FOURIER ANALYSIS OF NUMERICAL INTEGRATION IN MONTE CARLO RENDERING

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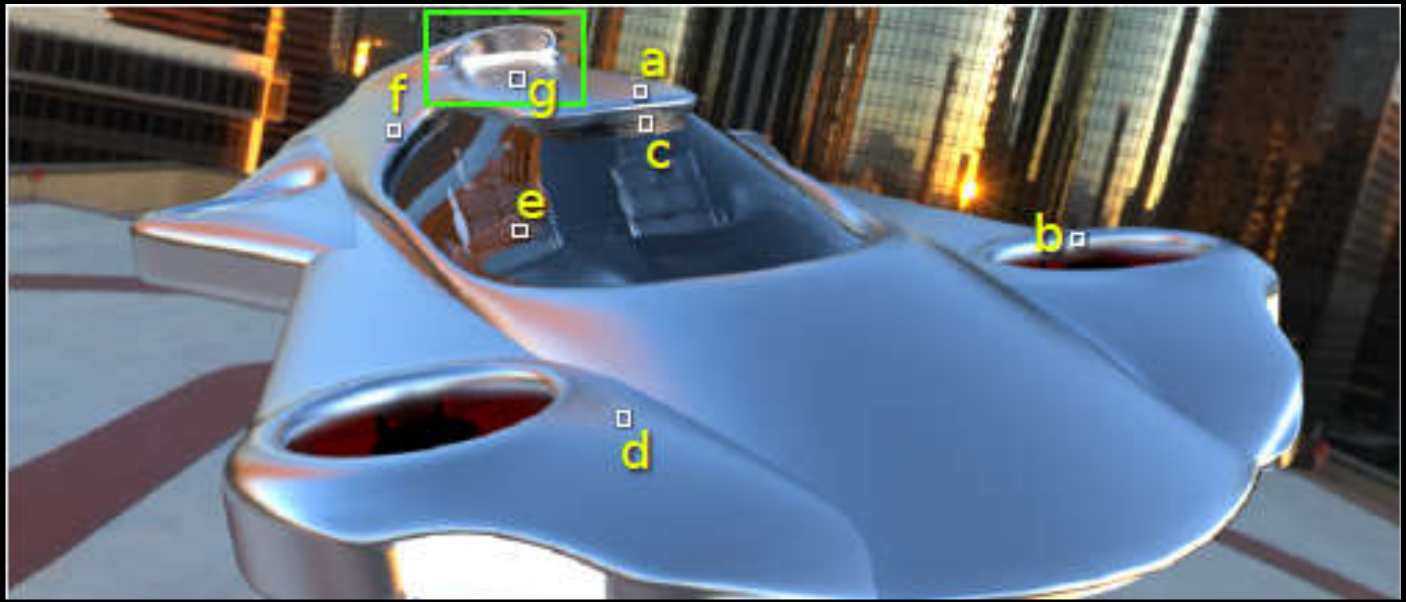
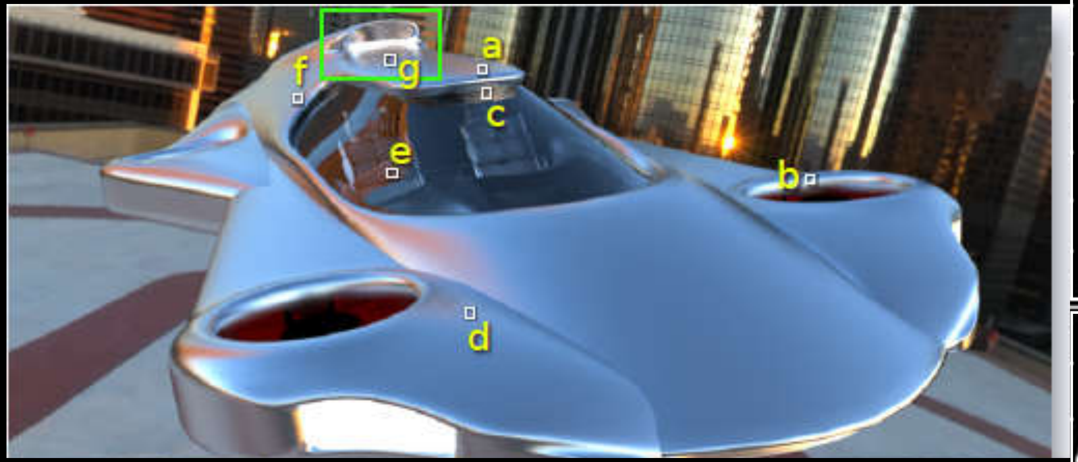
Motivation for analysis

- assess, compare existing methods for Monte Carlo rendering
- provide insight, inspire improvement

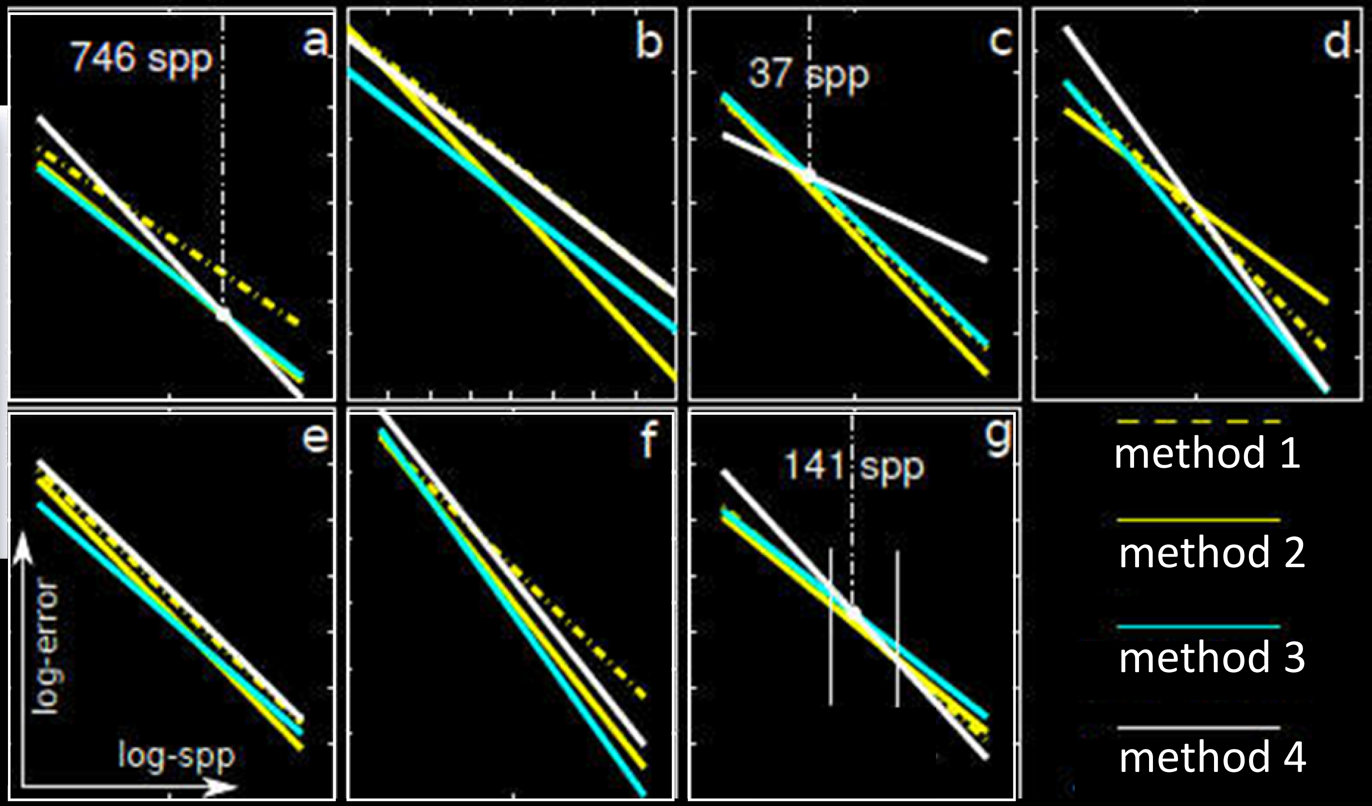


[Subr et al 2014]

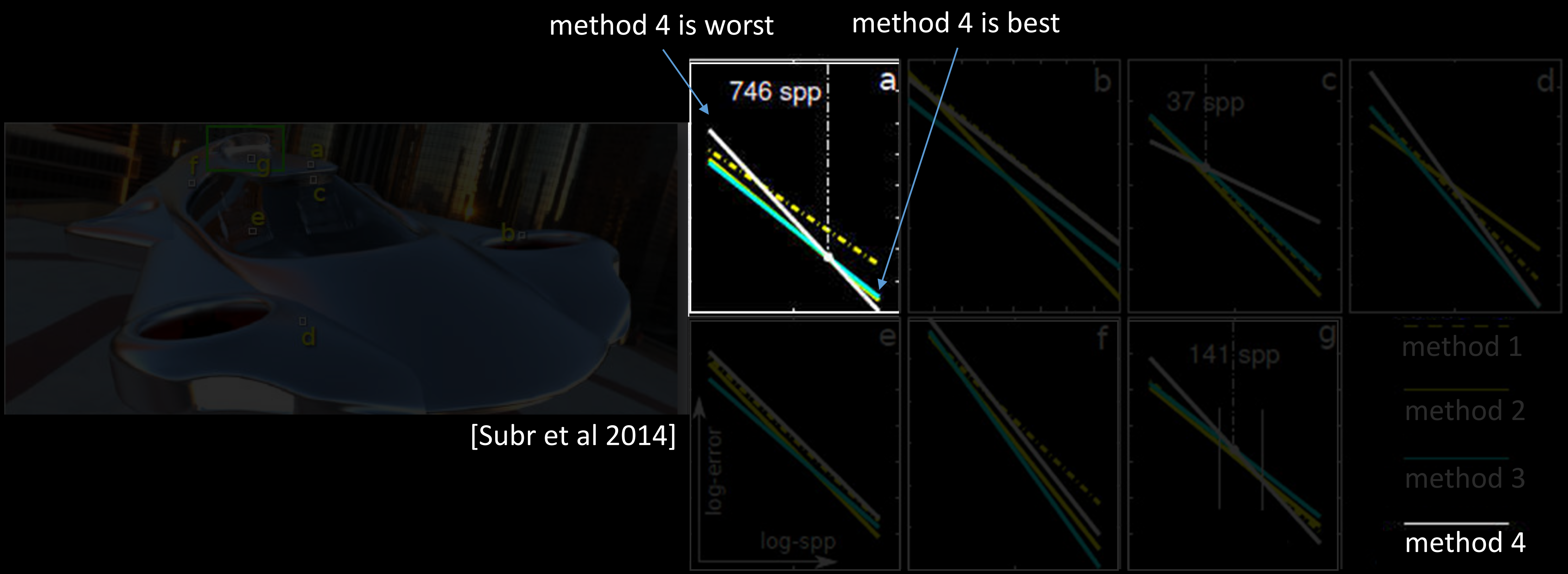
Error vs cost plots of rendering methods



[Subr et al 2014]



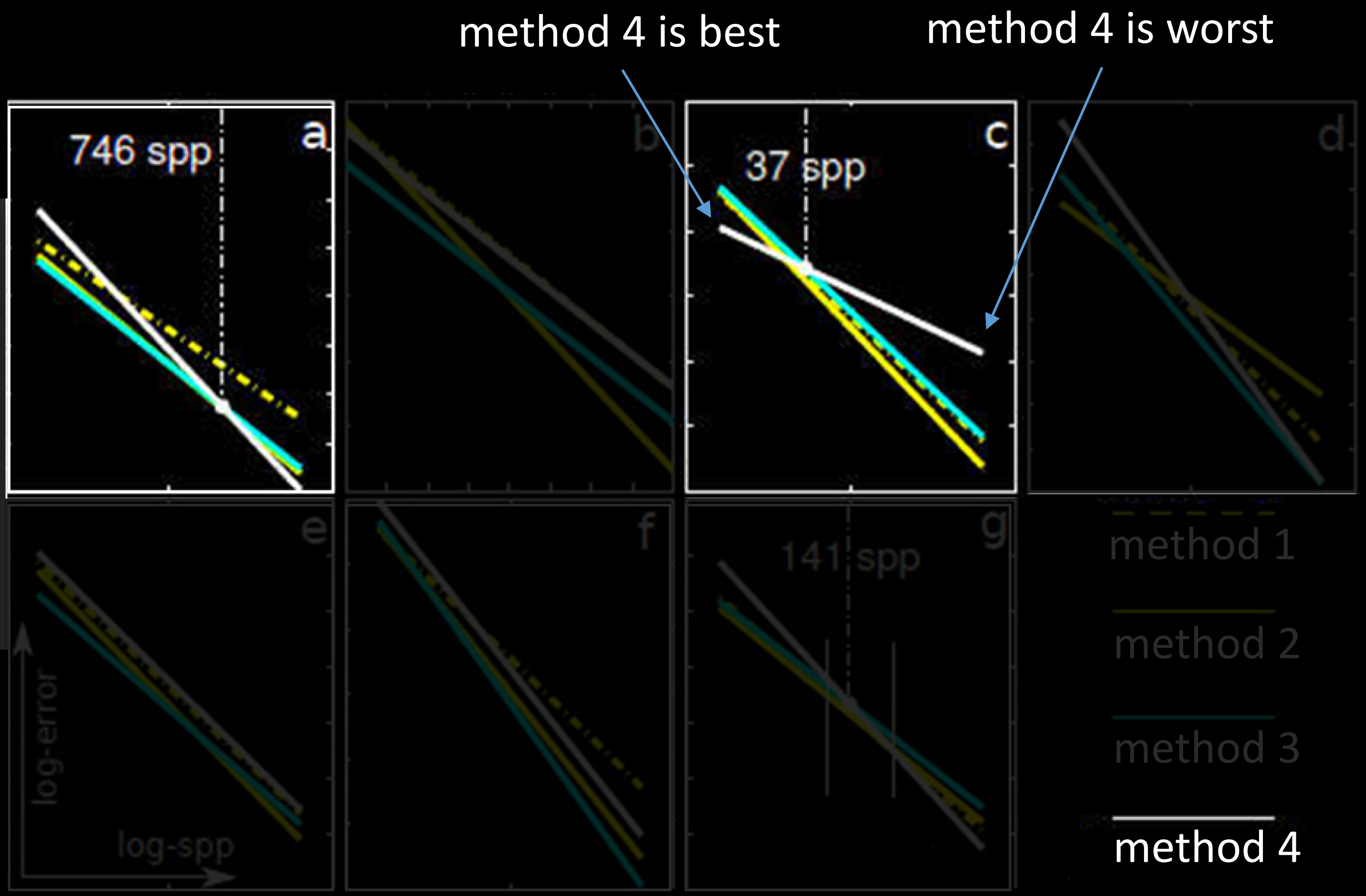
Error vs cost plots of rendering methods



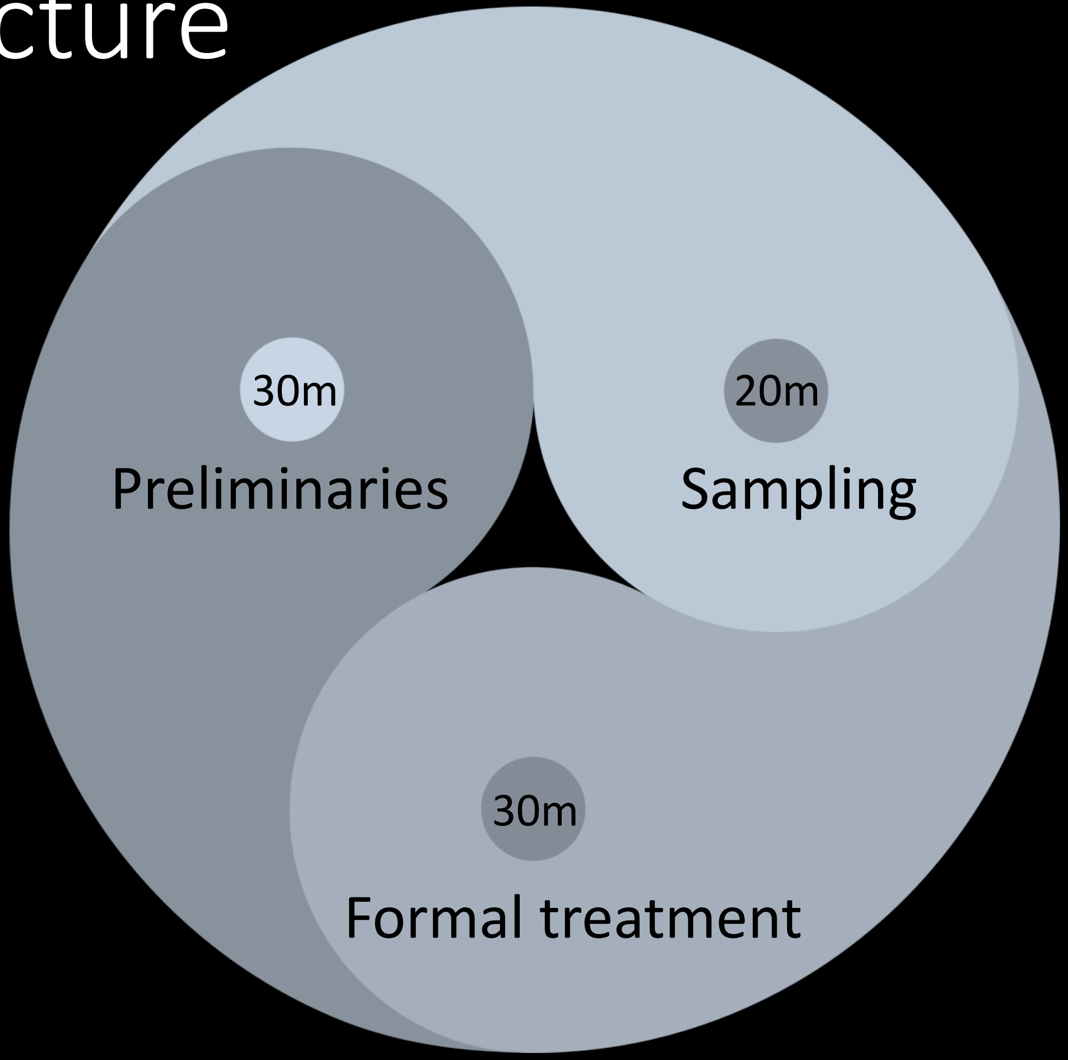
[Subr et al 2014]

Error vs cost plots of rendering methods

[Subr et al 2014]

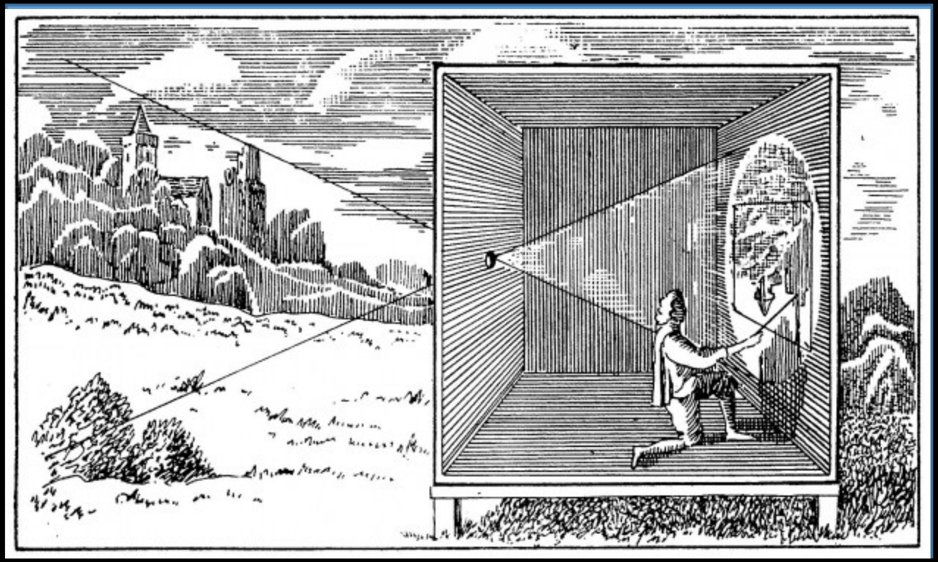


Course structure



Rendering = geometry + radiometry

geometry/projection
for pin-hole model known since 400BC



radiometrically accurate simulation
is important for photorealism

Rendering = geometry + radiometry

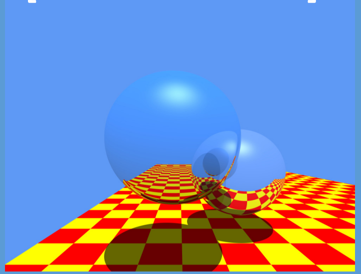
geometry/projection
for pin-hole model known since 400BC

OpenGL

[Stachowiak 2010]

Raytracing

[Whitted 1980]



radiometrically accurate simulation
is important for photorealism

[photo credit: videomaker.com June 2015]

Radiometric fidelity improves photorealism

photograph

Colourbox.com

manually painted

Pedro Campos

computer generated

Simulating the physics of light is challenging

lenses

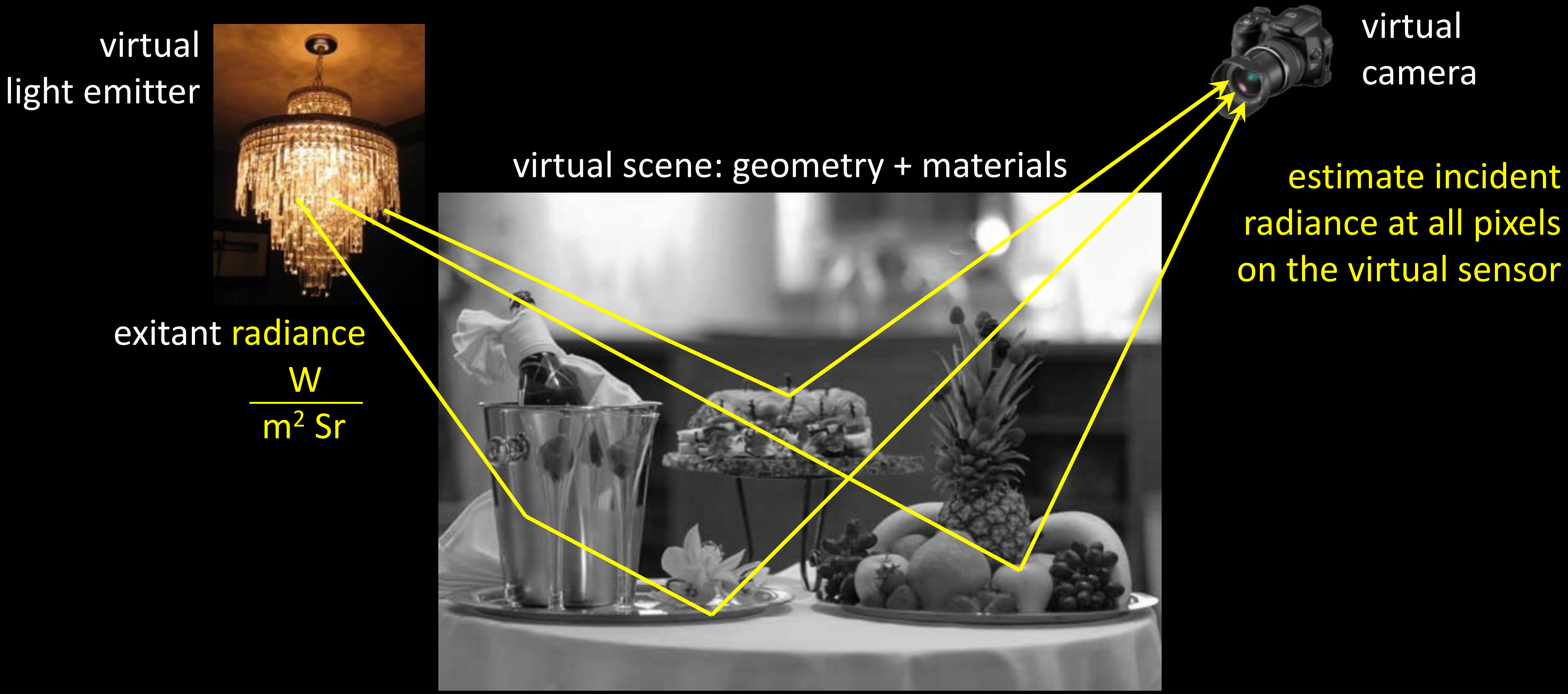
defocus

exposure time

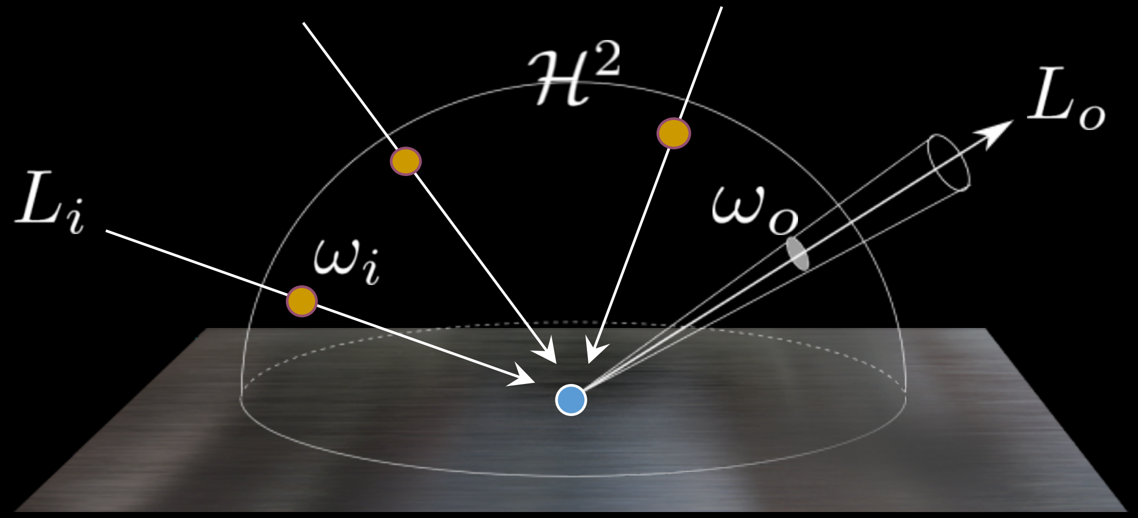
materials

light, media

Light transport

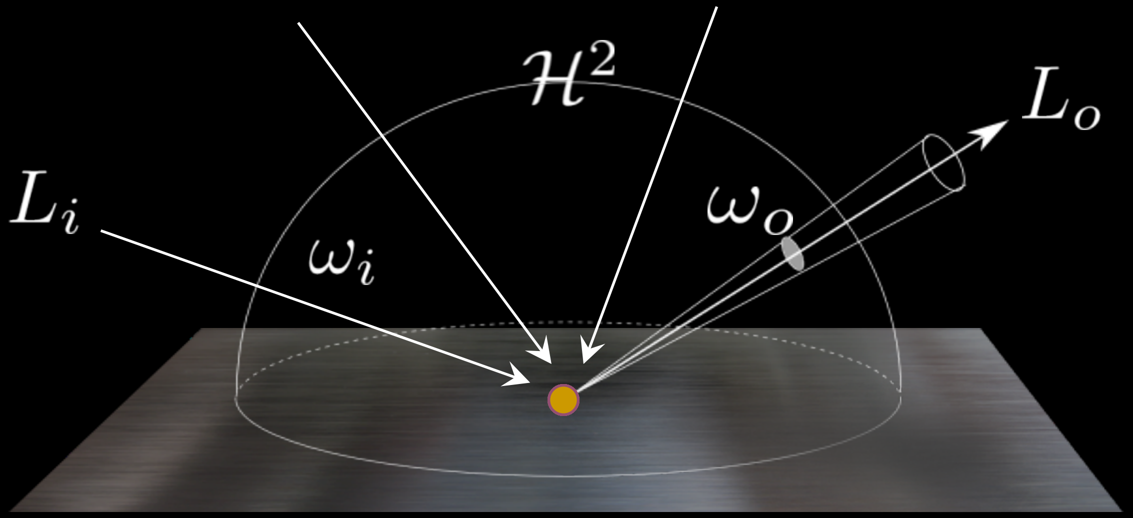


Each reflection is modeled by an integration



radiance: $$L_o = \int_{\mathcal{H}^2} L_i \, \rho(x, \omega_i, \omega_o) \, \mathrm{d}\mu(\omega_i)$$

Each reflection is modeled by an integration



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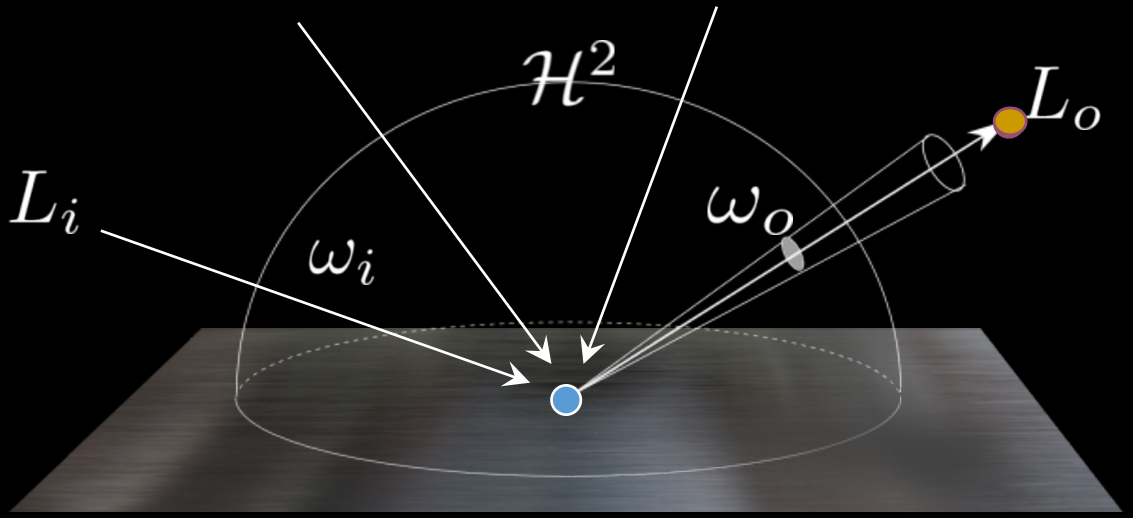
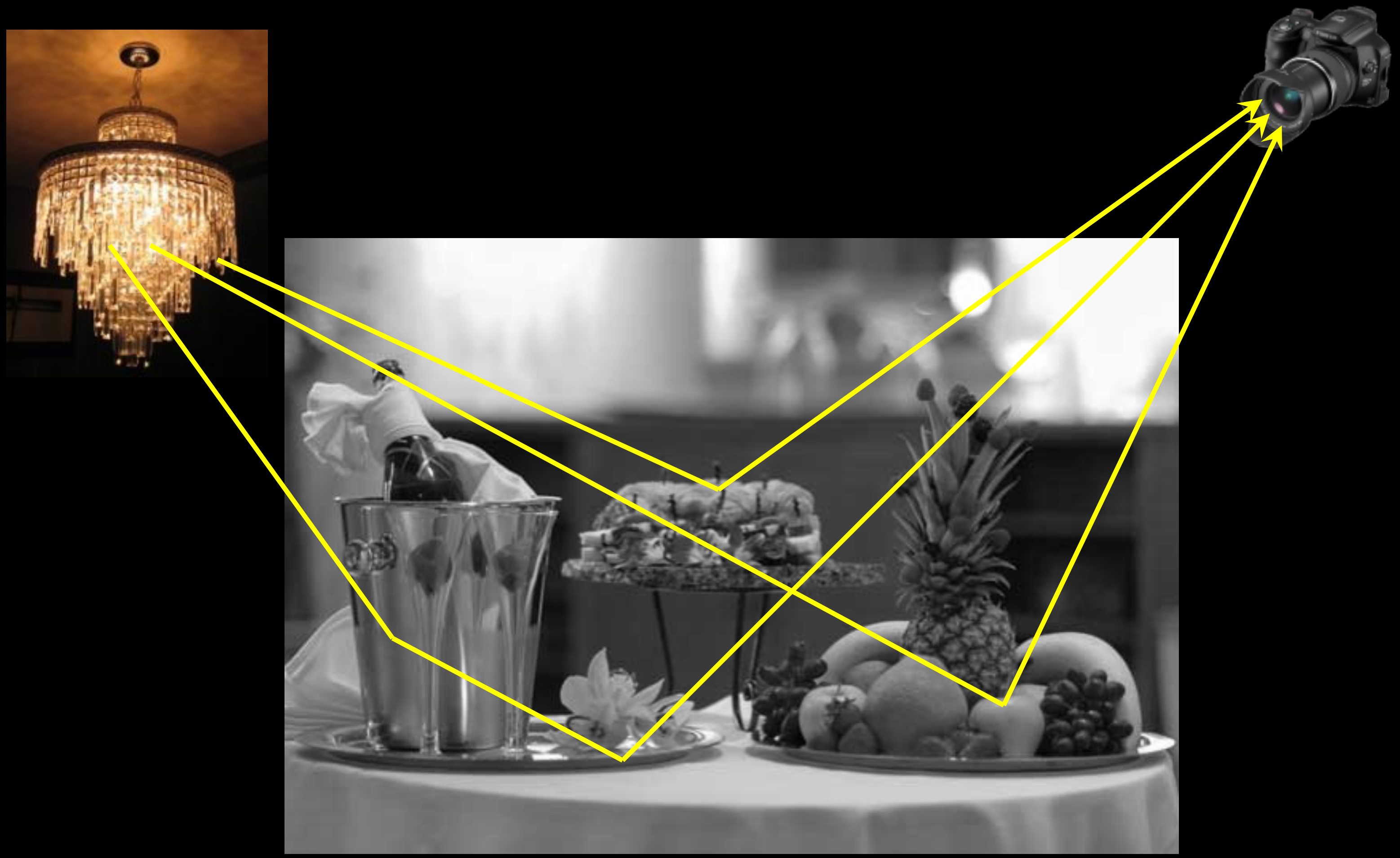


radiance: $$L_o = \int_{\mathcal{H}^2} L_i \, \rho(x, \omega_i, \omega_o) \, \mathrm{d}\mu(\omega_i)$$

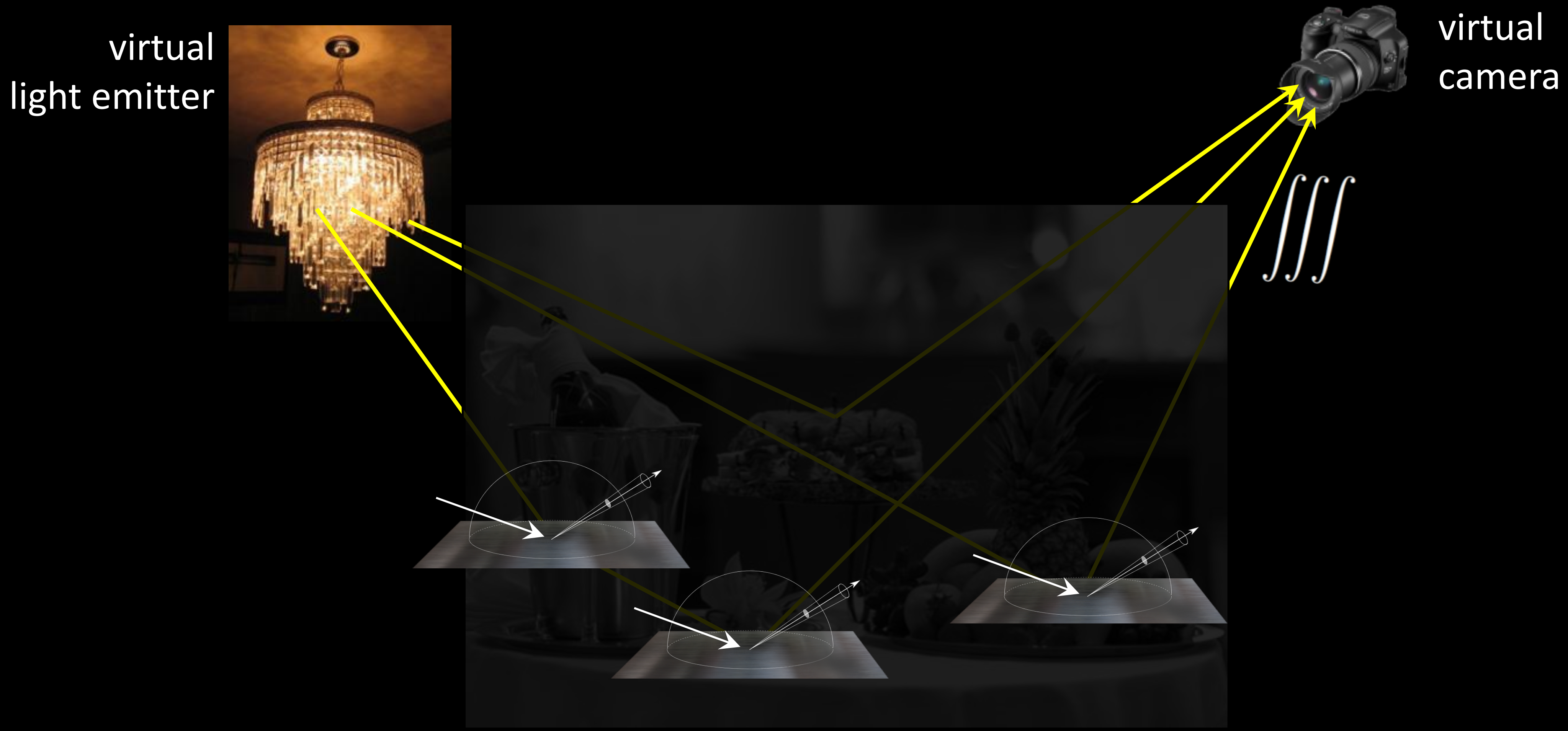
Recursive integrals

virtual
light emitter



virtual
camera

Recursive integrals



Light transport: recursive integral equation

radiance

$$L = E + KL$$

emitted radiance

integral operator

The Rendering equation [Kajiya 86]

Light Transport Operators [Arvo 94]

L is a sum of high-dimensional integrals

radiance

$$L = E + KL$$

integral operator

emitted radiance

$$IL - KL = E$$

$$(I - K)L = E$$

$$L = (I - K)^{-1}E$$

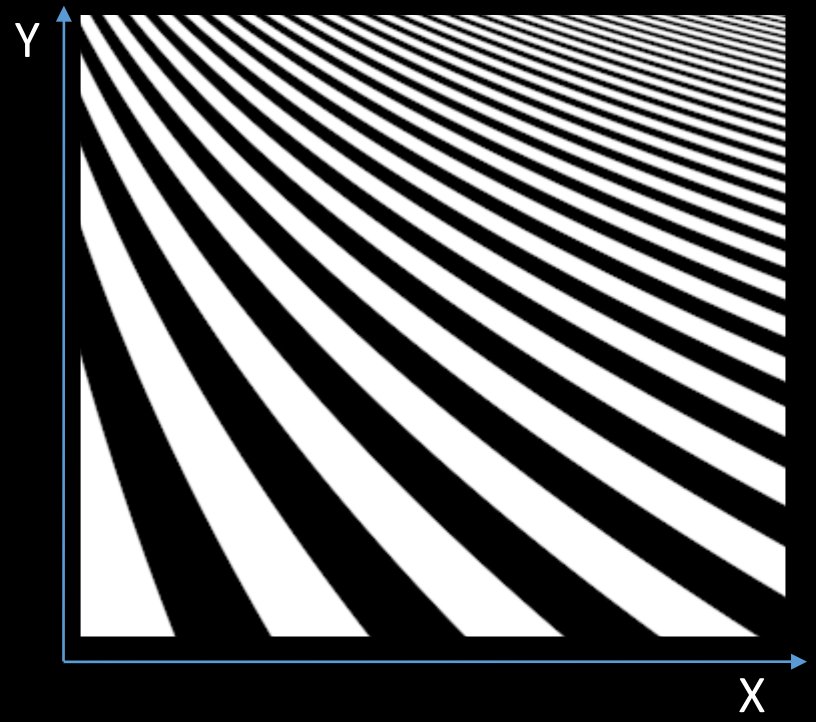
$$= (I + K + K^2 + K^3 + \ldots)E$$

$$= E + \underbrace{KE}_{\text{One bounce}} + K^2E + \underbrace{K^3E}_{\text{Three bounces}} + \ldots$$

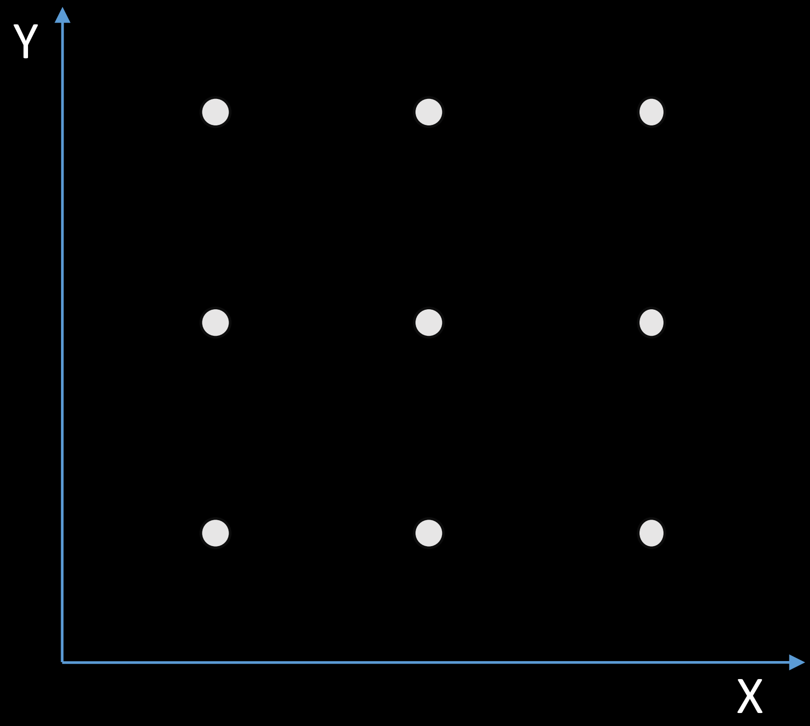
Reconstruction and integration in rendering

Reconstruction: estimate image samples

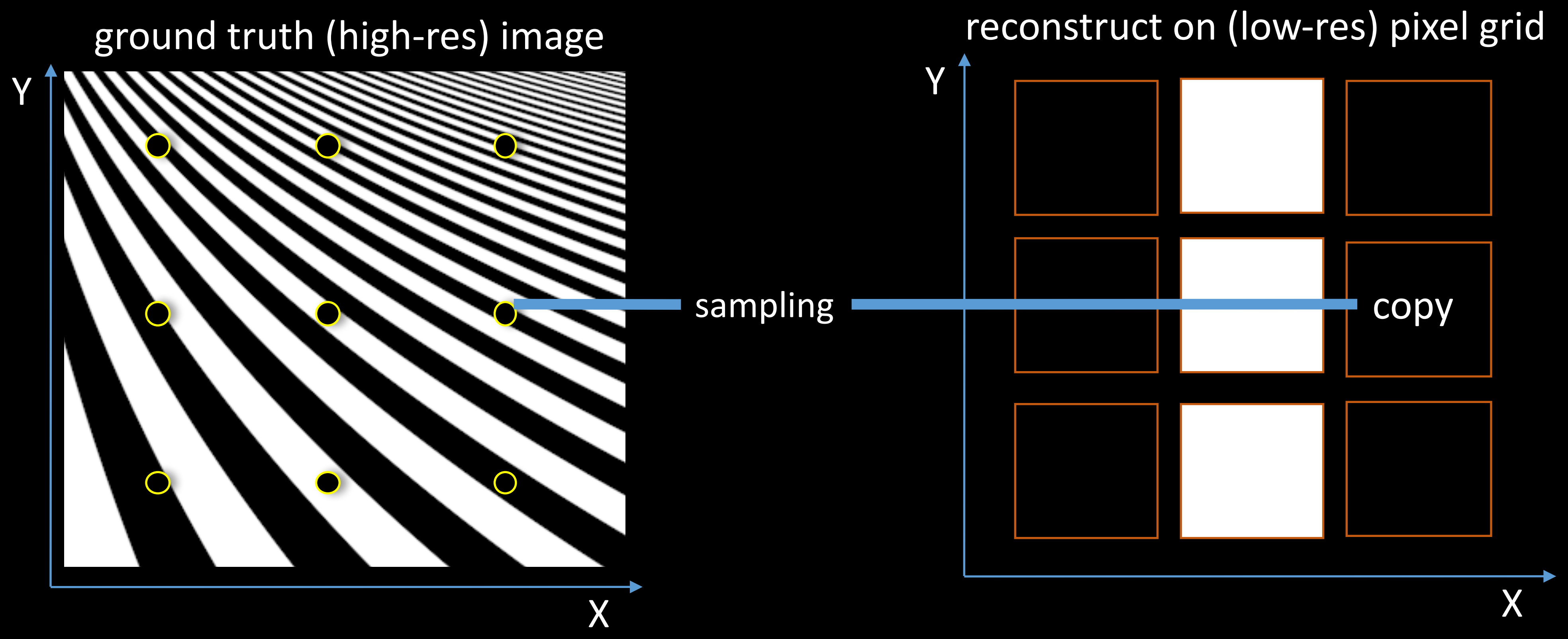
ground truth (high-res) image



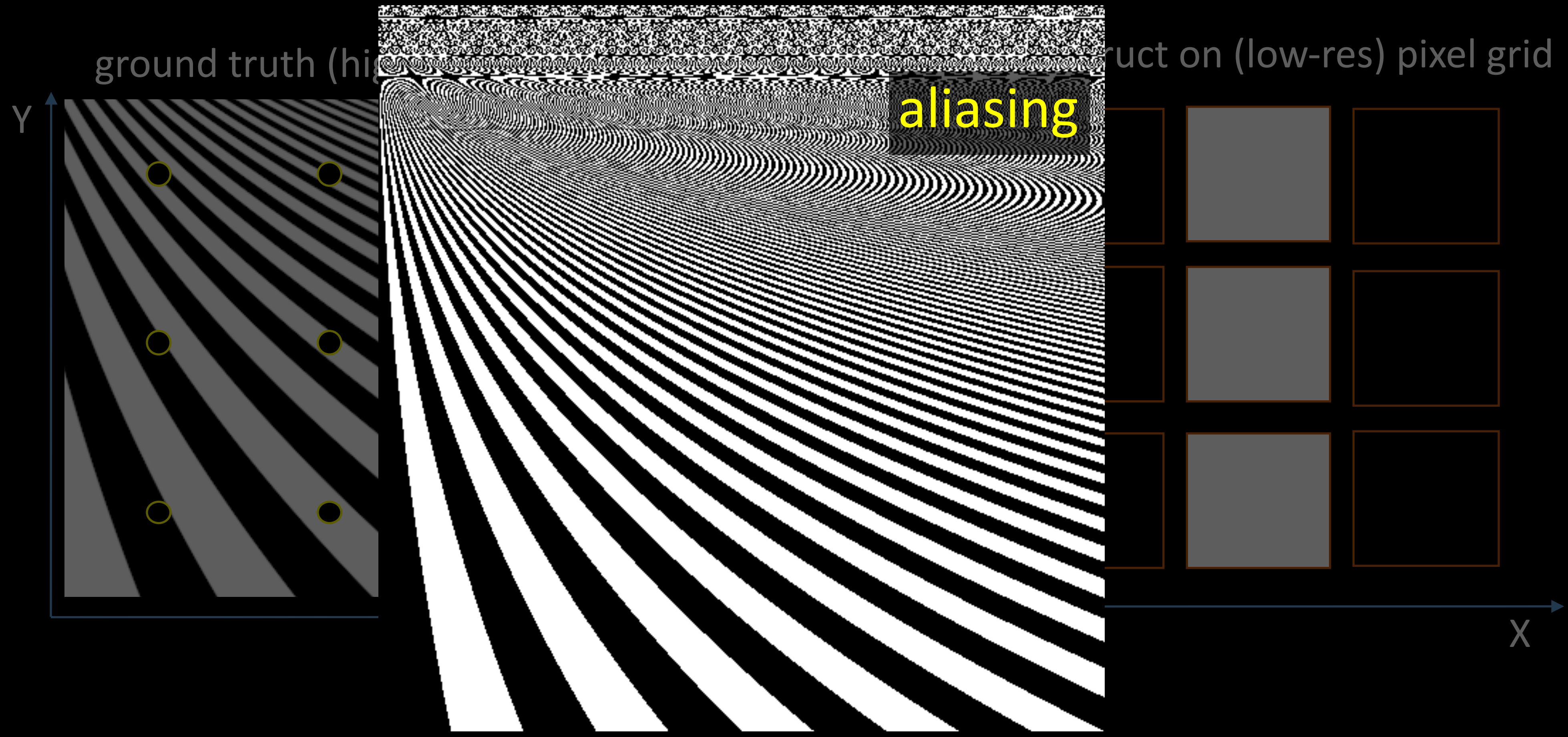
reconstruct on (low-res) pixel grid



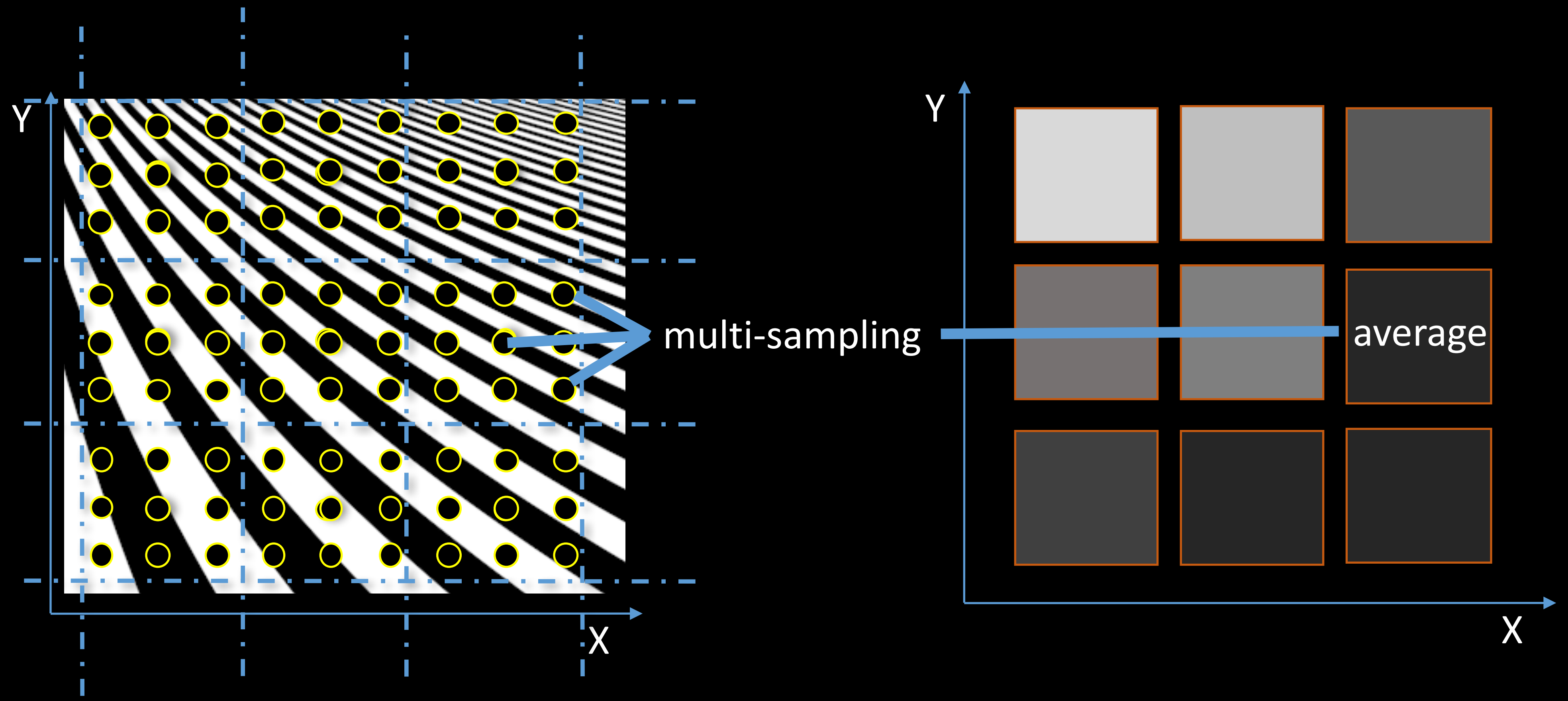
Naïve method: sample image at grid locations



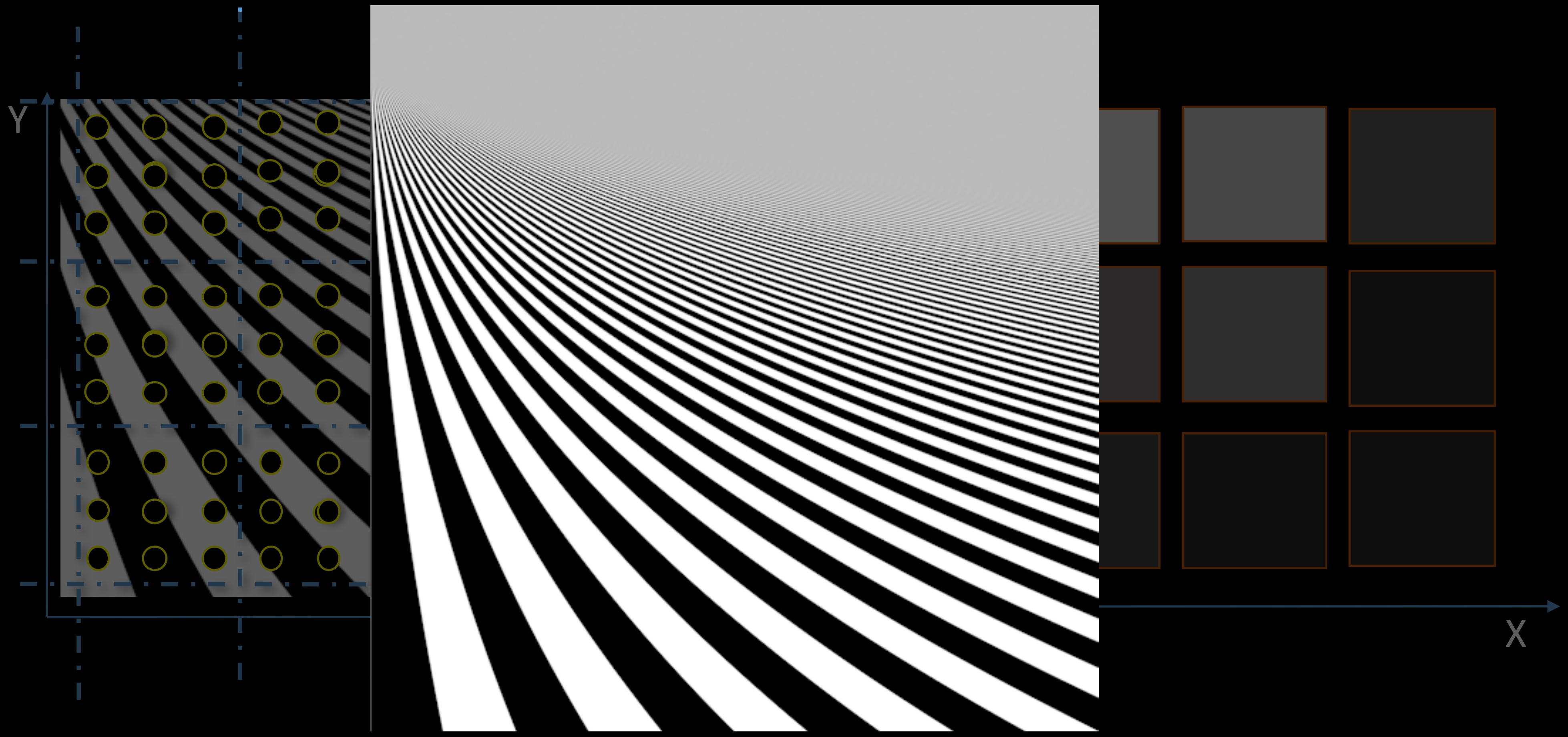
Naïve method: when sampling is increased



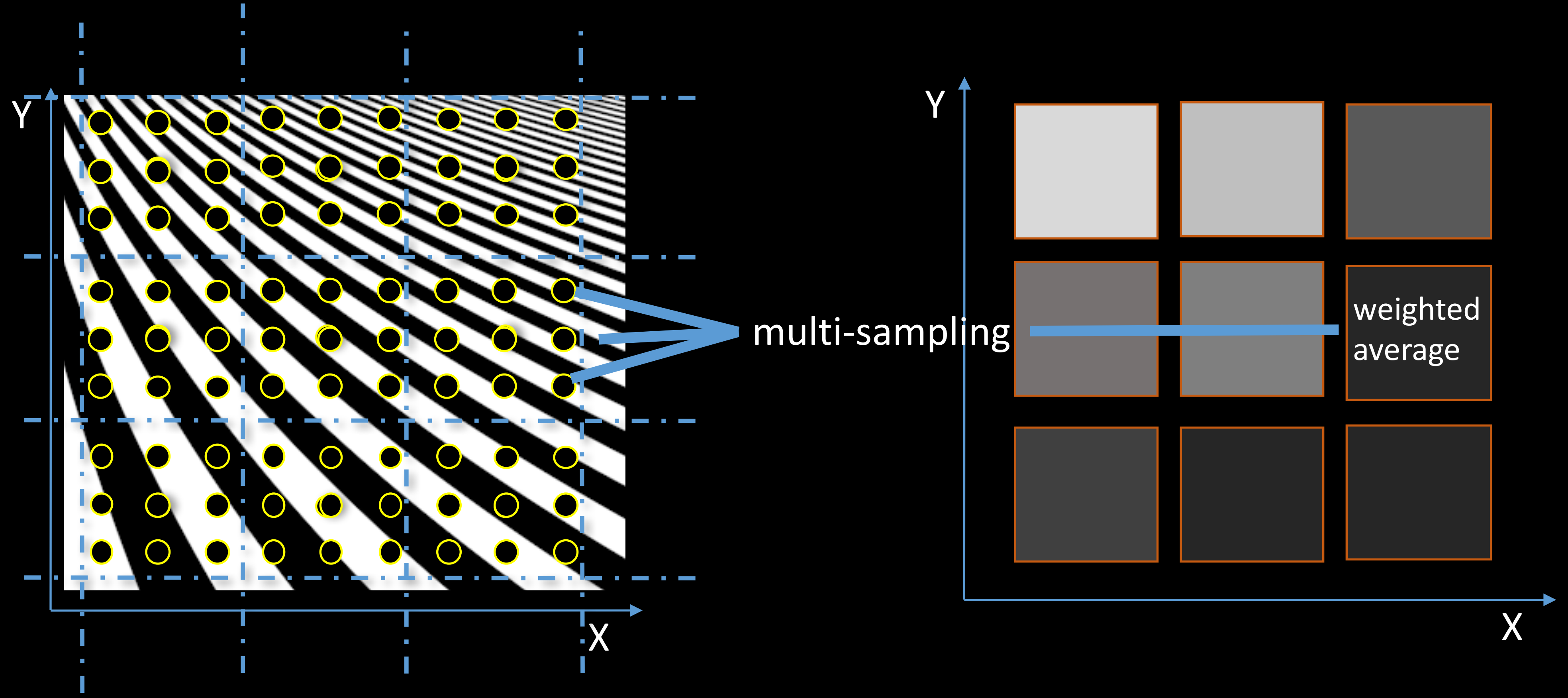
Antialiasing: assuming `square' pixels



Antialiasing is costly due to multi-sampling

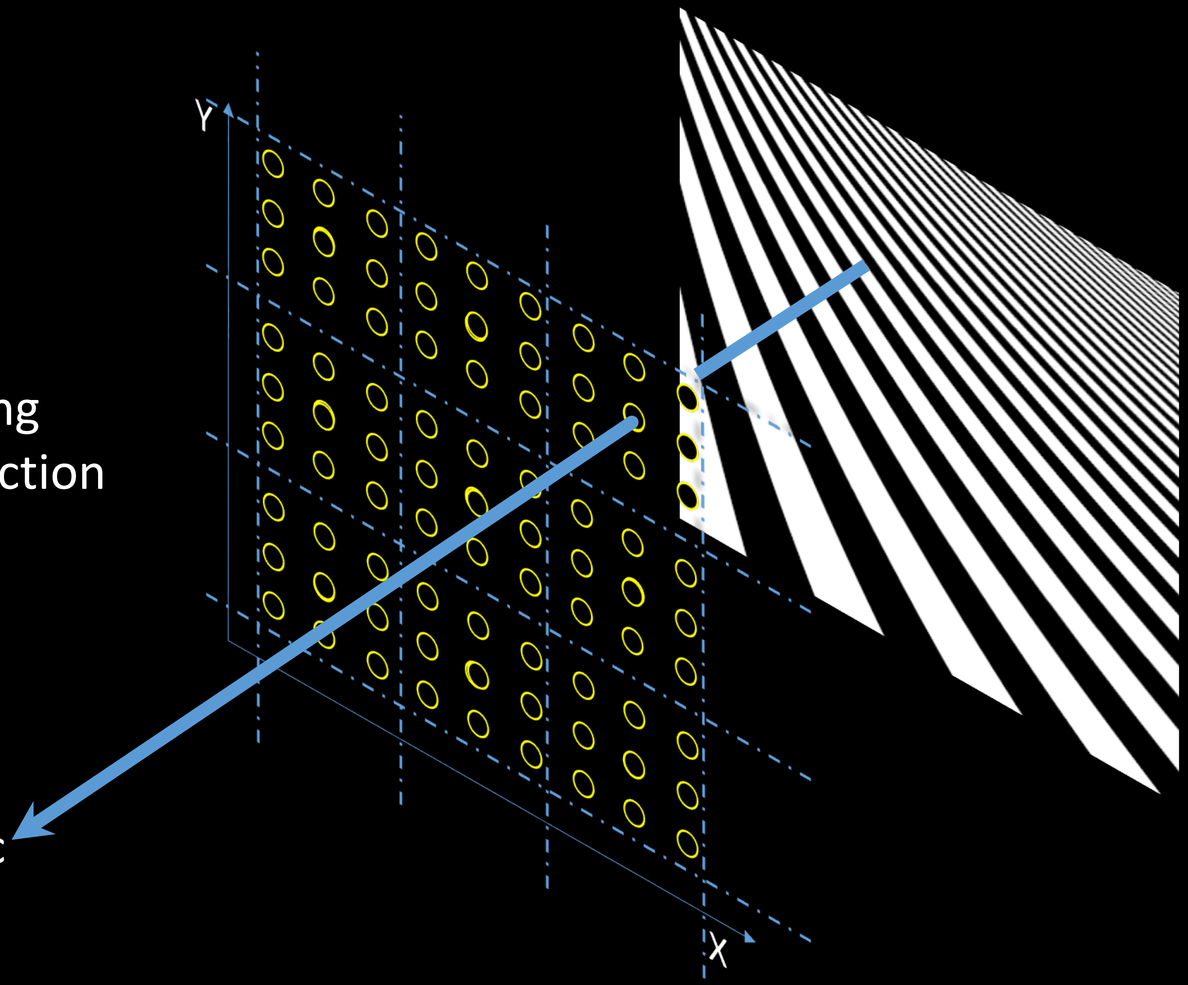


Antialiasing using general reconstruction filter



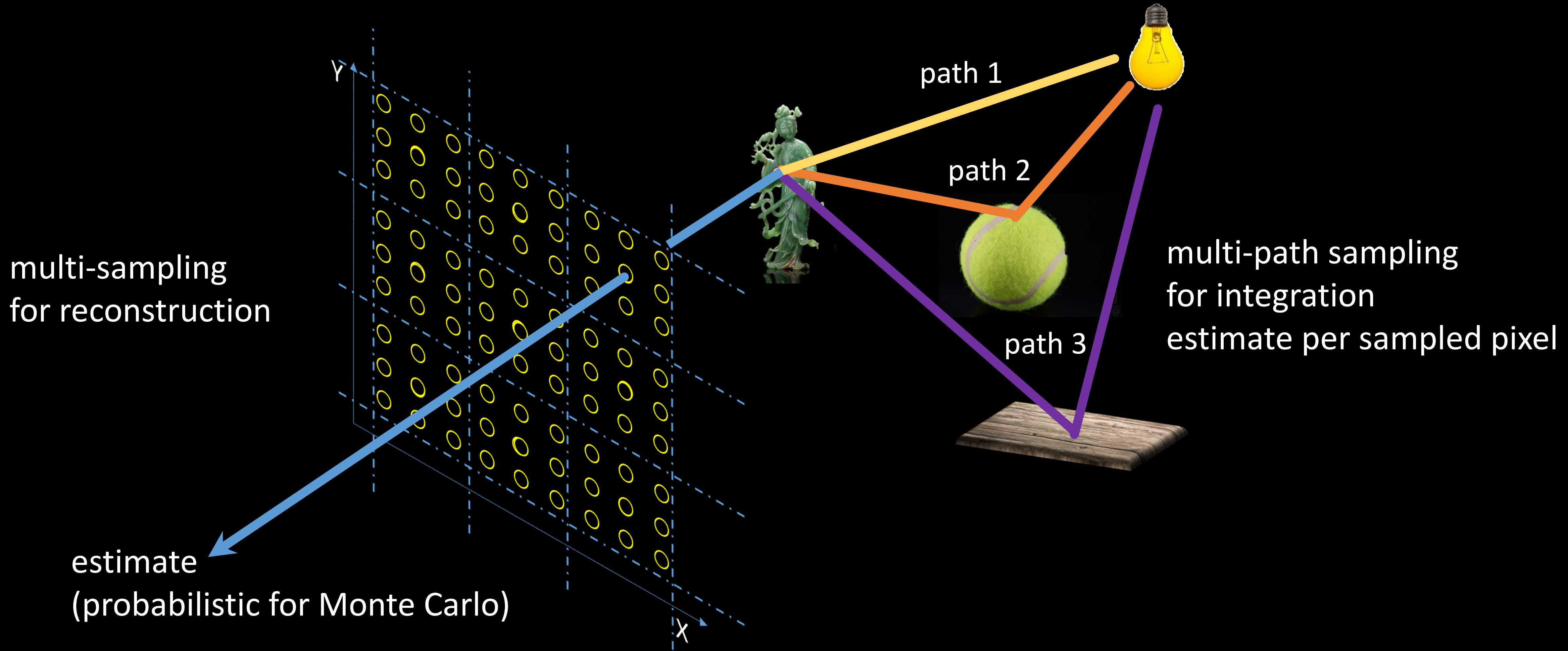
Rendering: Reconstructing integrals

multi-sampling
for reconstruction



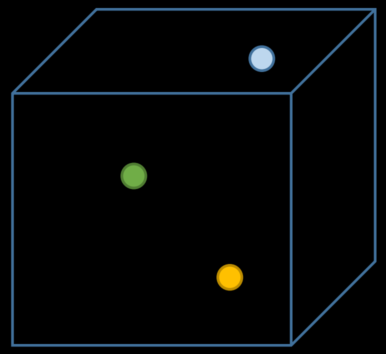
deterministic

Rendering: Reconstructing integrals

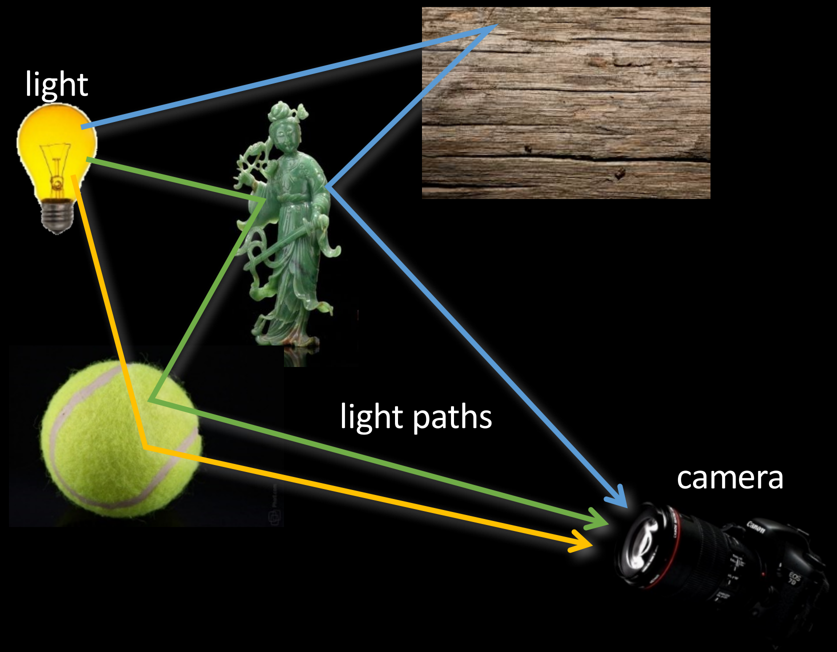


Function-space view: Sampling in path space

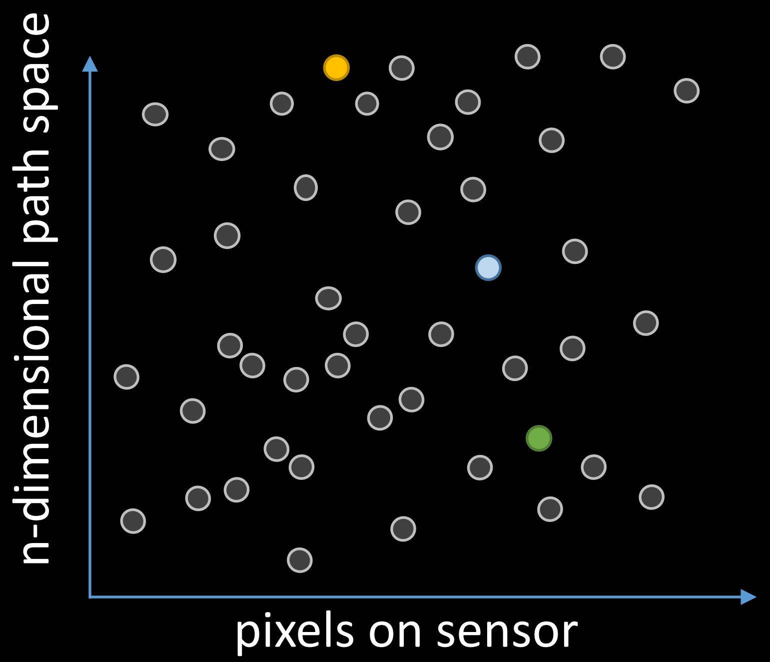
each sample represents a path
and has an associated radiance value



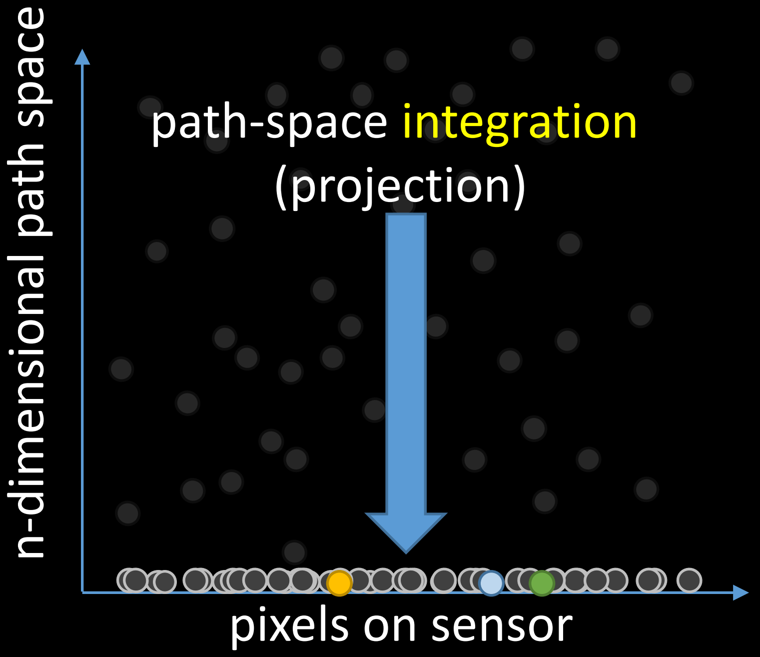
n-dimensional path space

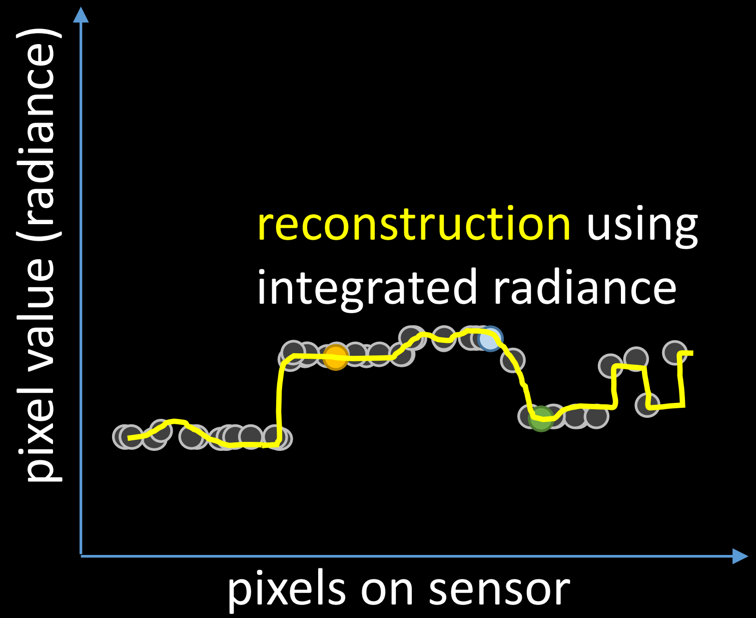


Sample locations shown in path-pixel space

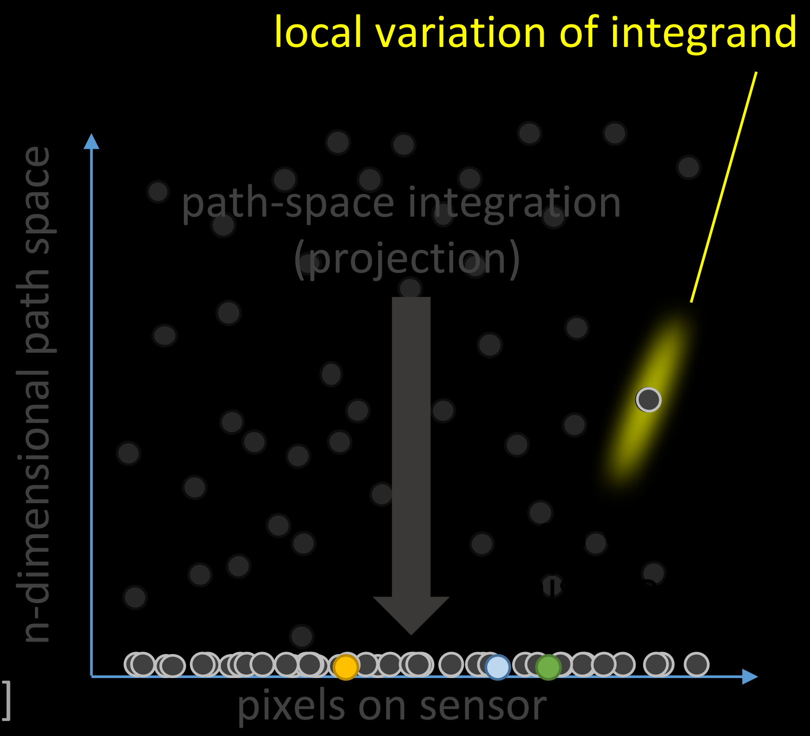


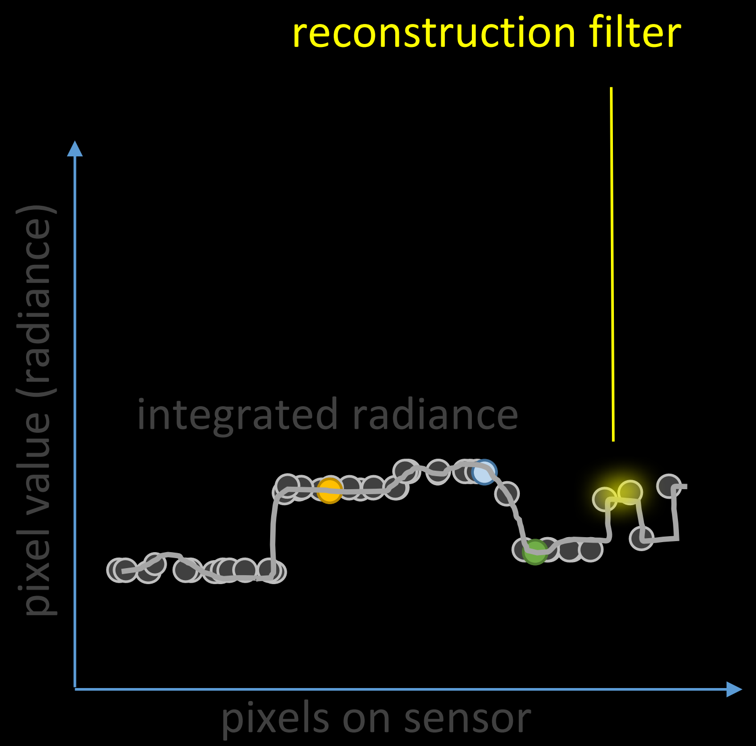
Rendering = integration + reconstruction





Frequency analysis of lightfields in rendering

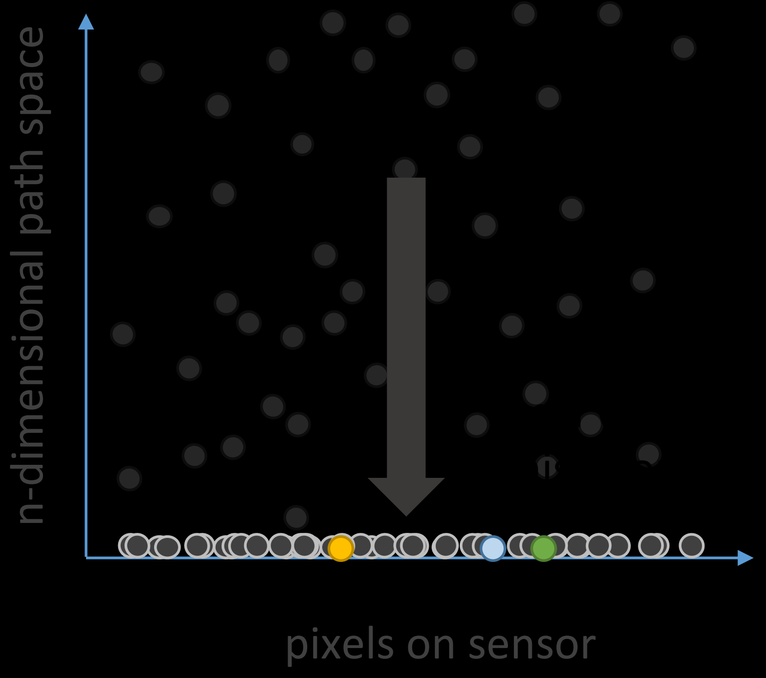




[Ramamoorthi et al. 04]
[Durand et al. 05]
[Soler et al. 2009]
[Overbeck et al. 2009]
[Egan et al. 2009, 2011]
[Ramamoorthi et al. 2012]

Freq. analysis of MC sampling: This course!

Assessing MSE, bias, variance and convergence of Monte Carlo estimators as a function of the Fourier spectrum of the sampling function.



[Durand 2011]
[Ramamoorthi et al. 12]
[Subr and Kautz 2013]
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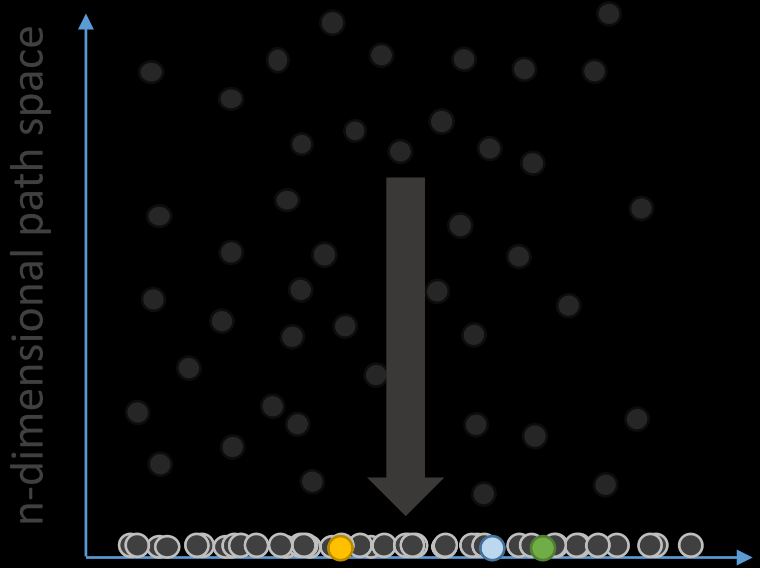

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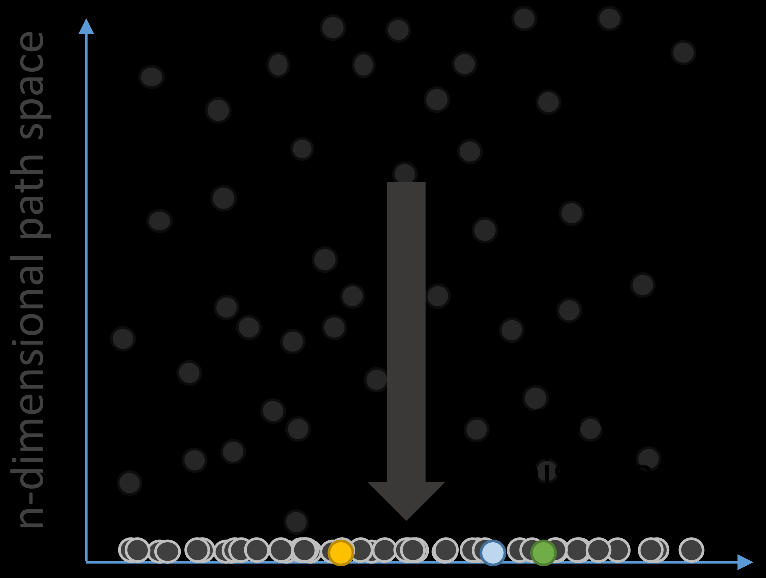

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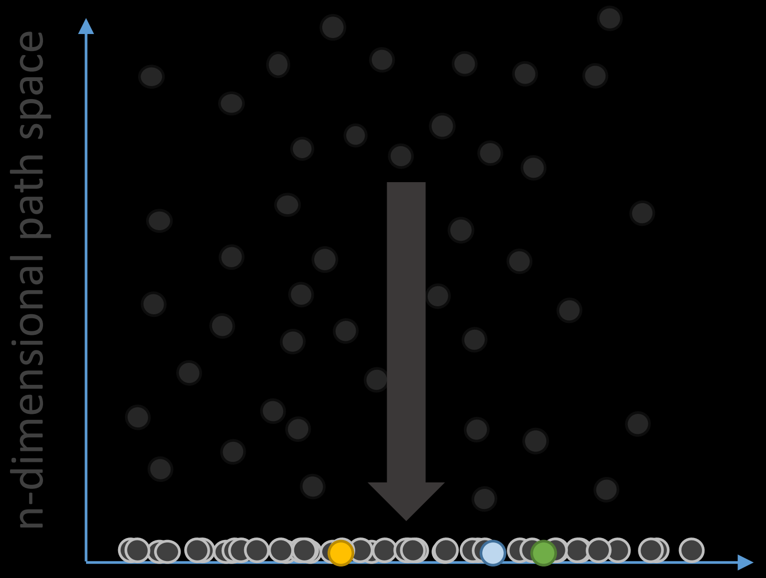

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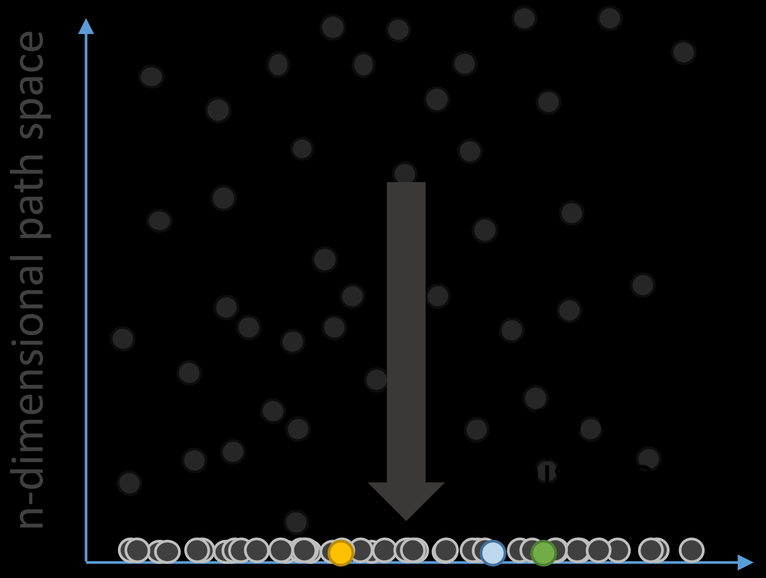

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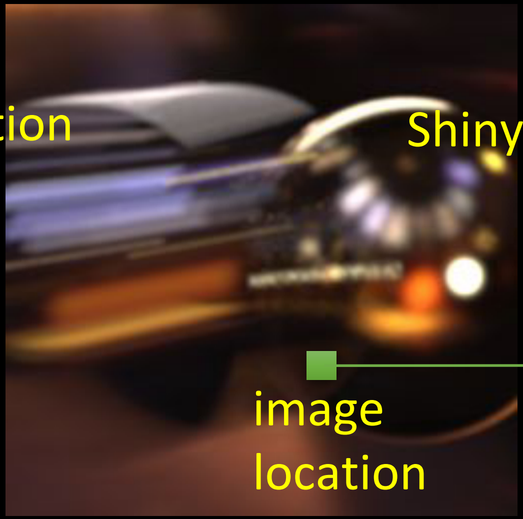
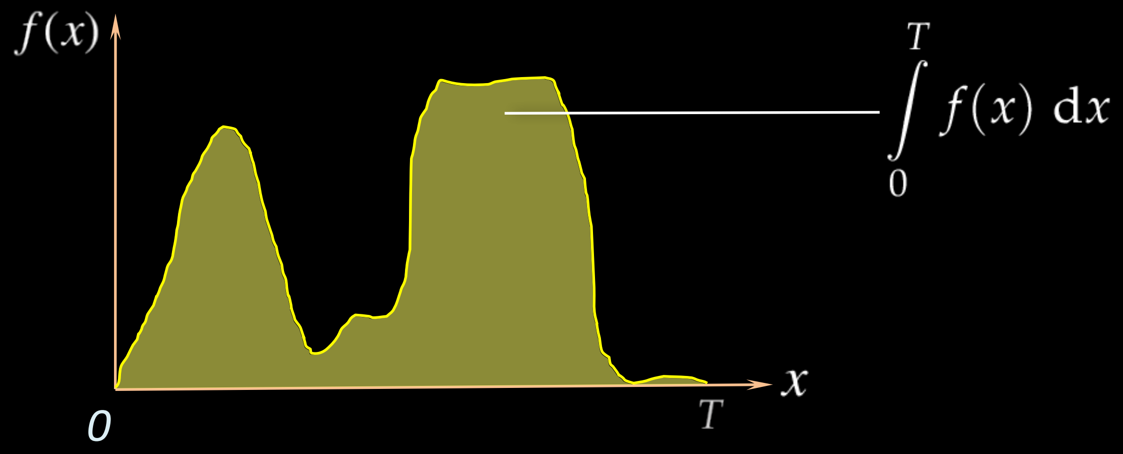
Rendering = integration + reconstruction



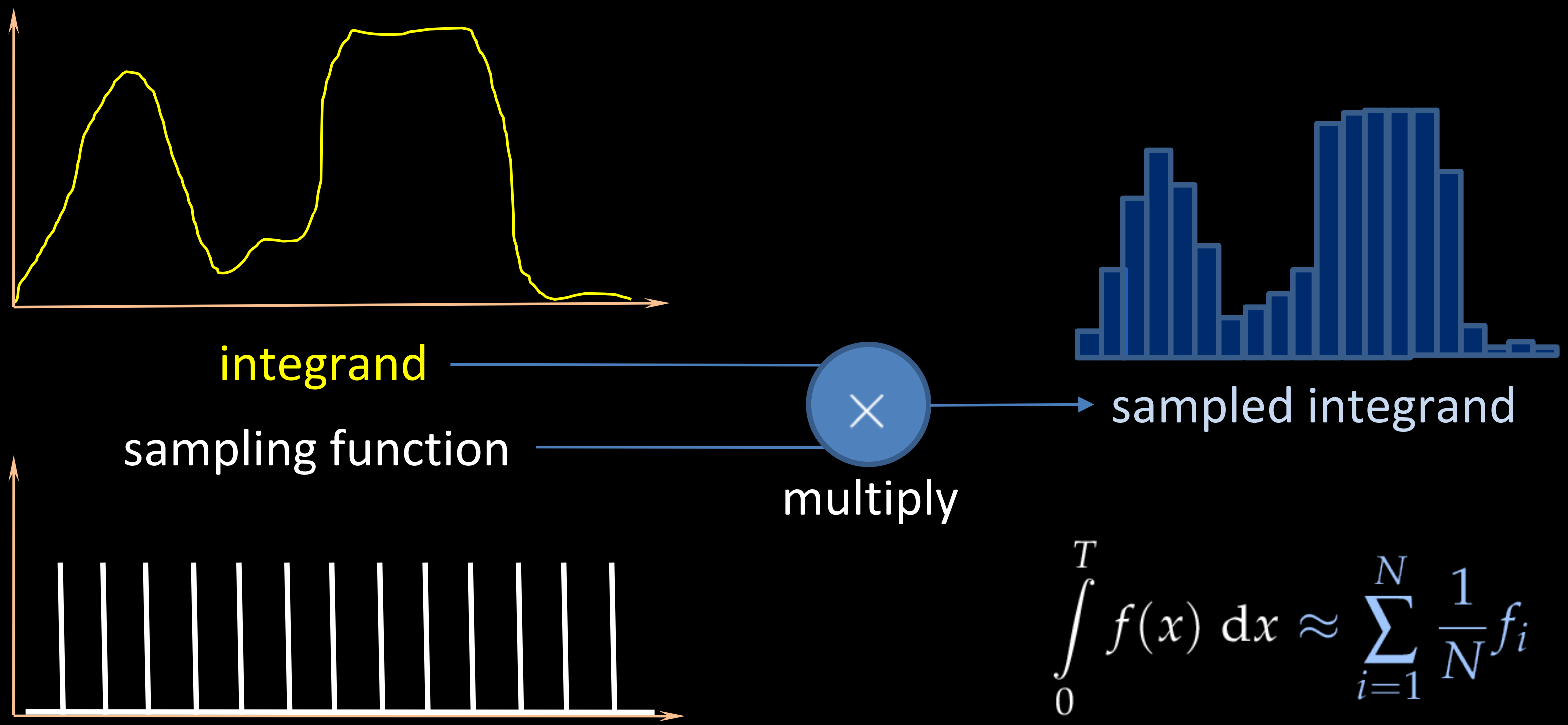
Integrand: radiance (W m^{-2} Sr^{-1})

Domain: shutter time x aperture area x 1st bounce x 2nd bounce …

The problem in 1D

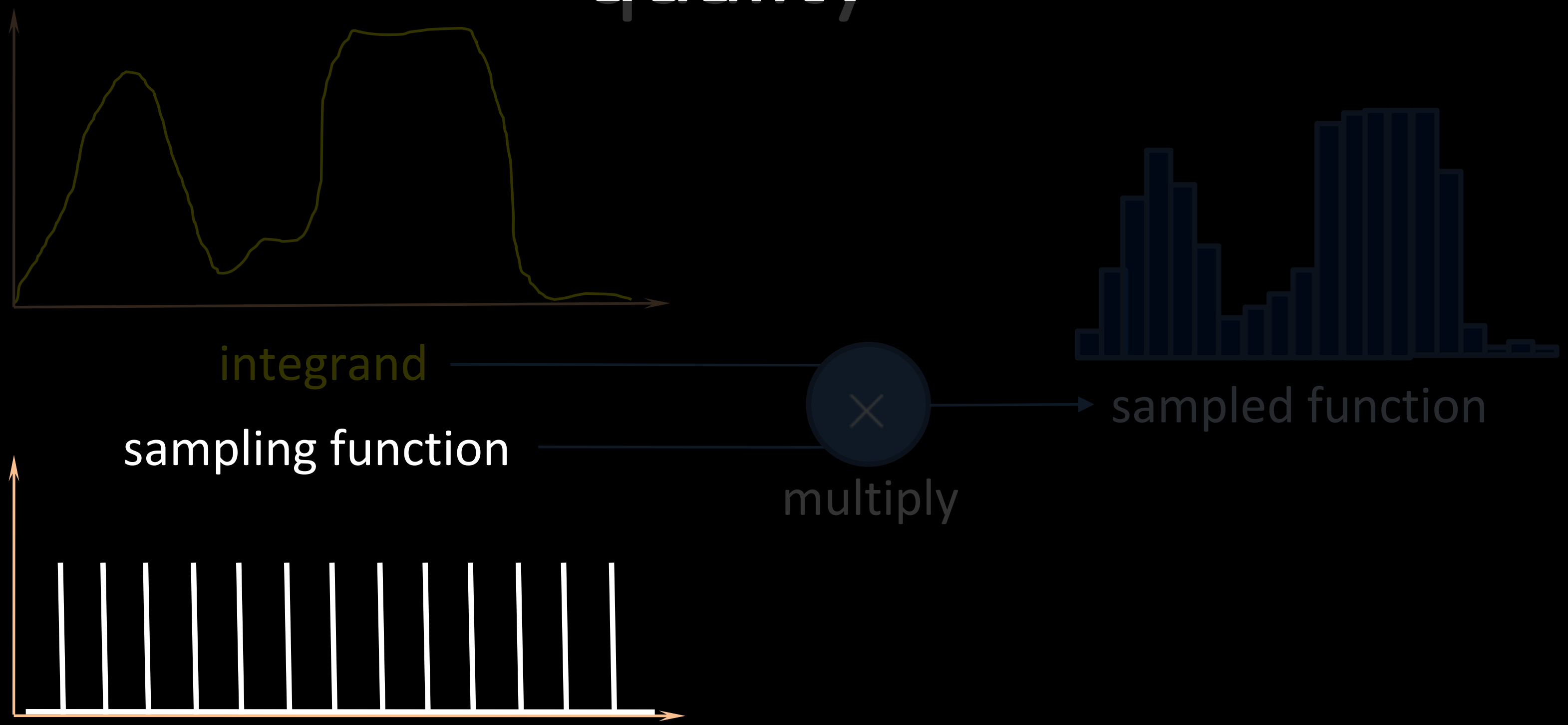


the sampling function



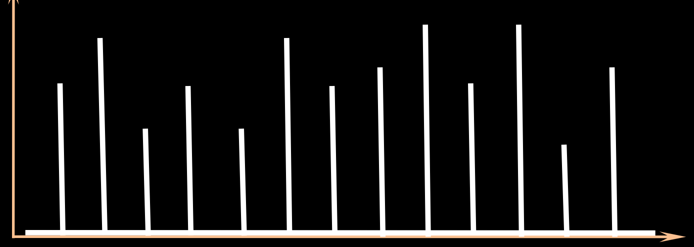
$$\int_0^T f(x)\,\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N} f_i$$

sampling function decides integration quality

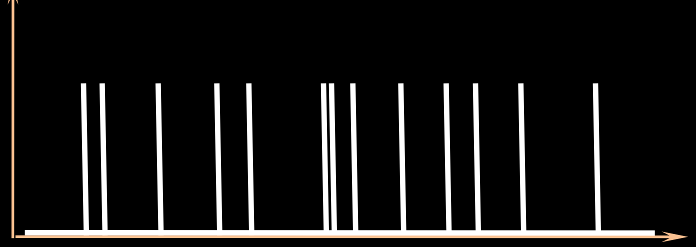


strategies to improve estimators

1. modify weights

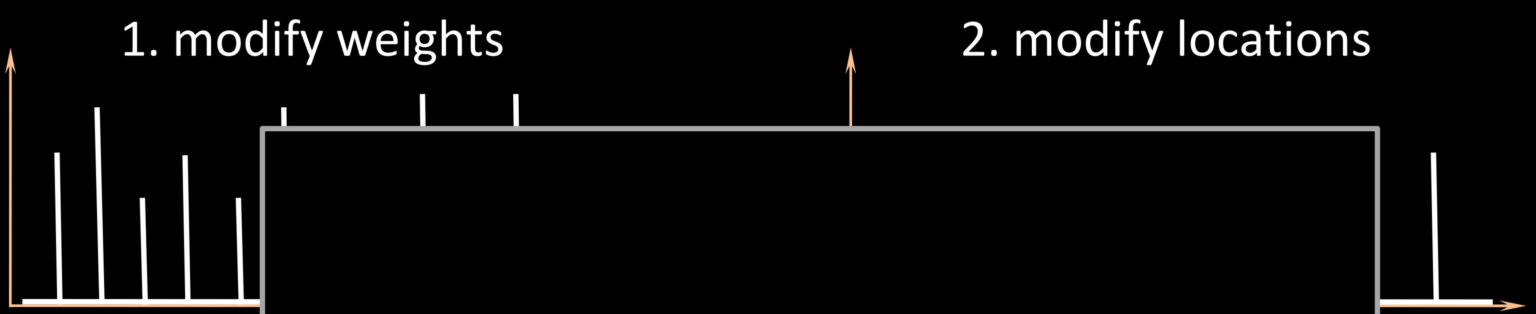


2. modify locations



eg. quadrature rules, importance sampling, jittered sampling, etc.

insight into impact: Fourier domain



analyse sampling function in Fourier domain

Fourier analysis: origin and intuition

- Eigenfunction of the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\lambda x} = \underbrace{\lambda}_{\text{scaling}} e^{\lambda x}$$

- Turns differential equations into algebraic equations

Fourier analysis: origin and intuition

- Eigenfunction of the differential operator

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\lambda x} = \underbrace{\lambda}_{\text{scaling}} e^{\lambda x}$$

- Turns differential equations into algebraic equations

- if $f(x) = \sum_{i=1}^{N} e^{\lambda_i x}, \quad \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \sum_{i=1}^{N} \lambda_i e^{\lambda_i x}$

projection

The Fourier domain

Image credit: Wikipedia

The continuous Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \imath \omega x} \mathrm{d}x$$

Fourier domain

primal (space, time, etc.) domain

The Fourier transform: `frequency' domain

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi \imath \omega x} \mathrm{d}x$$

$$\underbrace{\hat{f}(\omega)}_{\text{frequency domain}} = \int_{-\infty}^{\infty} f(x) \cos(2\pi \underbrace{\omega}_{\text{frequency}} x) \mathrm{d}x + \imath \int_{-\infty}^{\infty} f(x) \sin(2\pi \omega x) \mathrm{d}x$$

projection onto sin and cos

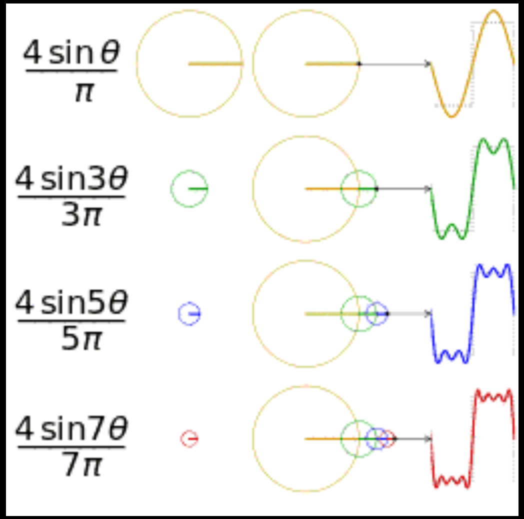
A single sample: $f(x) = \delta(x - x_k)$

$$\hat{f}(\omega) = \underbrace{}_{\text{amplitude = 1}} e^{-\underbrace{2\pi\imath x_k \omega}_{\text{phase}}}$$

$$\hat{f}(\omega) = \cos(2\pi\imath x_k\omega) + \imath \sin(2\pi\imath x_k\omega)$$

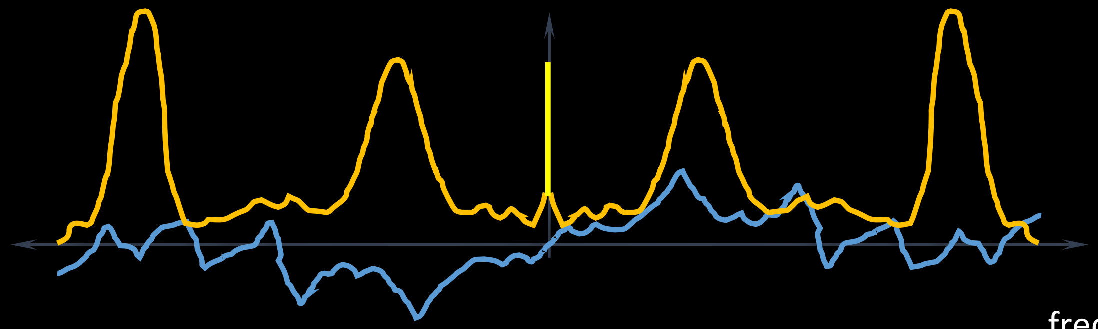
Fourier series: replace integral with sum

approximating a square wave using 4 sinusoids



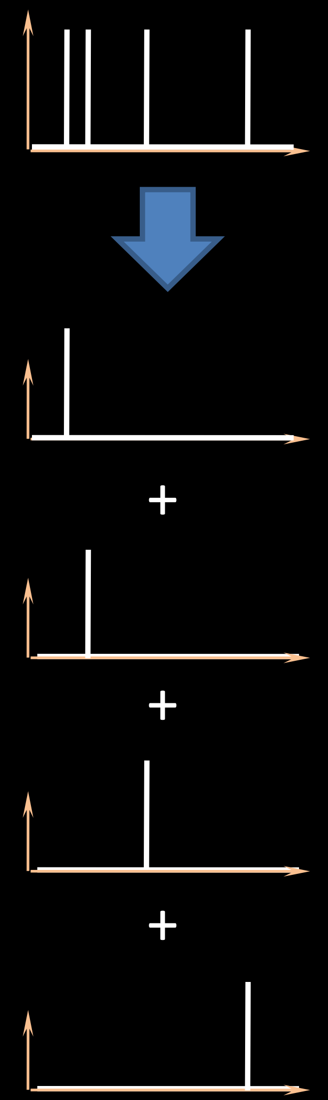
Fourier spectrum of the sampling function

$$S(x) = \sum_{k=1}^{N} \delta(x - x_k)$$

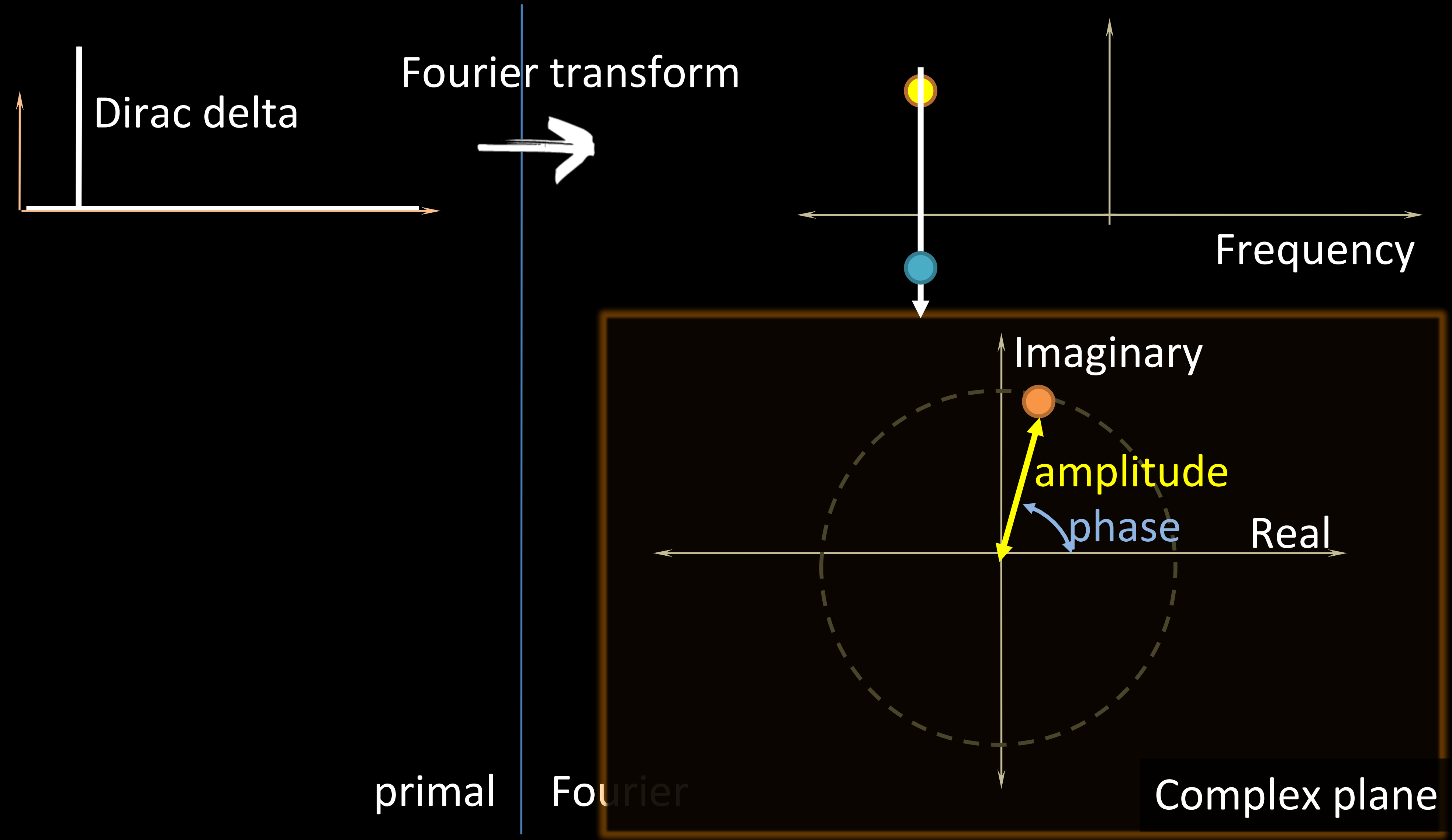


$$\hat{S}(\omega) = \sum_{k=1}^{N} e^{-2\pi \imath x_k \omega}$$

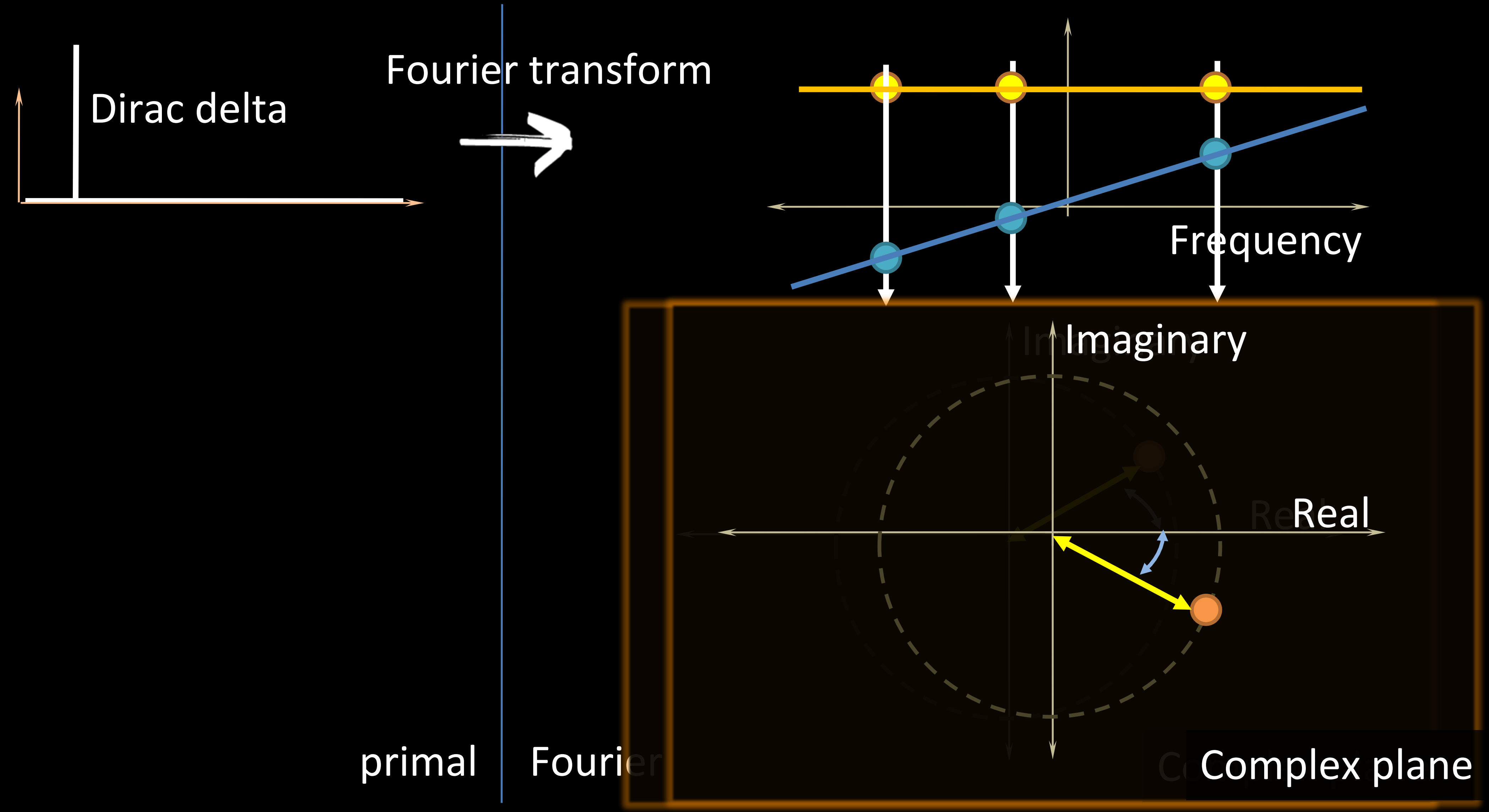
sampling function = sum of Dirac deltas



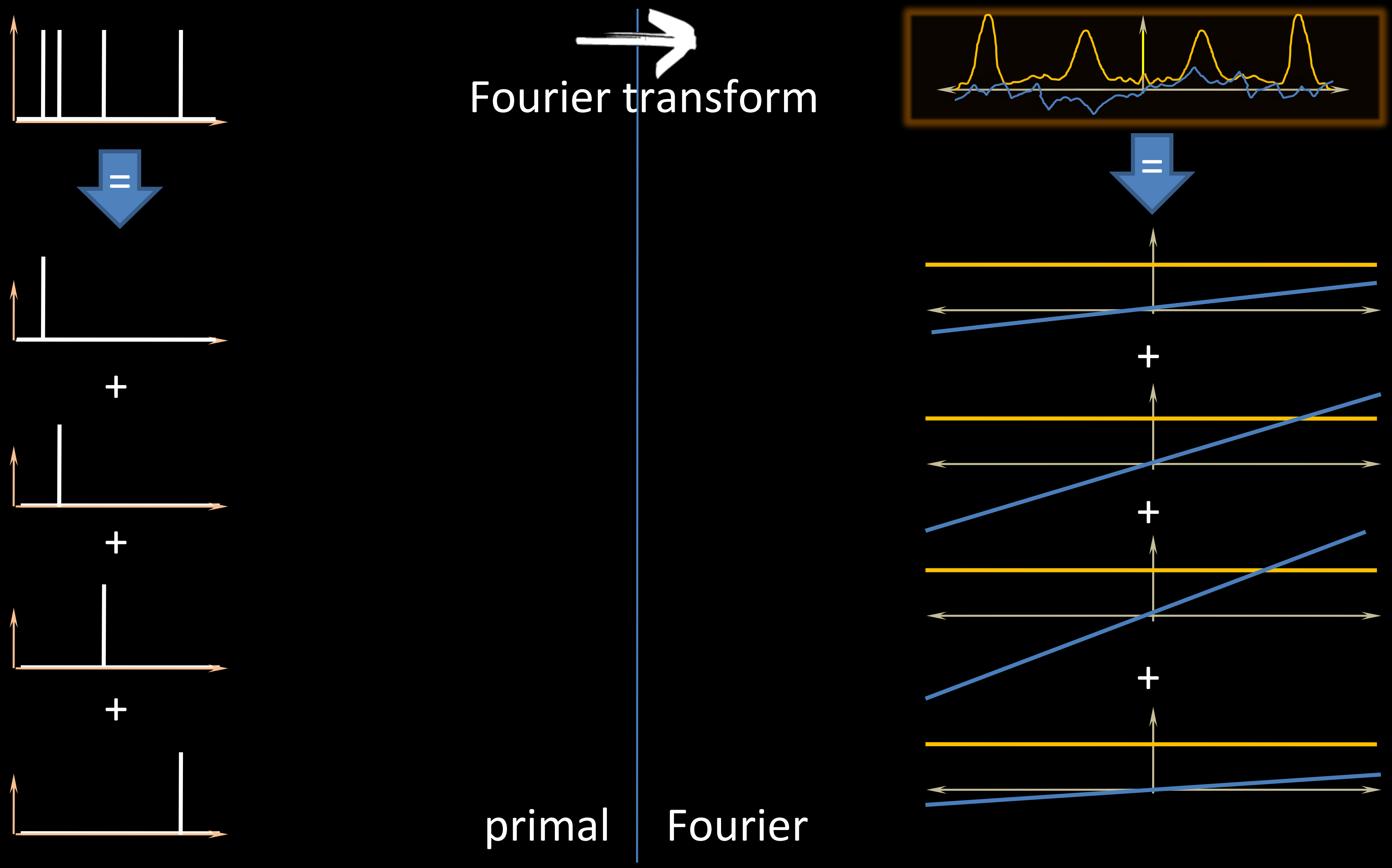
In the Fourier domain ...



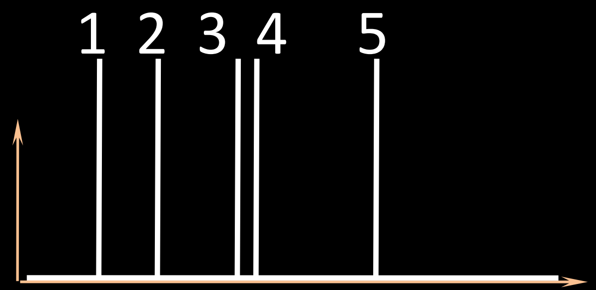
Review: in the Fourier domain …



amplitude spectrum is not flat

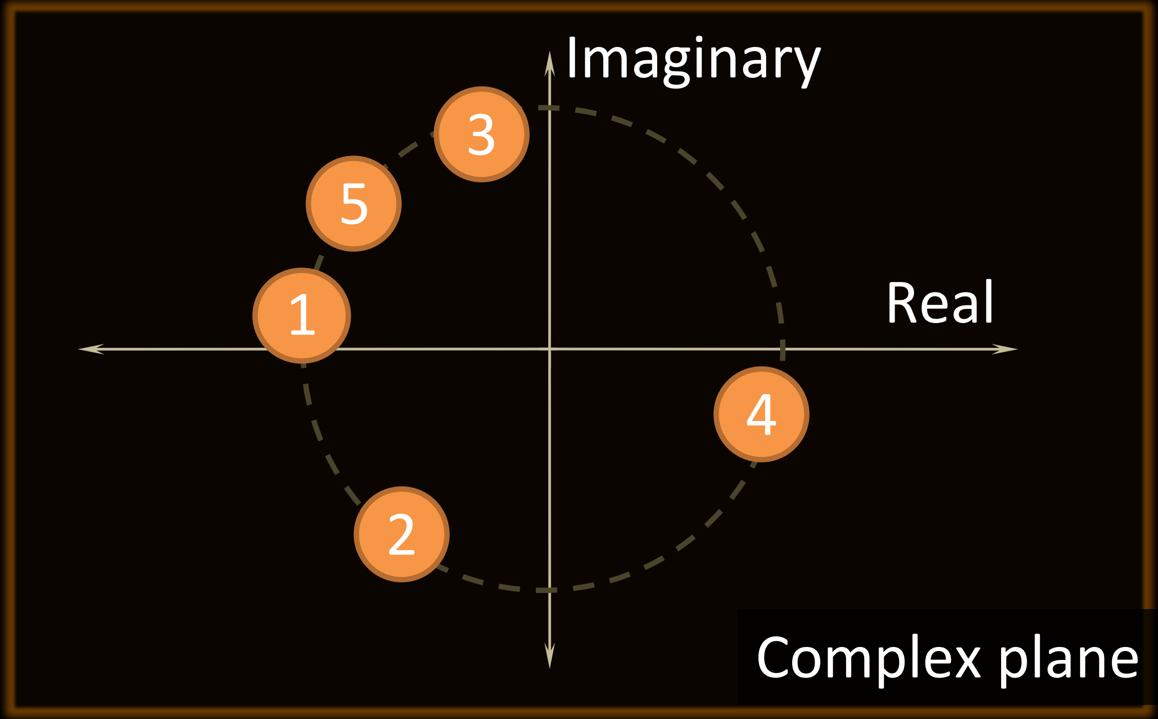


sample contributions at a given frequency

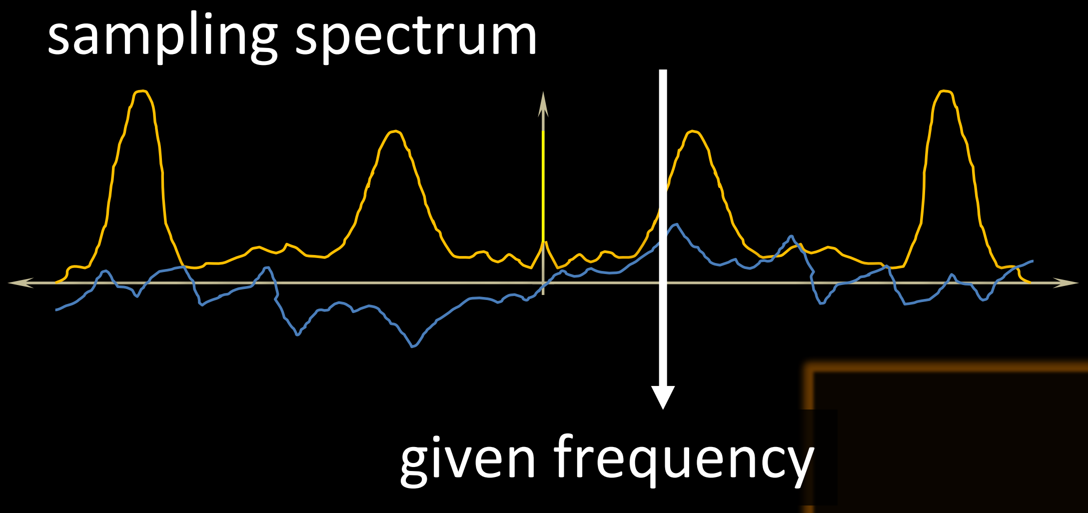


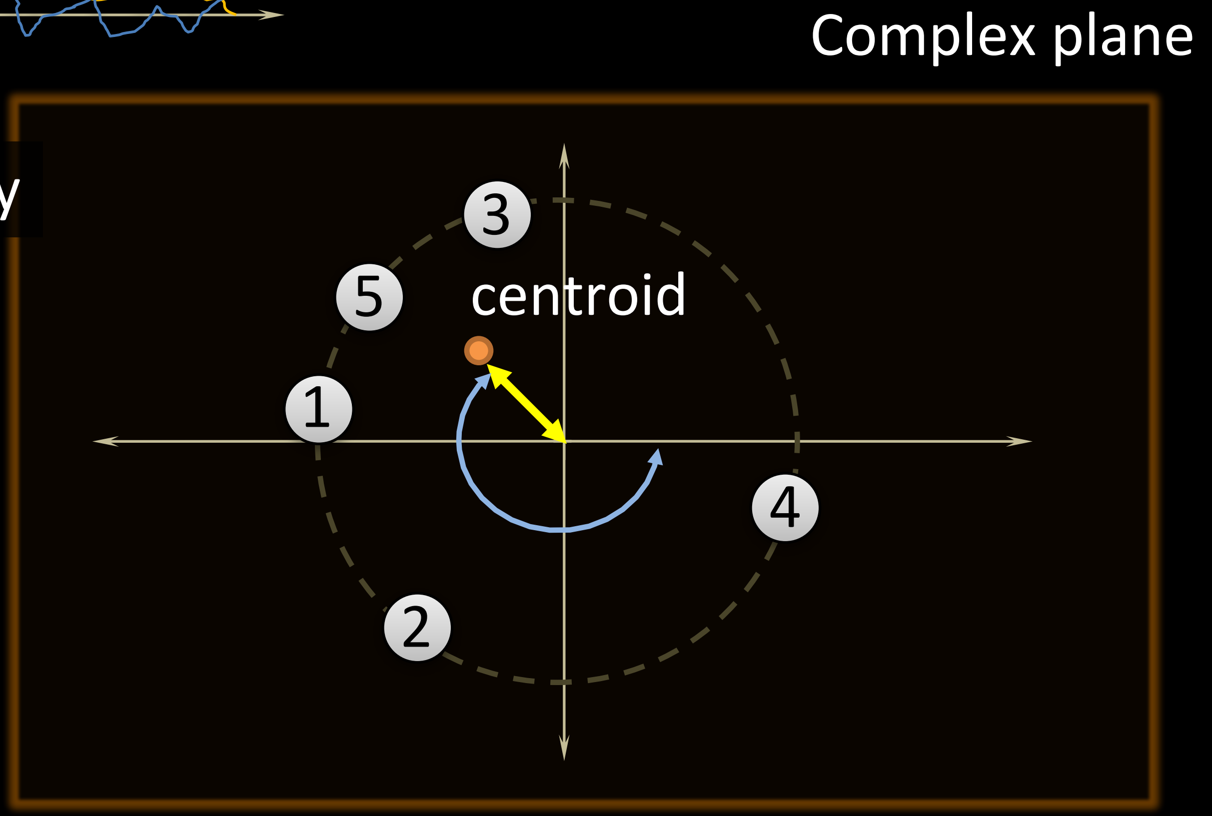
sampling function

At a given frequency

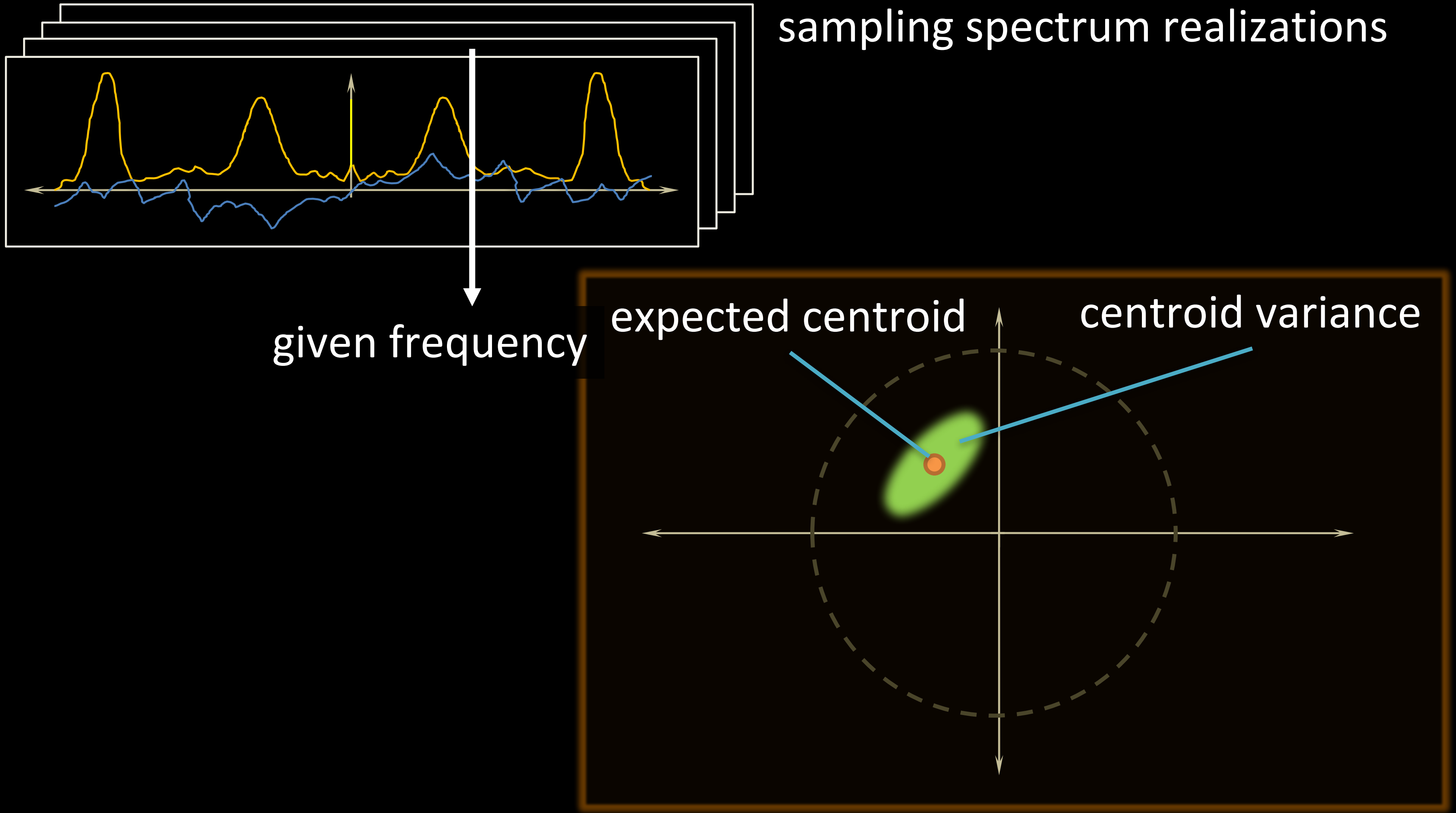


the sampling spectrum at a given frequency

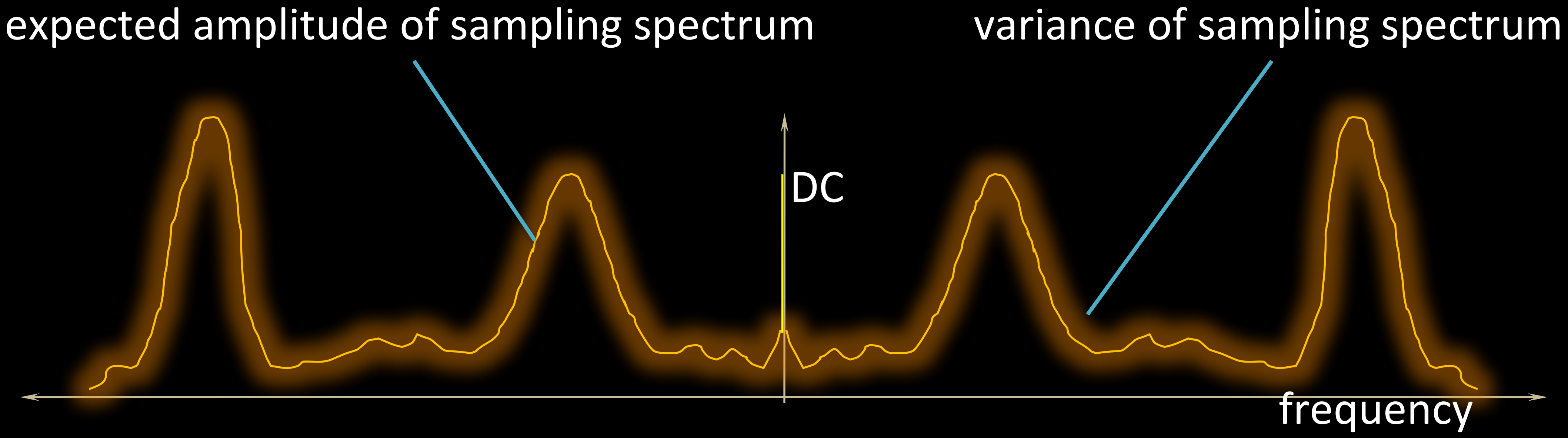




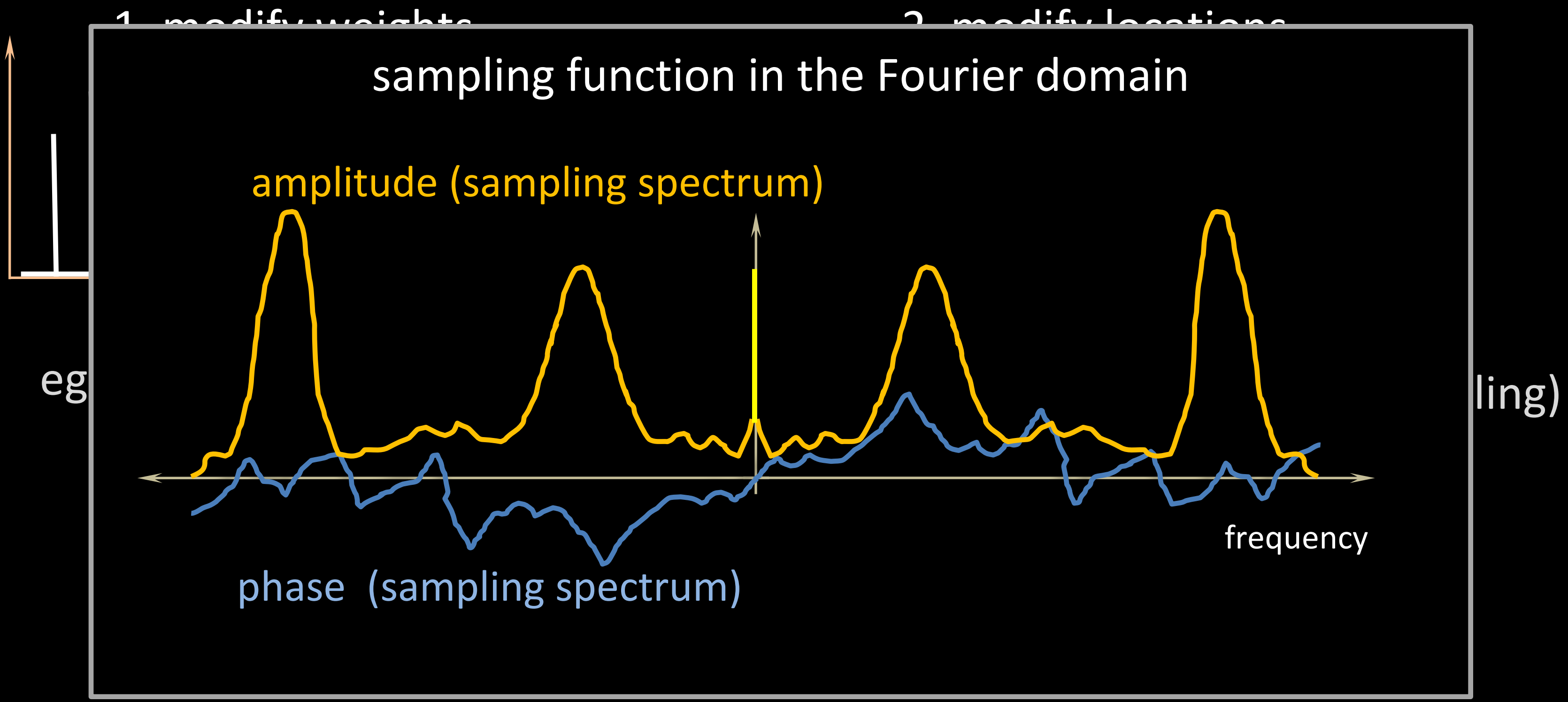
the sampling spectrum at a given frequency



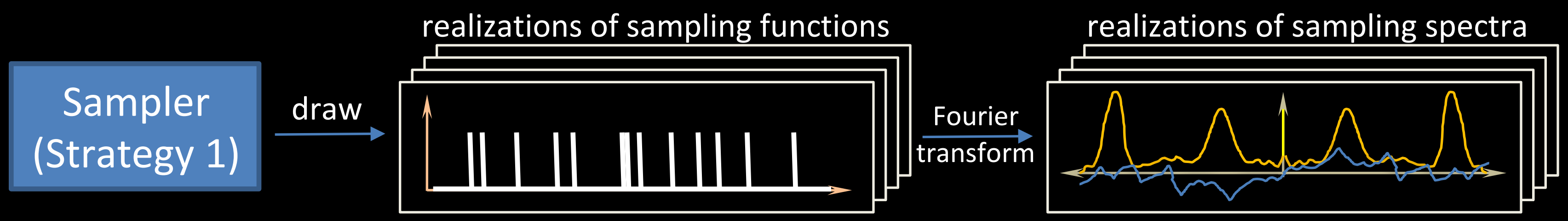
expected sampling spectrum and variance



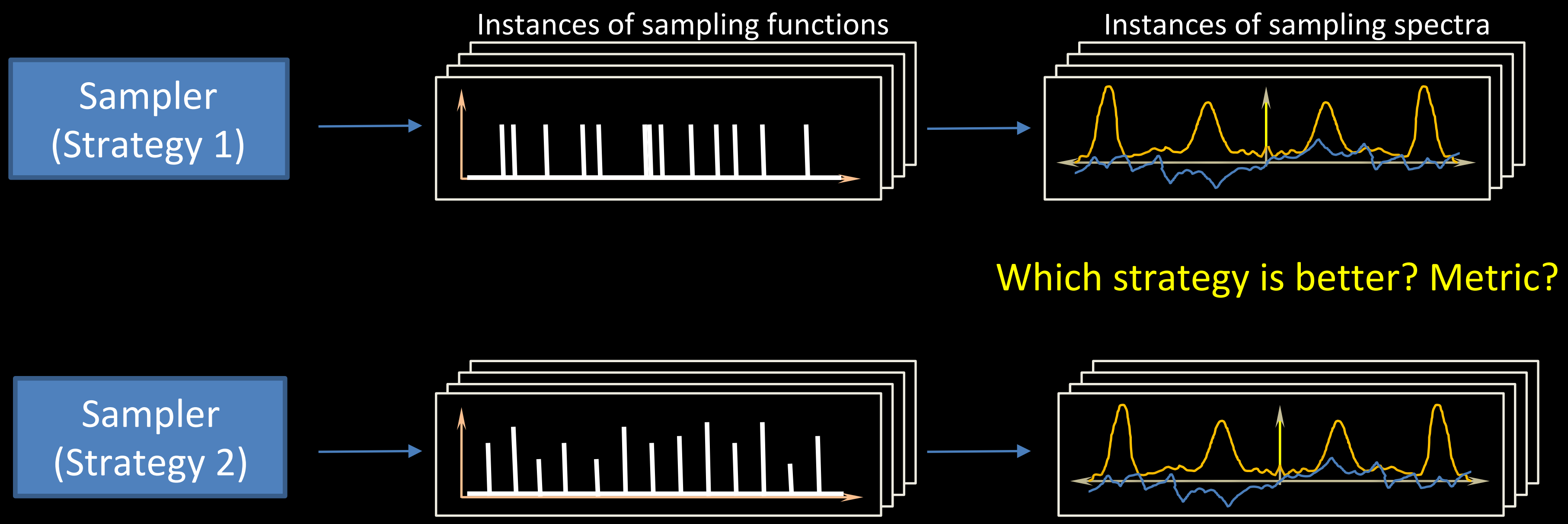
Abstracting sampling strategy using spectra



stochastic sampling & instances of spectra

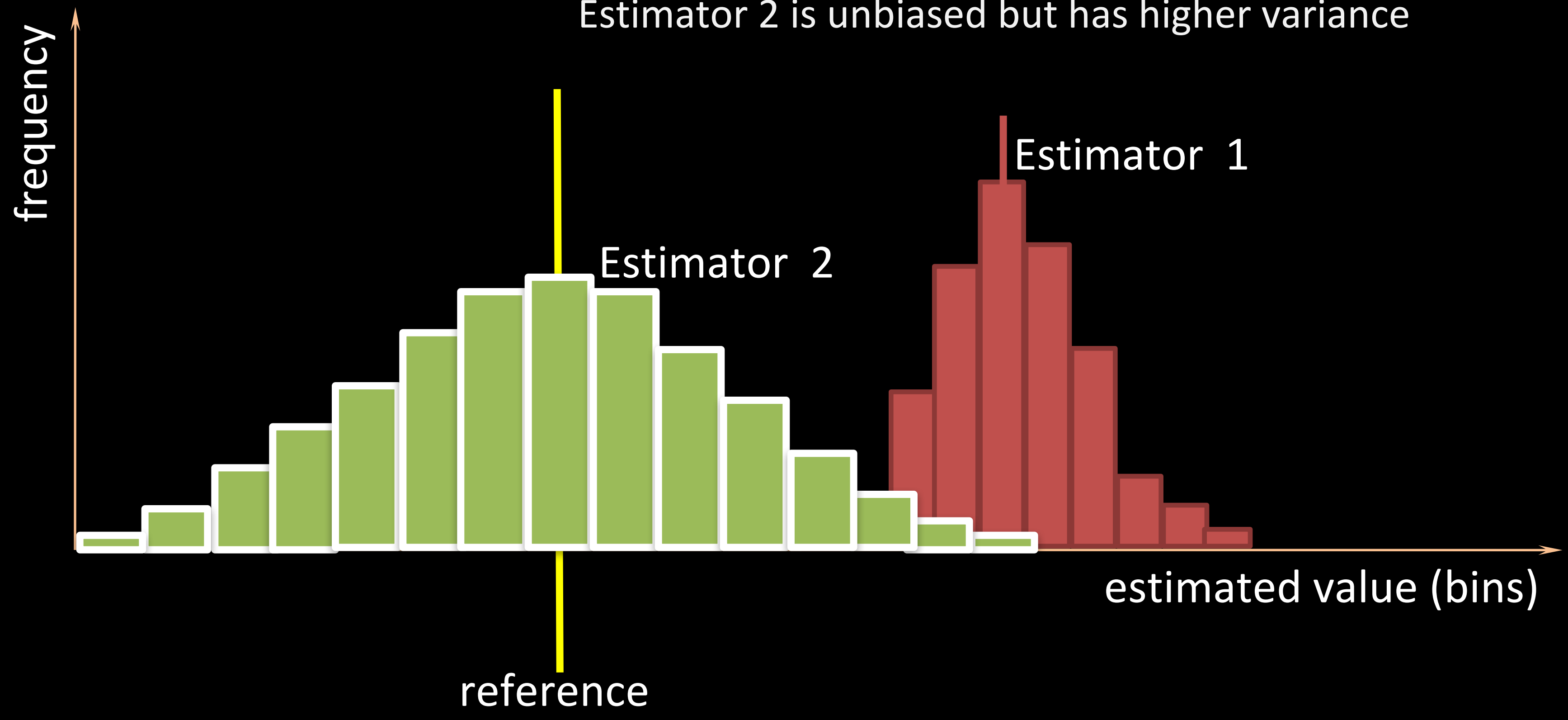


assessing estimators using sampling spectra

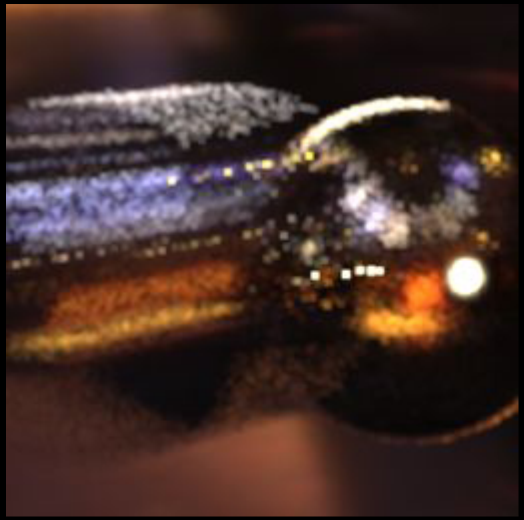


accuracy (bias) and precision (variance)

Estimator 2 is unbiased but has higher variance



Variance and bias



High variance

predict as a function of sampling strategy and integrand

High bias

Monte Carlo integration: summary and error

$$S(x) = \sum_{k=1}^{N} \delta(x - x_k), \quad x_k \sim [0, 1]$$

- Error
    - MSE, bias, variance
    - convergence rate: error as N is increased

Bird's-eye view of analysis

- Rewrite MC estimator in terms of sampling function

$$\frac{1}{N}\sum_{k=1}^{N} f(x_k) = \int_0^1 f(x)S(x)\,\mathrm{d}x \quad \text{where} \quad S(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$

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- Fourier transform preserves inner products, so

$$\int_0^1 f(x)S(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{S}(-\omega)\,\mathrm{d}\omega$$

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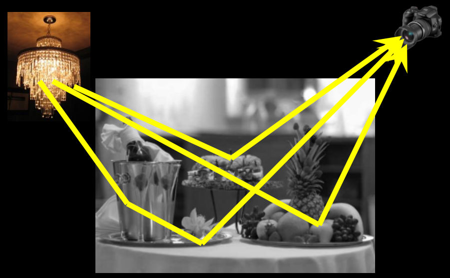
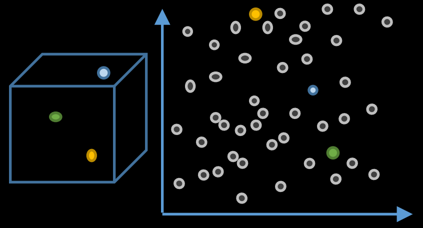
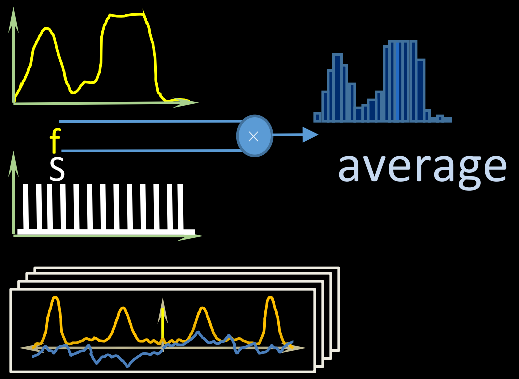
- Analyse MSE error, bias and convergence in terms of $\hat{S}(\omega)$

Summary

Summary







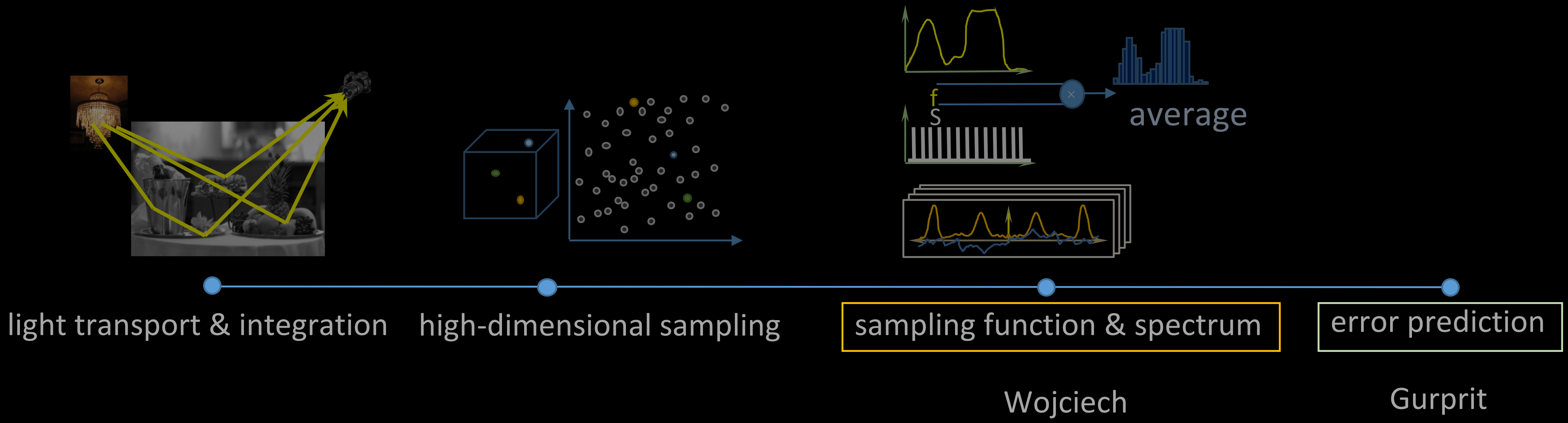
light transport & integration

high-dimensional sampling

sampling function & spectrum

error prediction

Next



f
s
average
light transport & integration
high-dimensional sampling
sampling function & spectrum
error prediction
Wojciech
Gurprit

Local variation is useful for adaptive sampling