19.321(202)	25.654(270)	6.435(6)	3.978(6)
tiger	52184 ✓	1173	99.286(1347)	16.073(210)	12.447(166)	5.274(6)	1.871(6)
tuebadz	8932 ✓	293	6.894(465)	1.925(133)	6.878(461)	0.477(7)	0.341(7)
wsj	31170 ✓	765	462.378(9787)	68.650(1439)	diverge	2.363(7)	3.616(8)

Table 1. Performance results for checking consistency of SCFGs derived from Penn Treebank. Time is in seconds. In parentheses is the number of iterations for the biggest SCC. Stopping condition: absolute tolerance $\epsilon = 10^{-12}$. SCFG was declared "consistent" if all nonterminals had termination probability $\geq (1 - 10^{-4})$. The SCFGs `brown` and `swbd` failed consistency by a wide margin.

Randomly generated RMCs and 1-RSSGs. We tested PReMo on randomly generated RMCs of different sizes, ranging from 10,000 to 500,000 nodes (variables). In random large instances, with very high probability most nodes are in one huge SCC with small diameter ("small world phenomenon"). Dense Newton's method did not work at all on these huge SCCs, because inverting such large matrices is too costly, but both Gauss-Seidel and Sparse Newton did very well.

[1] An SCFG is called *consistent* if starting at all nonterminals in the grammar, a random derivation terminates, and generates a finite string, with probability 1.

In particular, Sparse Newton handled instances with 500,000 variables in ~ 45 seconds. For random 1-RSSGs, although we have no Newton's method available for 1-RSSGs, value iteration performed well (see the appendix).

Quicksort. For expected termination time analyses, we considered a toy model of randomized Quicksort, using a simple hierarchical 1-RMC. The model has n components, Q_i, $i = 1, \ldots, n$, corresponding to invocations of Quicksort on arrays of size i. Component Q_i takes time i to pivot and split the entries, and then recurses on the two partitions. This is modeled by transitions of probability $1/(i-1)$, for each $d \in \{1, \ldots, i-1\}$, to two sequential boxes labeled by Q_d and Q_{i-d}. We computed expected termination time for various sizes n. We also tried letting the pivot be controlled by the *minimizer* or *maximizer*, and we computed optimal expected running time for such 1-RMDPs, in order to consider best-case and worst-case running times of Quicksort. As expected, the results fitted the well-known theoretical analysis of $\Theta(n \log n)$ and $\Theta(n^2)$ for running times of randomized/best-case, and worst-case Quicksort, respectively.

Future work. The next important step is to extend the RMC language to allow variables and conditional branching, i.e., probabilistic Boolean Programs. We are working toward implementation of a full-fledged linear-time model checker for RMCs. This is a major challenge because there are very difficult numerical issues that have to be overcome in order to enable general model checking.

PReMo 1.0 is available at: `http://homepages.inf.ed.ac.uk/s0571094/PReMo`

Acknowledgements. Thanks to Mihalis Yannakakis: the second author's work on analysis of RMCs/RSSGs, on which PReMo is based, is joint work with him. Thanks to Mark-Jan Neiderhof and Giorgio Satta for pointing us in the direction of large SCFG libraries used in NLP, and telling us about their own current work on implementing these methods [8]. Thanks to Amit Dubey and Frank Keller for providing us SCFGs from their NLP work.

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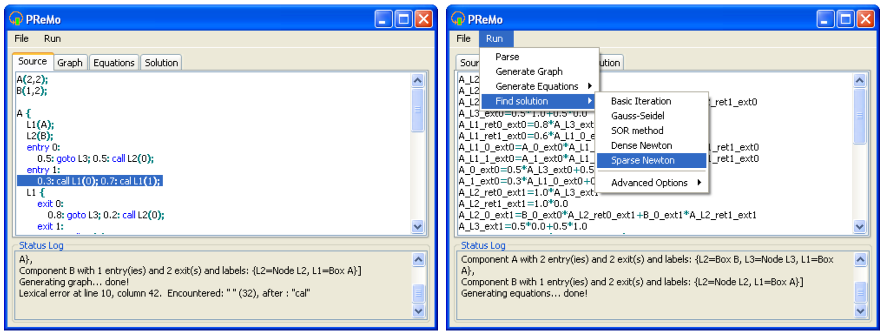
A Appendix

PReMo 1.0 is available at: `http://homepage.inf.ed.ac.uk/s0571094/PReMo`



```
A(2,2);
B(1,2);

A {
   L1(A);
   L2(B);
   entry 0:
      0.5: goto L3; 0.5: call L2(0);
   entry 1:
      0.3: call L1(0); 0.7: cal L1(1);
   L1 {
      exit 0:
         0.8: goto L3; 0.2: call L2(0);
      exit 1:
```

```
A},
Component B with 1 entry(ies) and 2 exit(s) and labels: {L2=Node L2, L1=Box A}]
Generating graph... done!
Lexical error at line 10, column 42.  Encountered: " " (32), after : "cal"
```

```
A_L2
A_L2
A_L2                                          _ret1_ext0
A_L3_ext0=0.5*1.0+0.5*0.0
A_L1_ret0_ext0=0.8*A_L3_ext
A_L1_ret1_ext0=0.6*A_L1_0_e
A_L1_0_ext0=A_0_ext0*A_L1_                    _ret1_ext0
A_L1_1_ext0=A_1_ext0*A_L1_                    _ret1_ext0
A_0_ext0=0.5*A_L3_ext0+0.5
A_1_ext0=0.3*A_L1_0_ext0+0
A_L2_ret0_ext1=1.0*A_L3_ext1
A_L2_ret1_ext1=1.0*0.0
A_L2_0_ext1=B_0_ext0*A_L2_ret0_ext1+B_0_ext1*A_L2_ret1_ext1
A_L3_ext1=0.5*0.0+0.5*1.0
```

```
Component A with 2 entry(ies) and 2 exit(s) and labels: {L2=Box B, L3=Node L3, L1=Box
A},
Component B with 1 entry(ies) and 2 exit(s) and labels: {L2=Node L2, L1=Box A}]
Generating equations... done!
```

Fig. 2. A screenshot of source code editing on the left, and running a solver on the system of equations generated from it on the right.

We can see a screenshot of PReMo in Fig. 2. PReMo has a source code editor that supports syntax highlighting, auto-indentation and error line highlighting for parsing errors. The user can save code source, generate visual depiction of RMCs, generate the corresponding equations, find solutions using various methods, obtain performance data, and export these to external files.

A.1 Experimental results, continued.

In Table 1, on page 4, size is indicated in # of productions, and the maximum number of variables in any SCC after decomposition of the nonlinear equations. Time is measured in seconds. In parentheses is the number of iterations needed for the biggest SCCs. The stopping condition for the iteration was that absolute tolerance with $\epsilon = 10^{-12}$. There is a ✓if the grammar was found to be be consistent (to within 10^{-4} error), and a ✘ otherwise. In fact, in the two inconsistent SCFGs, many nonterminals had termination probability as low as 0.3. Note that both dense and sparse versions of decomposed Newton's method are by far the fastest. Since the largest SCCs are no bigger than 1000 vertices, dense Newton also worked on these examples. Most of the time for Newton's method was in fact taken up by the initialization phase, for computing all the entries of the Jacobian $F'(\mathbf{x})$. We thus optimized the computation of the Jacobian in several ways.

Randomly generated RMCs and 1-RSSGs. We tested PReMo on randomly generated RMCs of different sizes, ranging from 10,000 and 500,000 nodes (variables). Random generation was done in two ways. First we generated 1-RMCs (SCFGs) in a normal form by directly generating their nonlinear equations of three possible kinds: (1) $x_i = x_j x_k$, (2) $x_i = px_j + (1-p)x_k$, and (3)

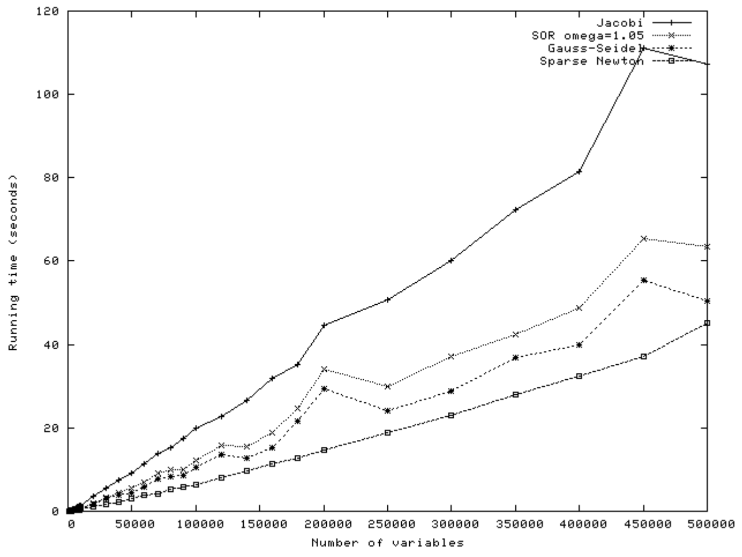


Fig. 3. Random 1-RMCs. For each size n that was tried, $(2 \times 10^6)/n$ instances of size n were generated and the running time was averaged. The termination condition was absolute tolerance $\epsilon = 10^{-12}$.

$x_i = p_1 x_j + p_2 x_k + p_3$, where $p_1 + p_2 + p_3 = 1$. We chose type (1) with 0.2 probability, type (2) with 0.6 probability (and then $p \in [0, 1]$ uniformly), and (3) with 0.2 probability (and p_1, p_2, p_3, were chosen uniformly in $[0, 1]$ and then normalized, dividing by $p_1 + p_2 + p_3$, so they sum to 1). See Fig. 3 for the running times. Size was measured in number of variables, n. The number of random instances generated for each size n, was $(2 \times 10^6)/n$, and running time was averaged over all instances of size n.

On these random large instances, with very high probability most nodes are in one huge SCC with small diameter (by the so called "small world phenomenon"). Dense Newton's method did not work at all on these huge SCCs, because inverting such large matrices is too expensive. On the other hand, as can be seen both Gauss-Seidel and Sparse Newton's method did very well.

Newton's method does not apply to 1-RSSGs, with nonlinear min-max equations, but Gauss-Seidel does. We applied a similar random generation technique to generate 1-RSSGs (this time with min and max nodes as well) and obtained the results in Figure 4.

Finally, for multi-exit RMCs, we had a difficult time finding a direct random generation scheme that was simple to define. We instead chose to randomly generate more general monotone nonlinear polynomial equations, where equations can be of the form $x_i = x_j x_j + x_r$, in addition to the possible equations we generated for 1-RMCs. These equation systems could potentially have no LFP solution, in which case the methods diverge to infinity, and create overflow error (or may not be defined, in the case of Newton's method, because the jacobians

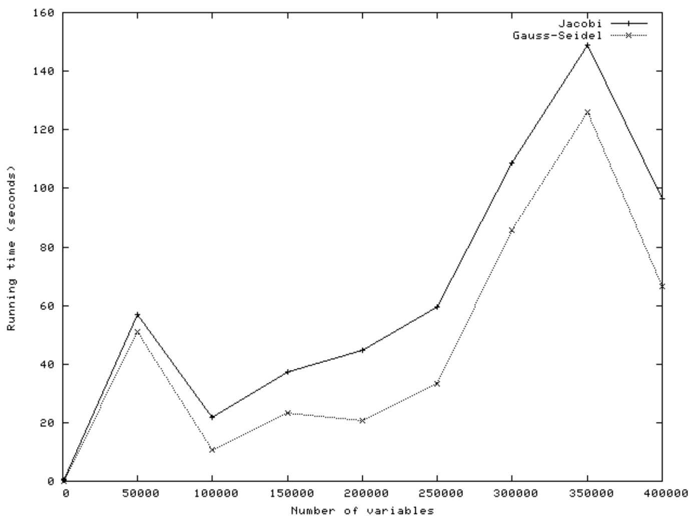


Fig. 4. Random 1-RSSGs. Average running time for computing termination probability of 2 instances for each size n, $n = 50,000 * i$, $i = 1, \ldots, 7$, with absolute tolerance $\epsilon = 10^{-12}$.

may not be invertible). But the equations generated form a superset of the equations for multi-exit RMCs. We generated these equations and tested to see if the methods do converge. Two generated instances were discarded because the results were diverged to infinity. The results for the instances that did converge are in Figure 5.

A.2 Quicksort

We modeled the performance of the Quicksort algorithm, using a simple hierarchical 1-RMC. In our model we have n components, Q_i, $i = 1, \ldots, n$, corresponding to invocations of Quicksort on arrays of size i. Each such component takes time i to pick the pivot and split the entries. We modeled this as "expected" time i, by having a self-loop at the entry node of component i with probability $(1 - 1/i)$. and transitioning to the "pivot choice" node with probability $\frac{1}{i}$. After the transition, the pivot d is chosen, uniformly at random. We have to recursively solve two instances of Quicksort, of sizes d and $i - d$. We modeled this by random transitions of probability $1/i$, for each $d \in \{1, \ldots, i-1\}$, to a sequence of two boxes labeled by Q_d and Q_{i-d}, and then to the terminal exit of the component.

We computed the *expected time for termination* for these models, to see whether the excepted running time of the algorithm matches the known theoretical average-case analysis of $\Theta(n \log n)$. We also tried letting the pivot node be controlled by the *minimizer* or *maximizer*, and generated the corresponding linear min-max equations for expected running time for such 1-RMDPs, in order to consider best-case and worst-case running times of Quicksort. Our models, as

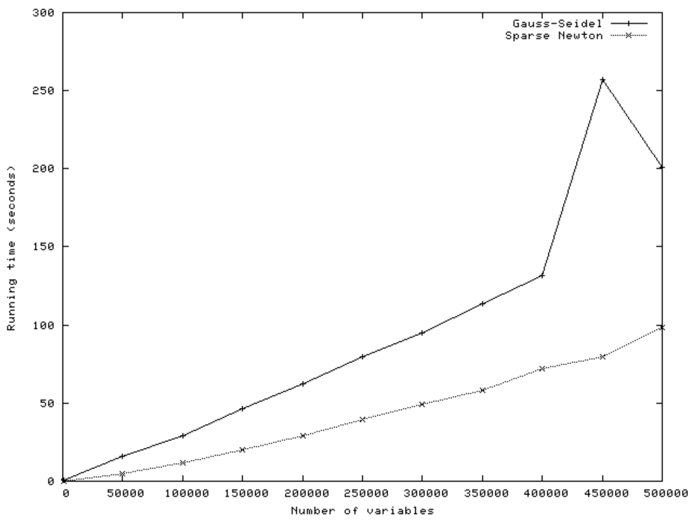


Fig. 5. Random RMCs and general monotone polynomial systems. Average running time for computing termination probability of 3 instances for each size n, $n = 50,000 * i$, $i = 1, \ldots, 10$, with absolute tolerance $\epsilon = 10^{-12}$. Two generated instances that diverged were discarded.

would be expected, matched the known theoretical analysis of running times for (randomized) Quicksort. Namely, for constants the expected running time for random pivot choices is $cn \log n$, the expected running time best pivot choice is $c'n \log n$, and the expected time for the worst pivot choice is $c''n^2$. In our model we found $c = 2.76$ and $c' = 2.18$. This would suggest that random pivot choice runs 1.26 times slower than the optimal possible choice of pivots. It would be interesting to know whether the exact analytically derived constants for Quicksort can confirm this relationship.

A.3 Long chains

It is easy to construct examples of simple, even finite state Markov chains, where the behavior of Newton's method can in principle be exponentially better than Gauss-Seidel iteration. This occurs, for example, when a finite Markov chain consists of a "long chain" with n nodes $v_1, \ldots, v_n$, where v_n is the terminal state, and where there are transitions of the form $v_i \overset{1/2}{\rightarrow} v_{i+1}$ and $v_i \overset{1/2}{\rightarrow} v_1$, for $i = 1, \ldots (n-1)$. In these examples, clearly the probability of termination from all nodes v_i is 1. Let x_i^j be the value of basic value iteration (jacobi) after j iterations, starting at 0, for a variable x_i representing the probability of termination from v_i. It is easy to show that, in order that $x_1^j \geq 1/2$, it must be the case that $j \in 2^{\Omega(n)}$. On the other hand, Newton iteration on Markov chains converges in 1 iteration, because the Jacobian is a constant, invertible, matrix. We ran

Jacobi, Gauss-Seidel, and both Dense and Sparse Newton iteration on such long chains. Interestingly, although Dense Newton performed as expected, solving the required single iteration in very little time (for the sizes where it could handle the matrix inversion), Sparse Newton iteration encountered numerical problems with all the sparse linear solvers we tried for one iteration of Newton. It appears that the small probabilities that arise in solving this linear system causes problems for the available sparse linear solvers. The results of the running times for Jacobi, Gauss-Seidel, and Dense Newton, are in Figure 6.

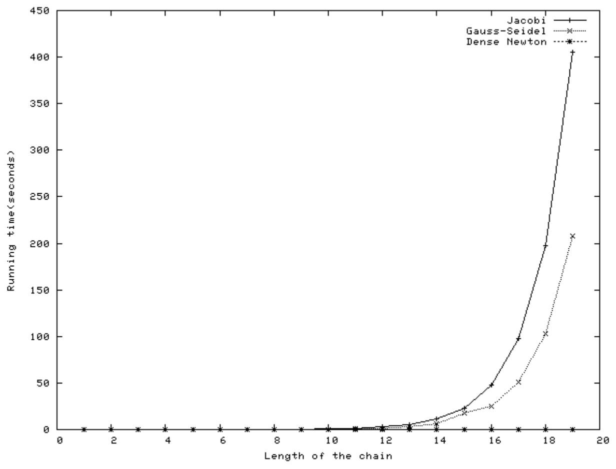


Fig. 6. Running times for long chains, absolute tolerance $\epsilon = 10^{-12}$. Note that dense Newton takes essentially no time for these sizes.