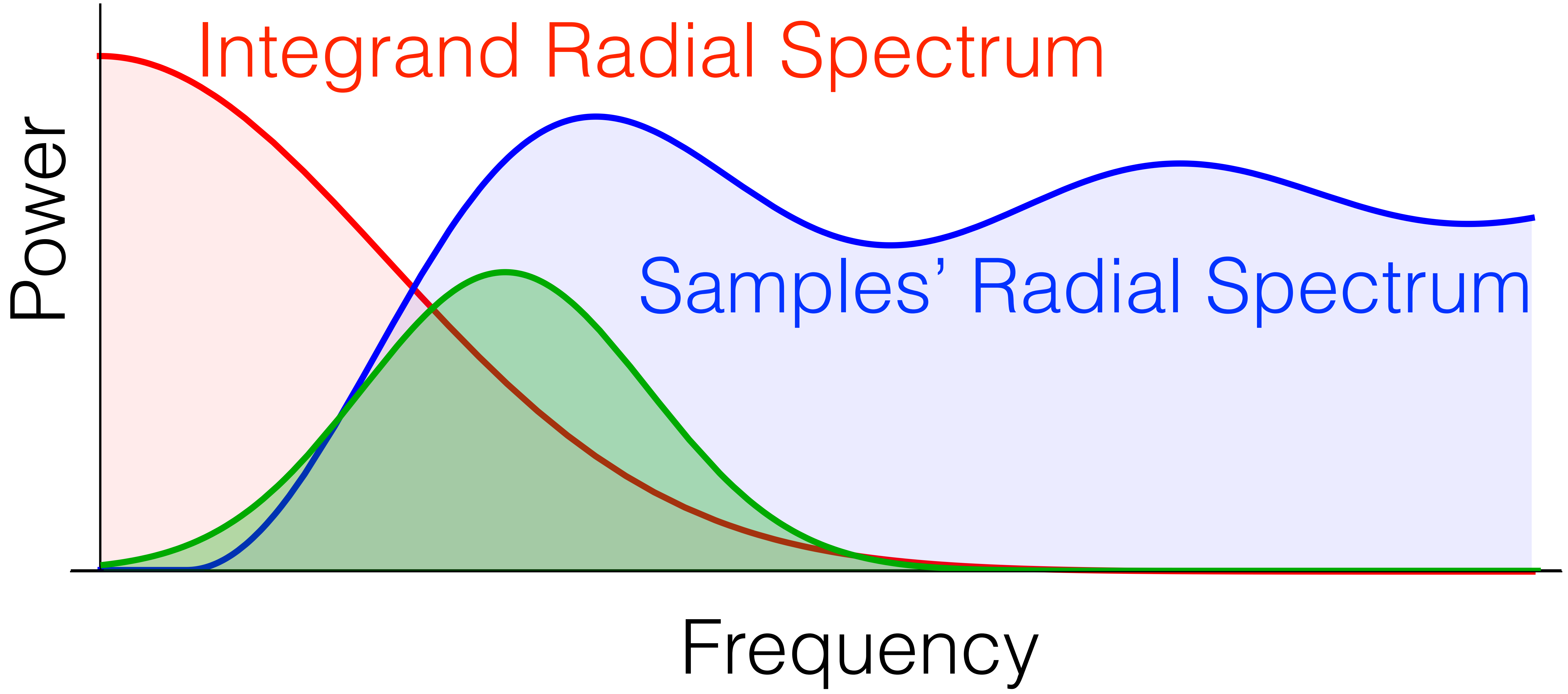
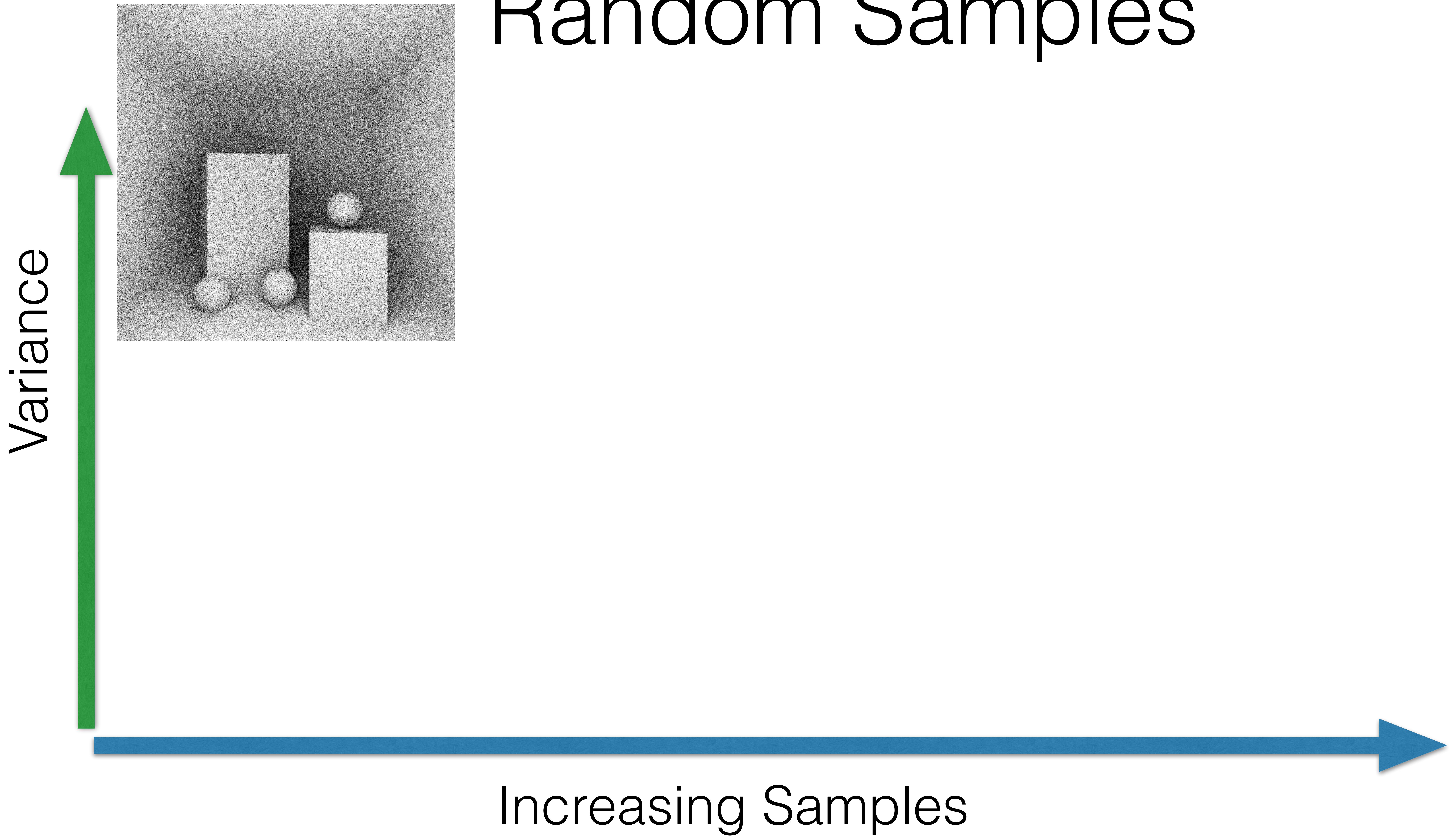
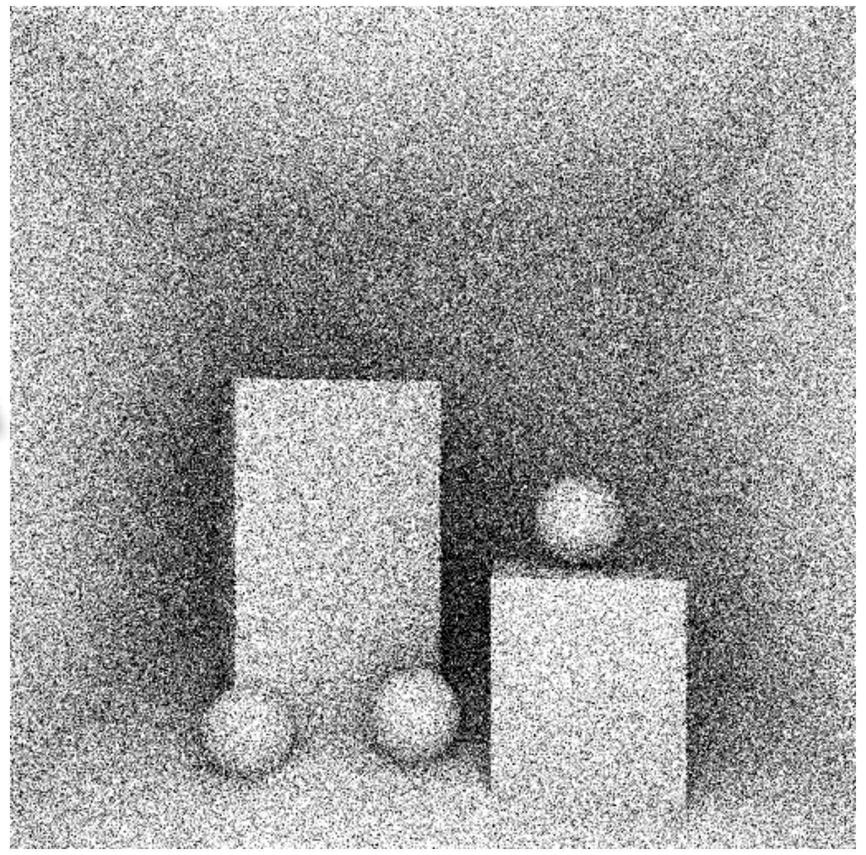
Part 3: Formal Treatment of MSE, Bias and Variance

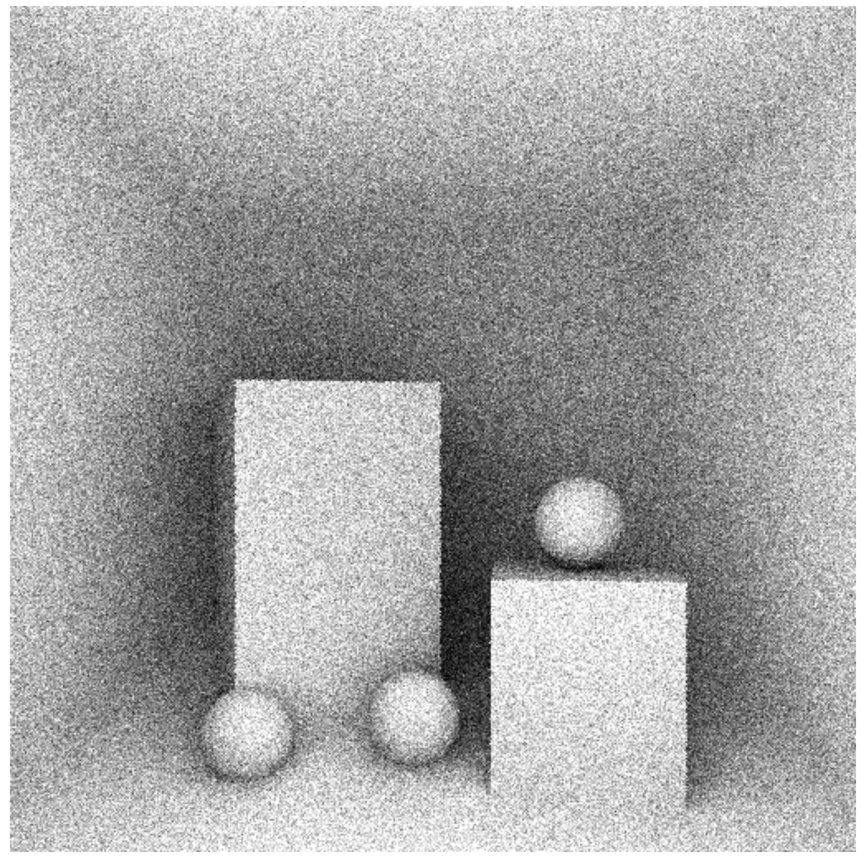


Convergence rate for Random Samples



Convergence rate for Random Samples

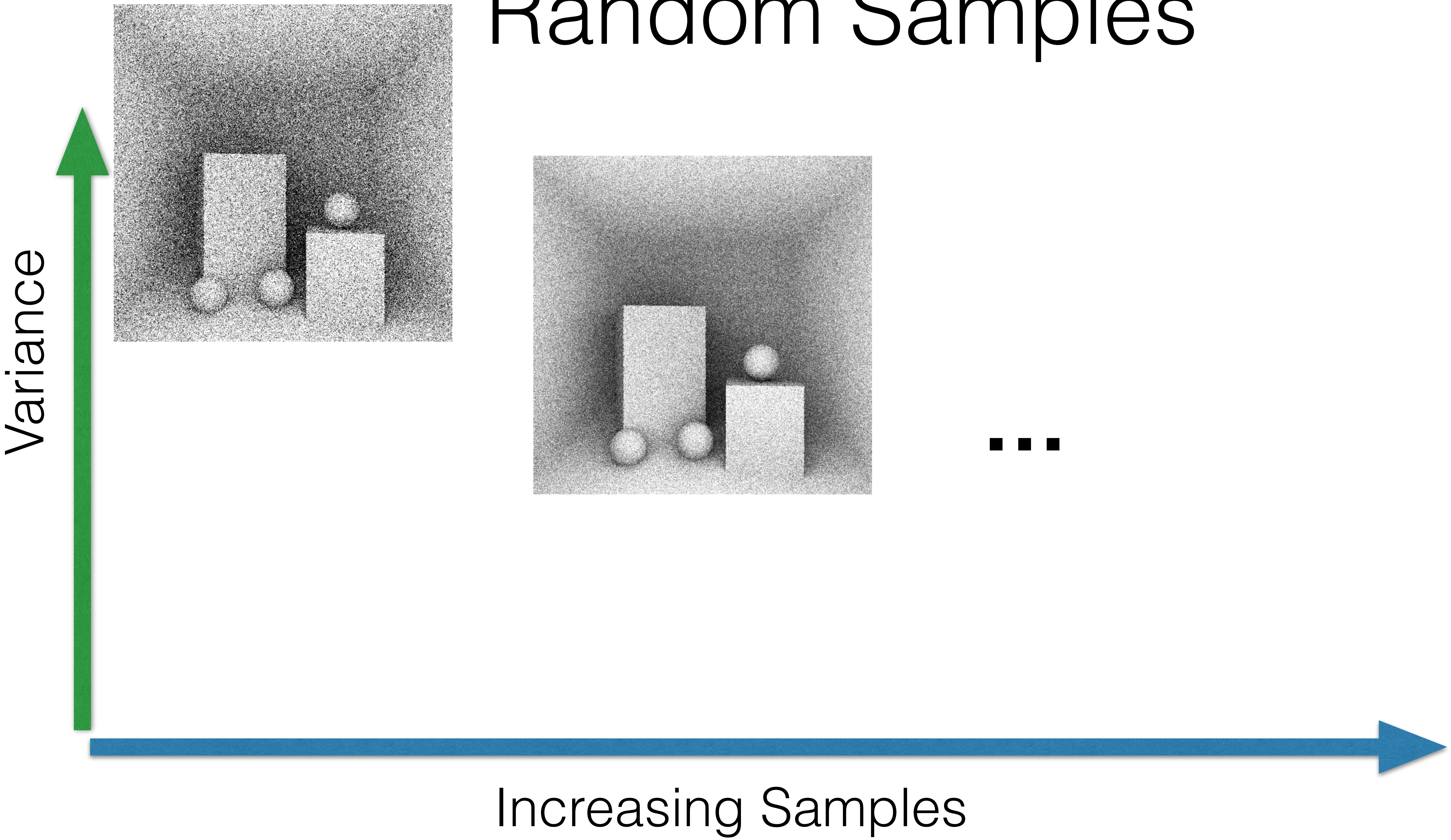




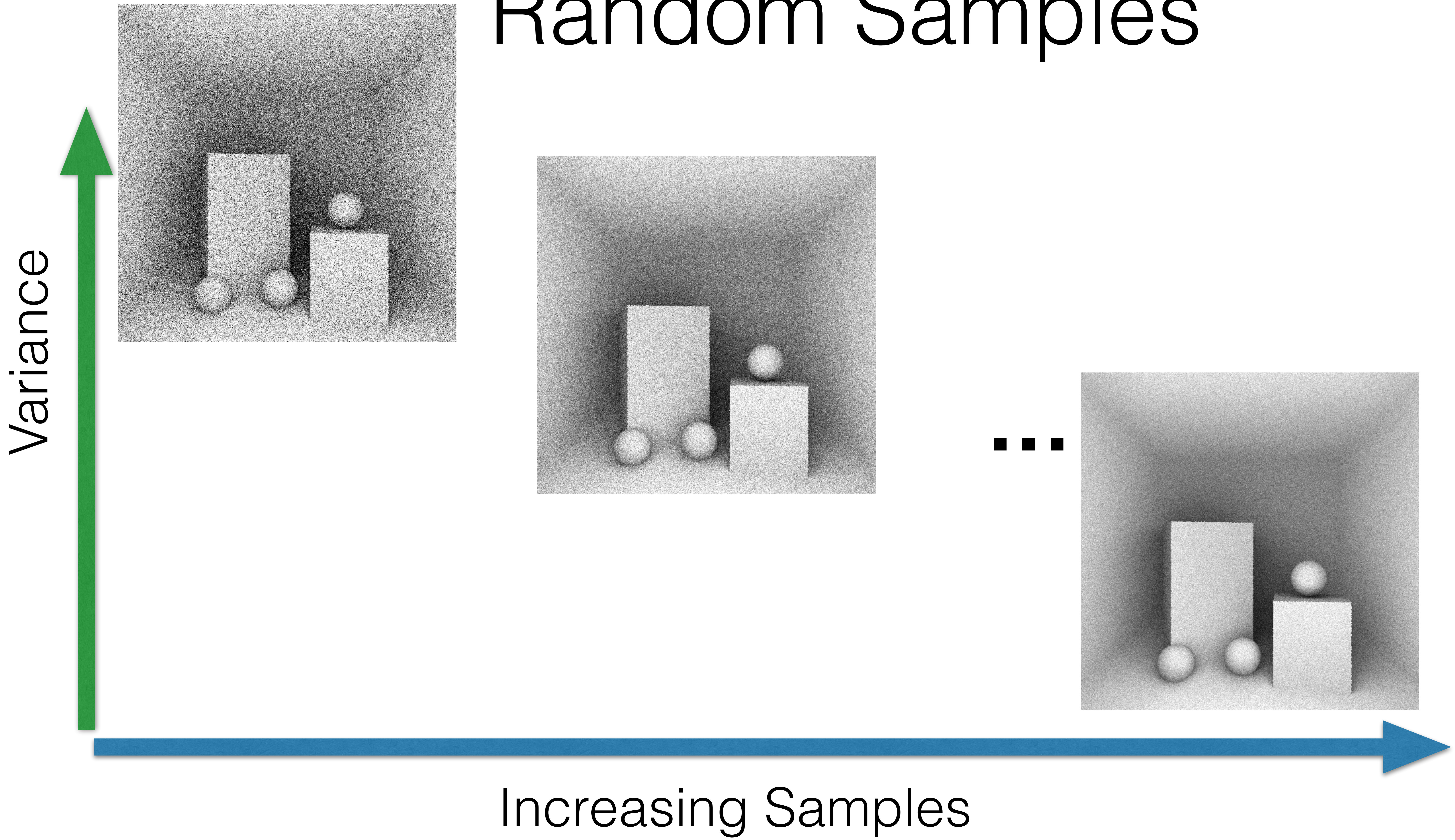
Variance

Increasing Samples

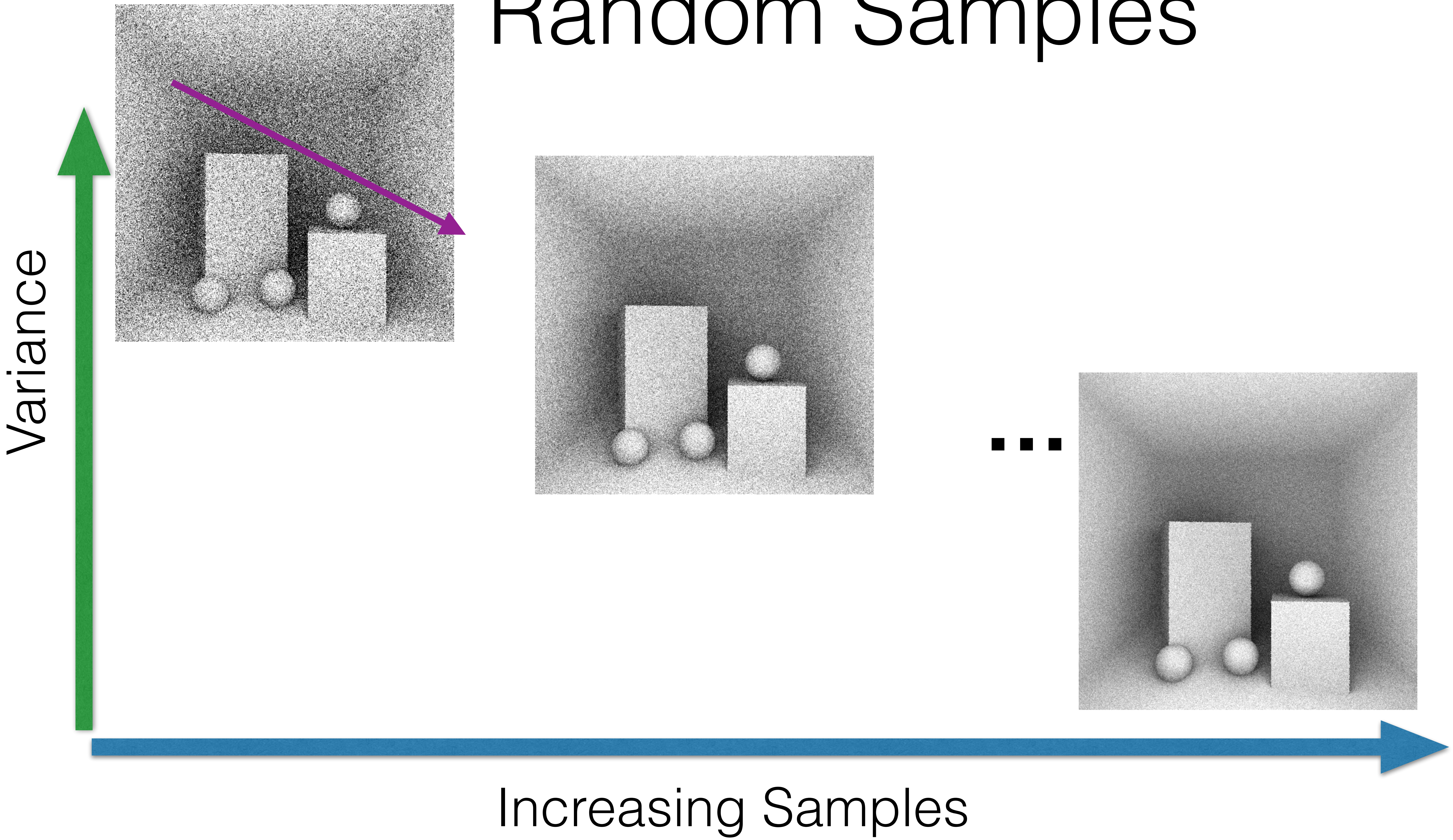
Convergence rate for Random Samples



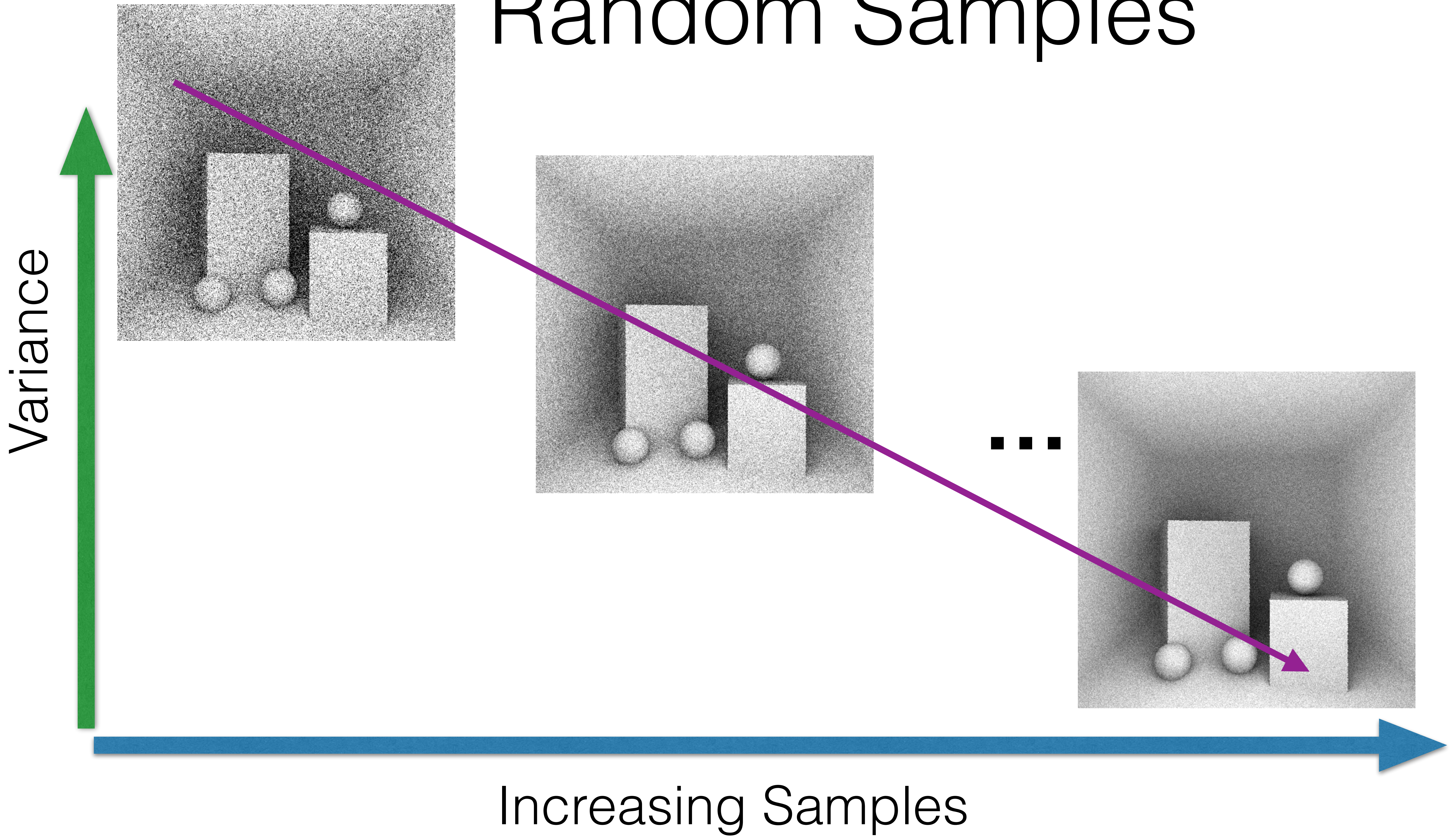
Convergence rate for Random Samples



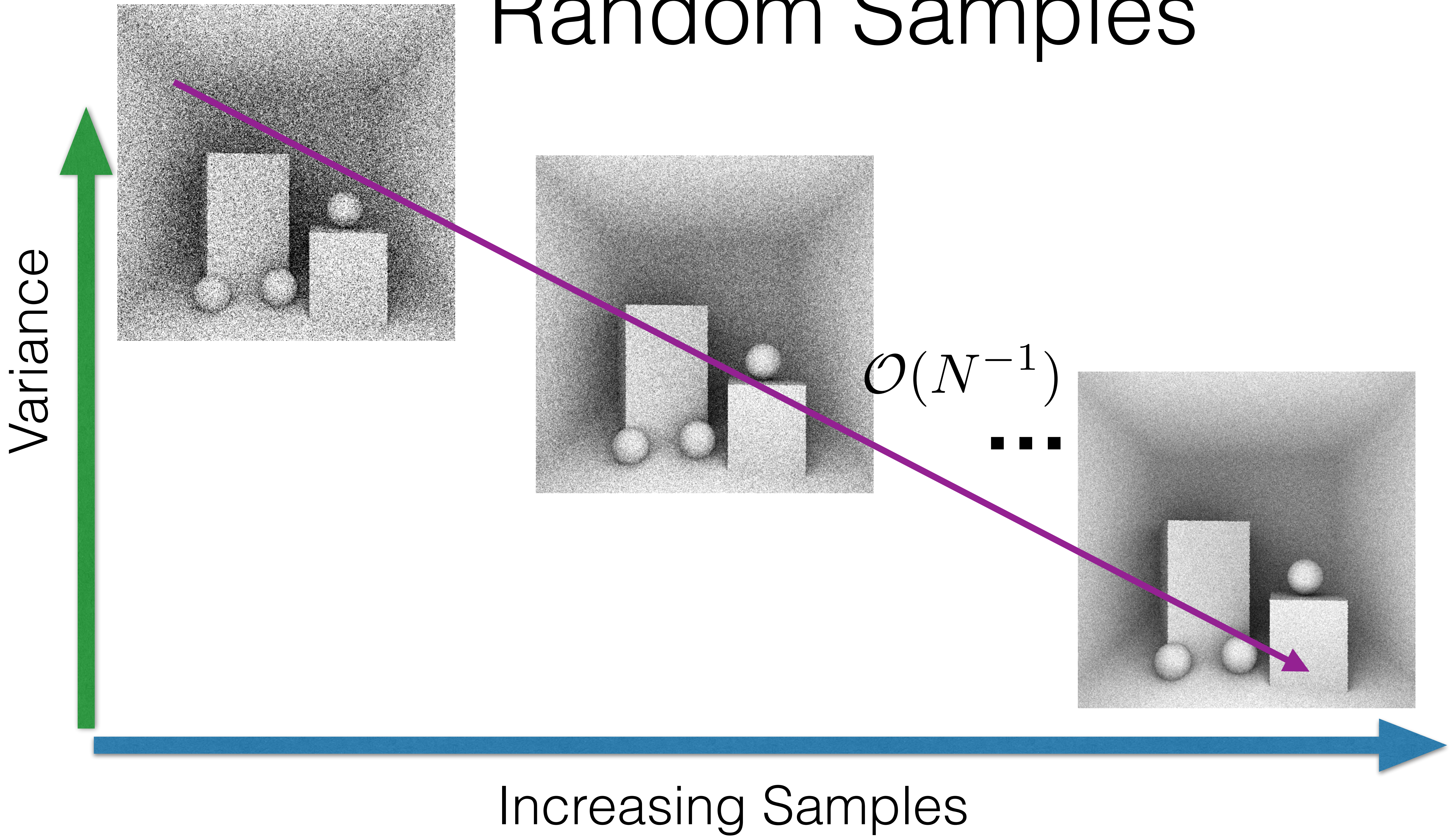
Convergence rate for Random Samples



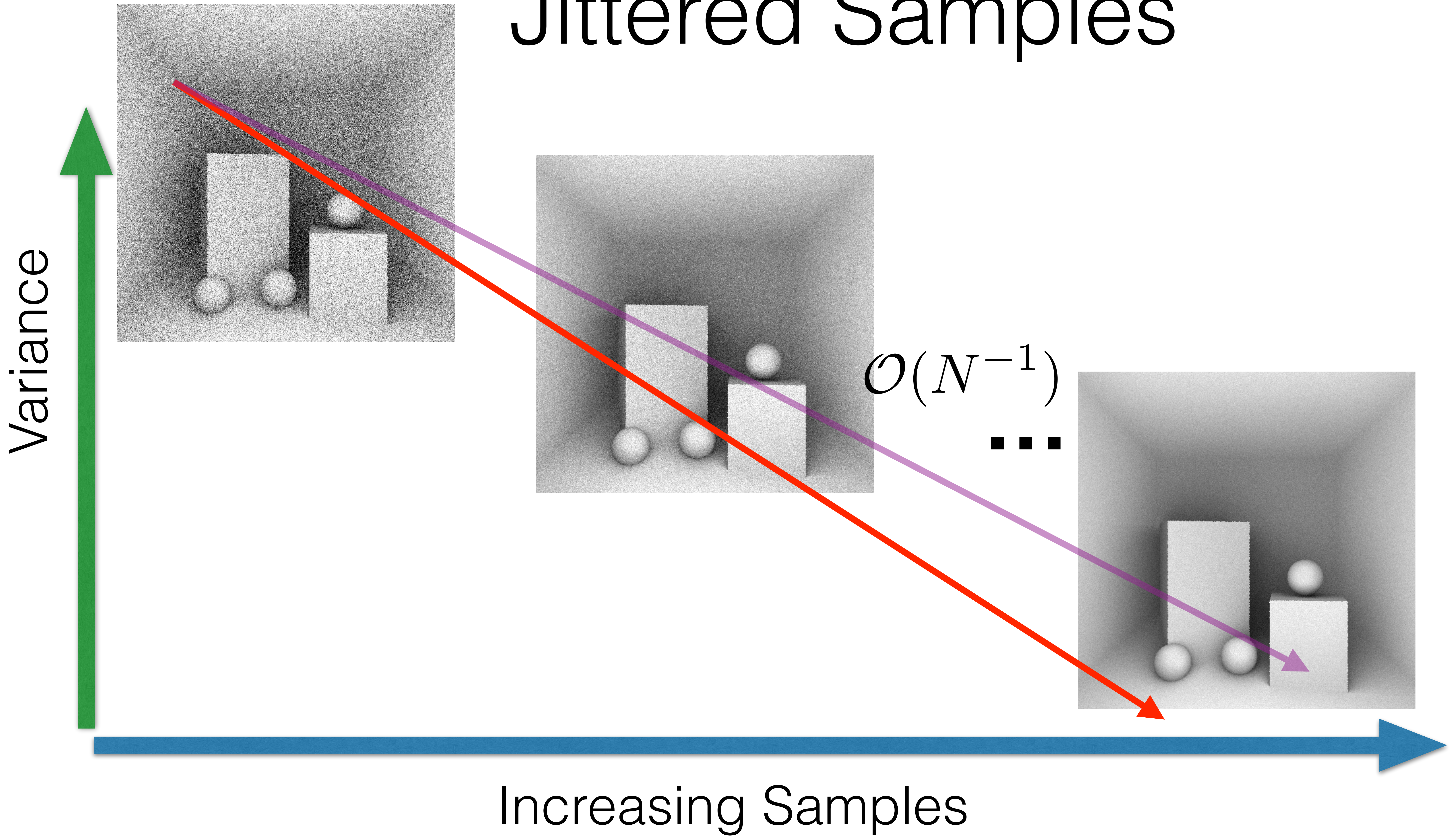
Convergence rate for Random Samples



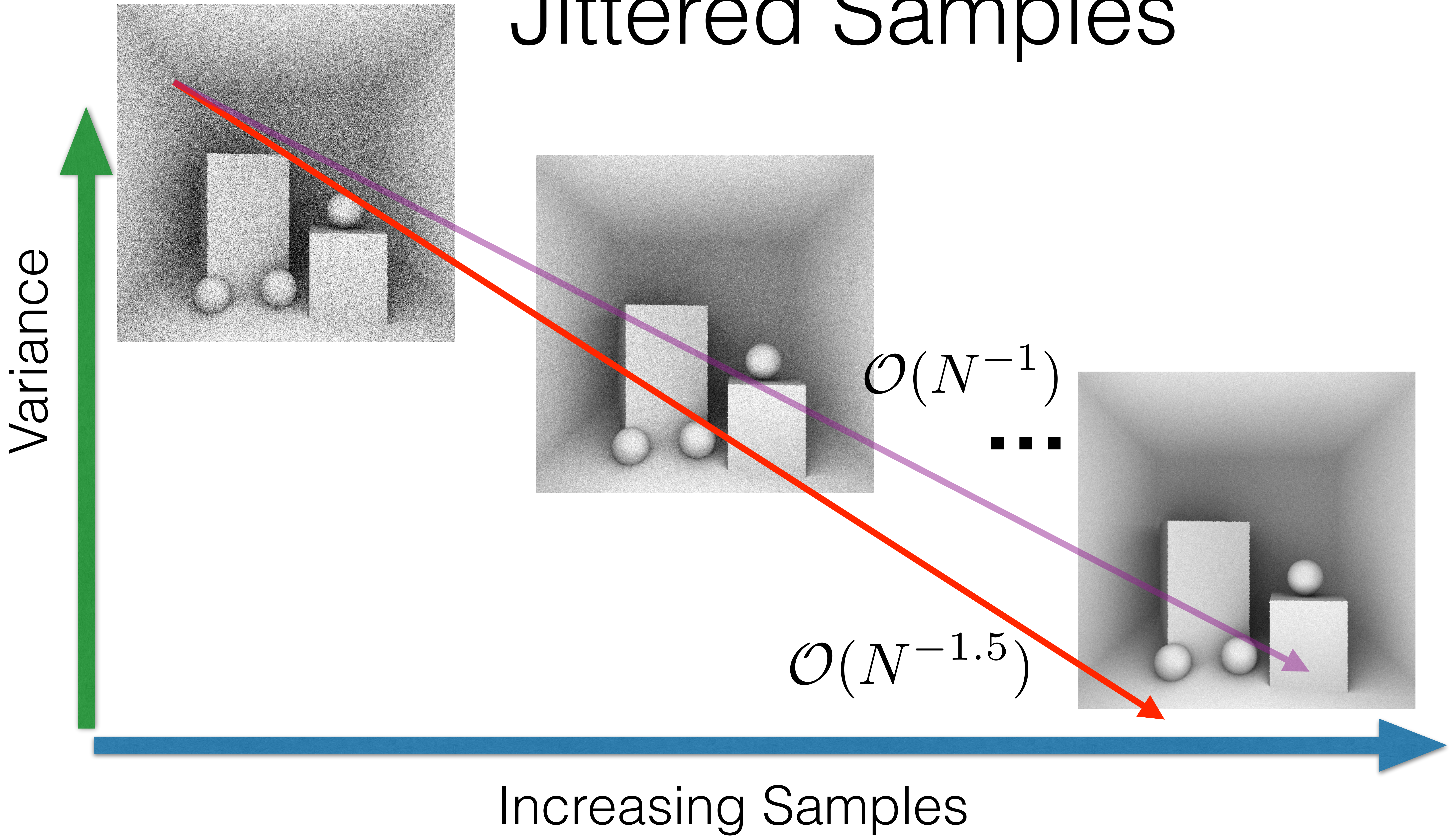
Convergence rate for Random Samples



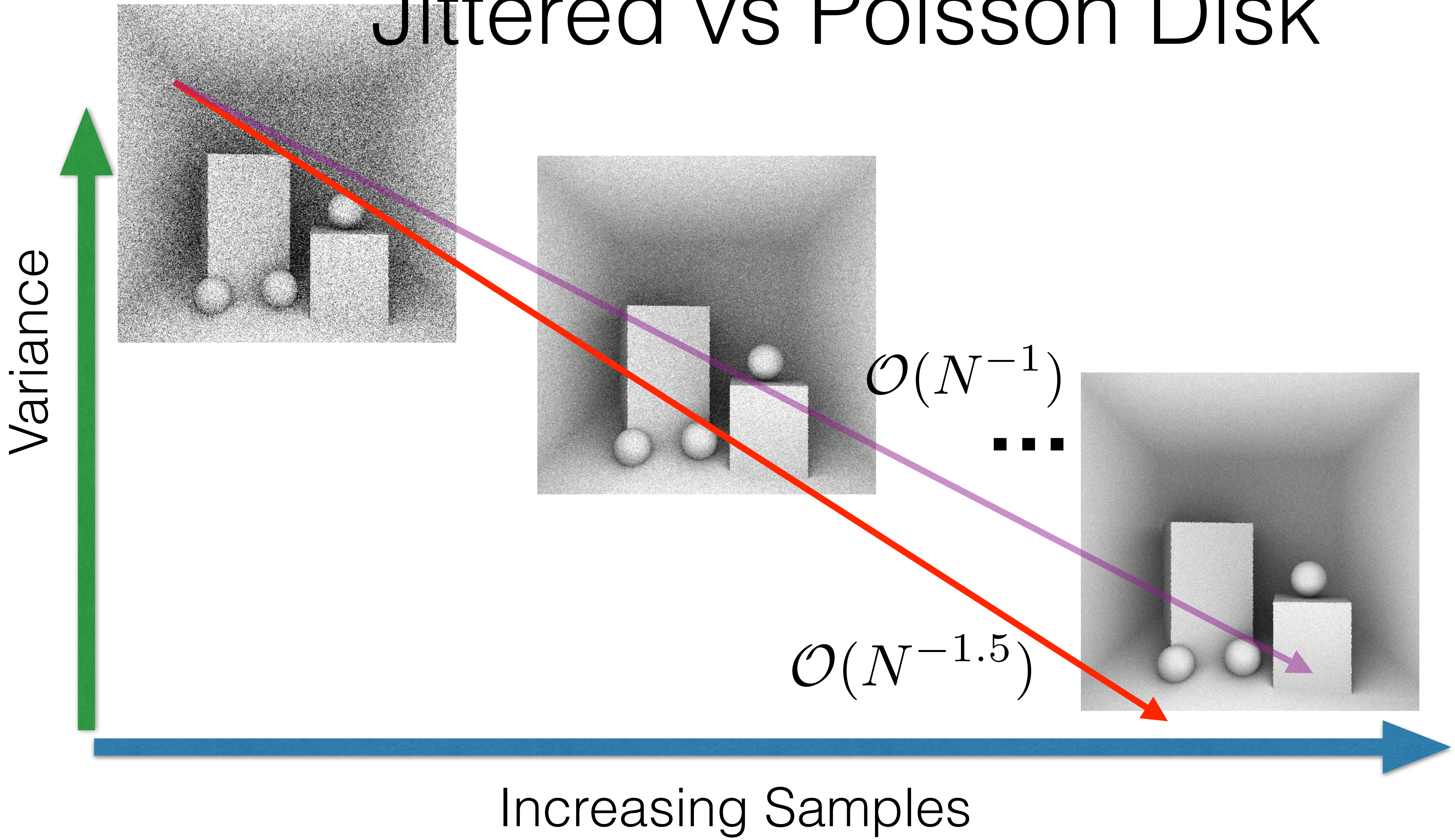
Convergence rate for Jittered Samples



Convergence rate for Jittered Samples



Convergence rate
Jittered vs Poisson Disk



Convergence rate
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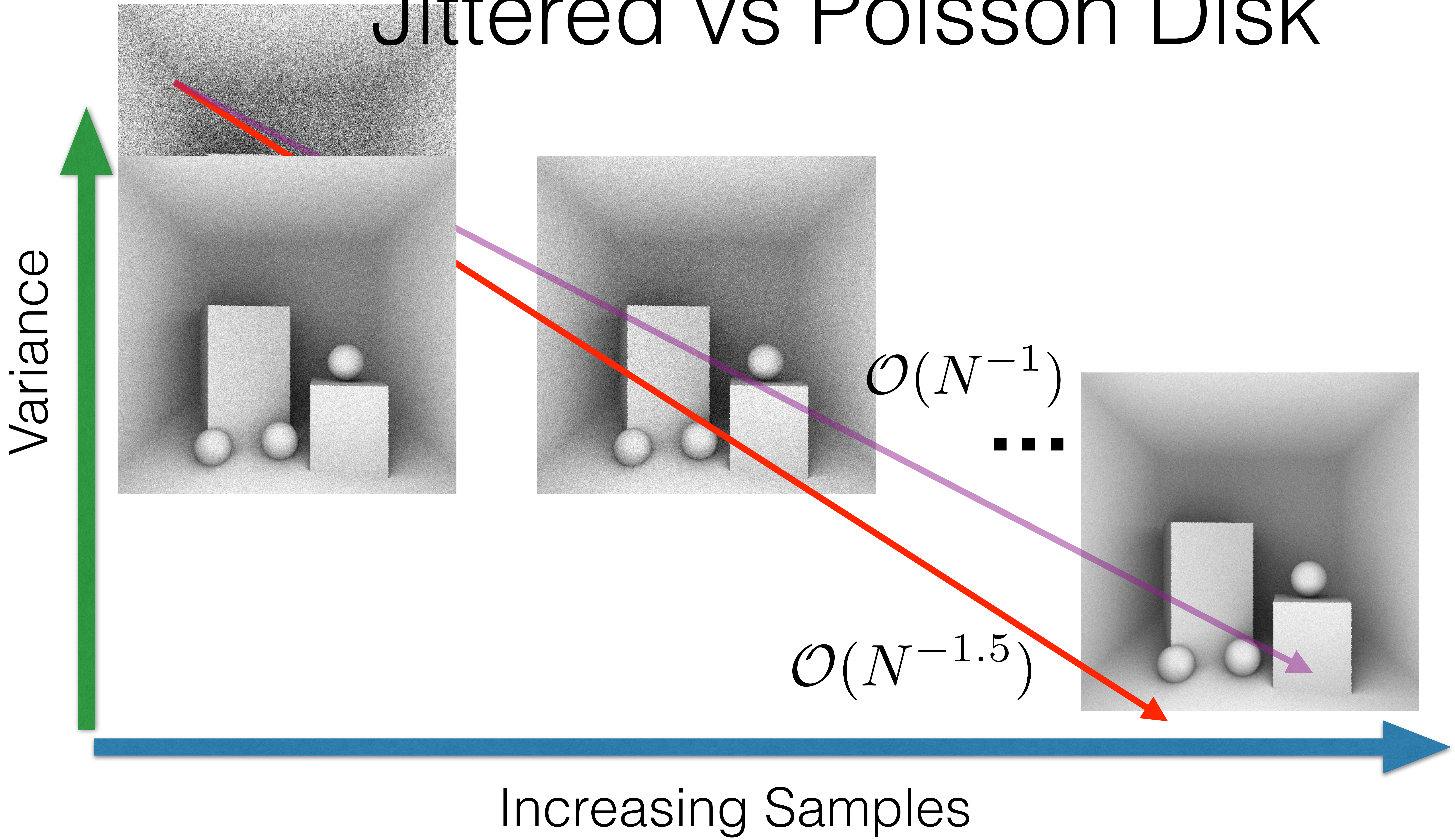
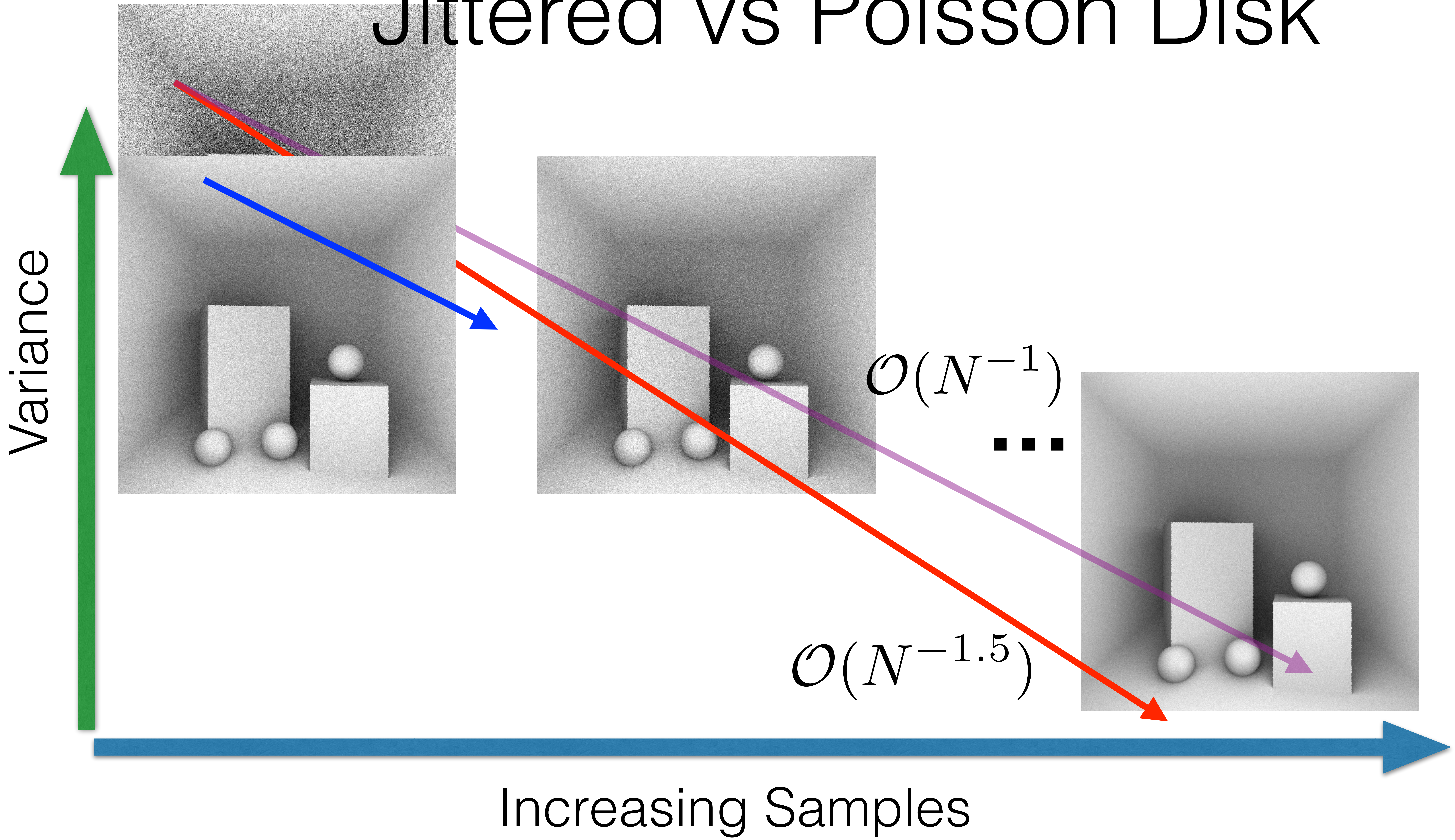
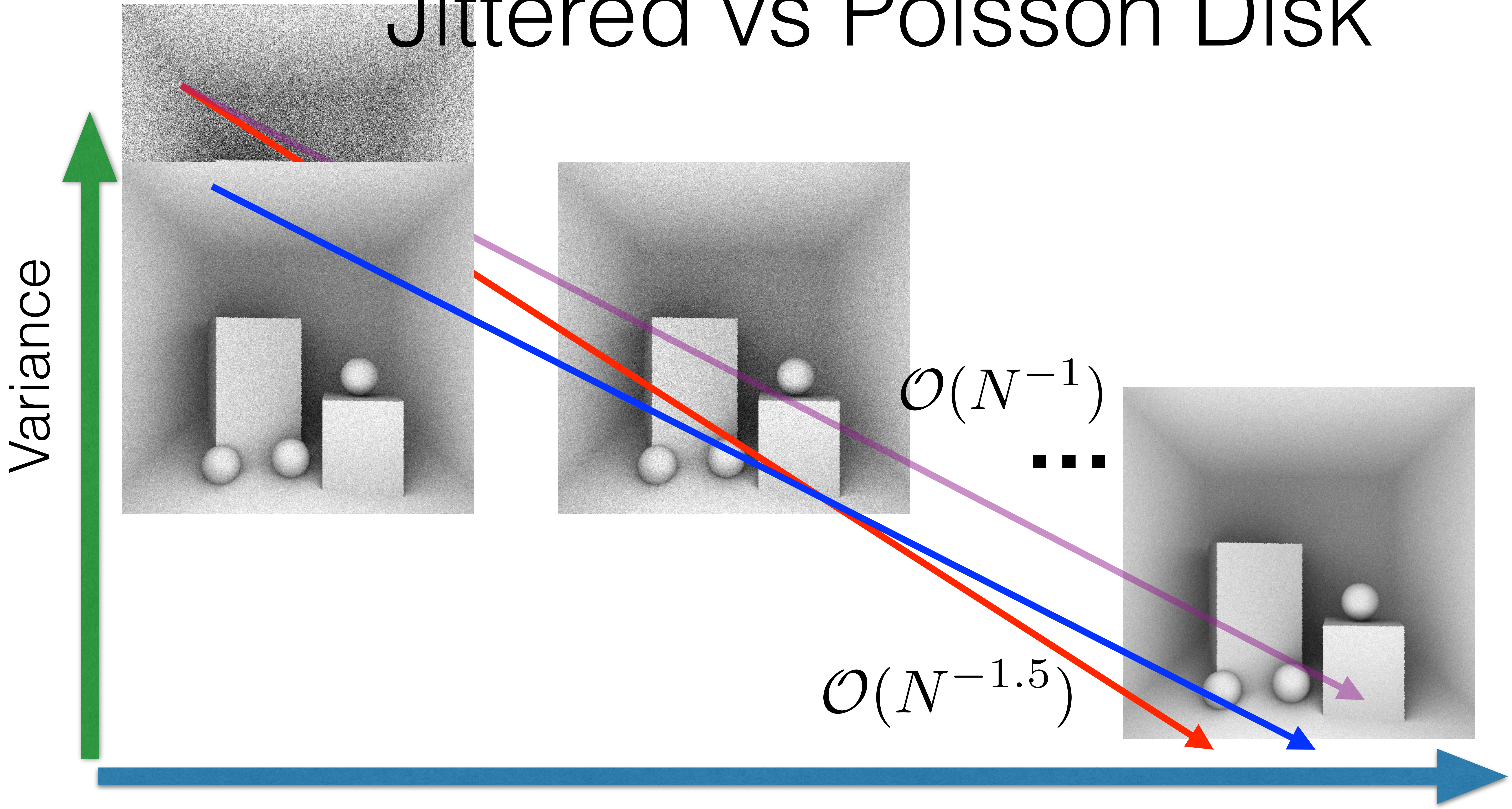


Convergence rate
Jittered vs Poisson Disk

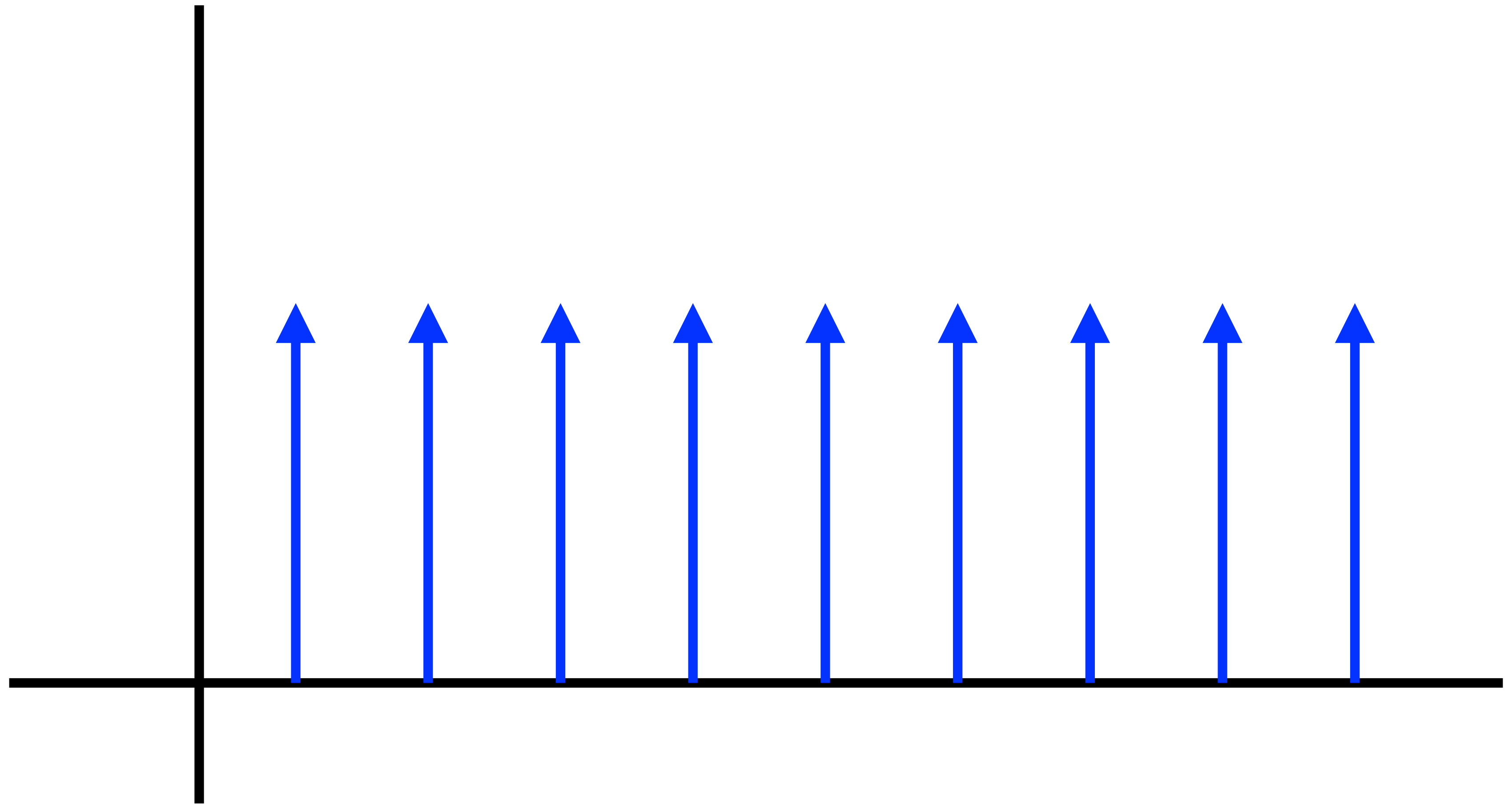


Convergence rate
Jittered vs Poisson Disk



Samples and function in Fourier Domain

Spatial Domain

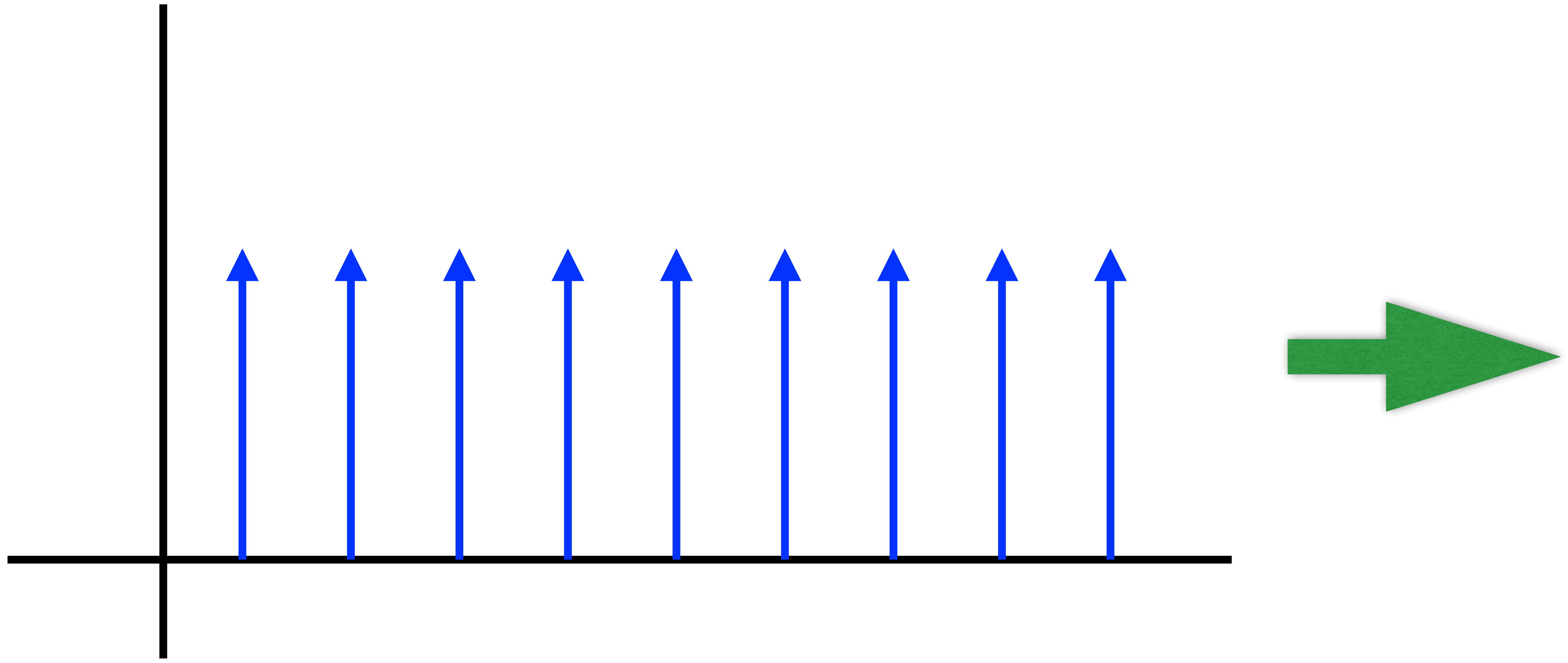


Fourier Domain

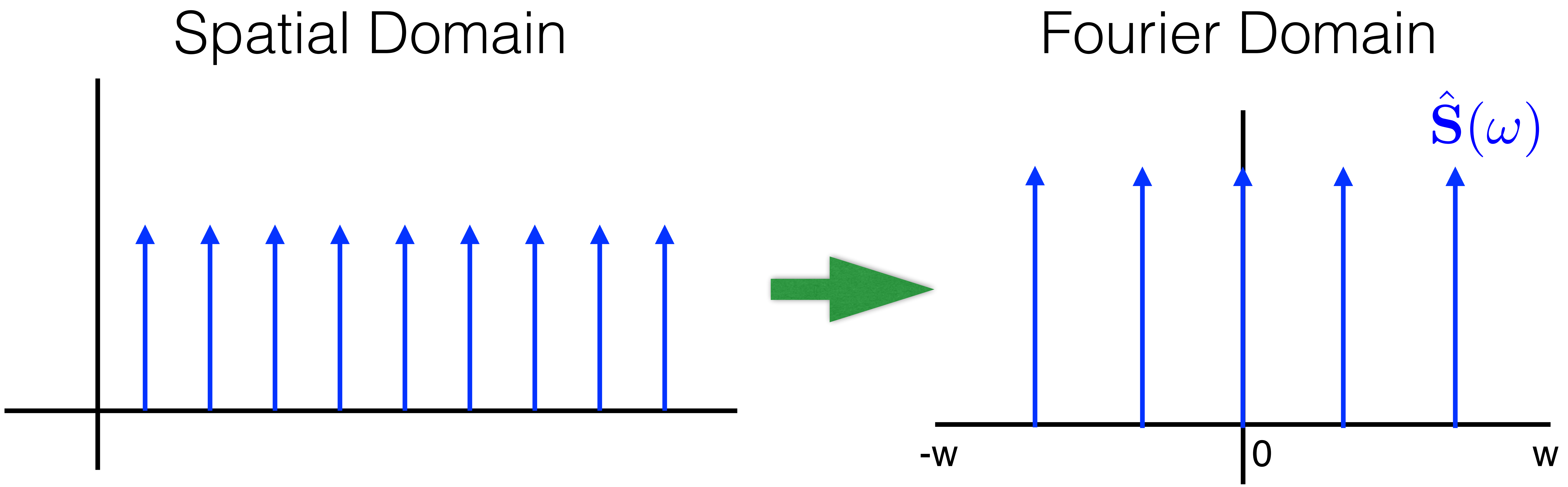
Samples and function in Fourier Domain

Spatial Domain

Fourier Domain



Samples and function in Fourier Domain



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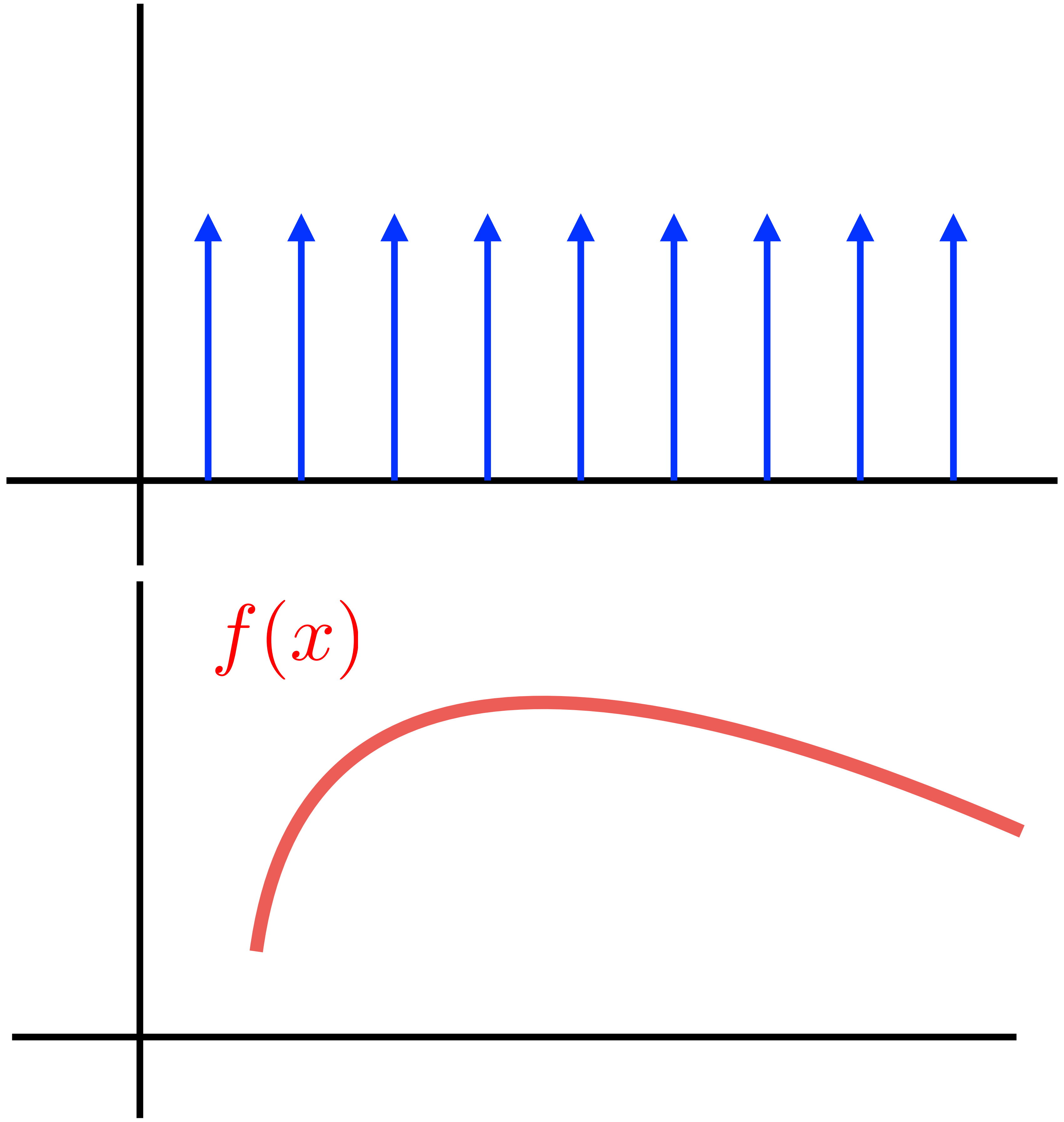
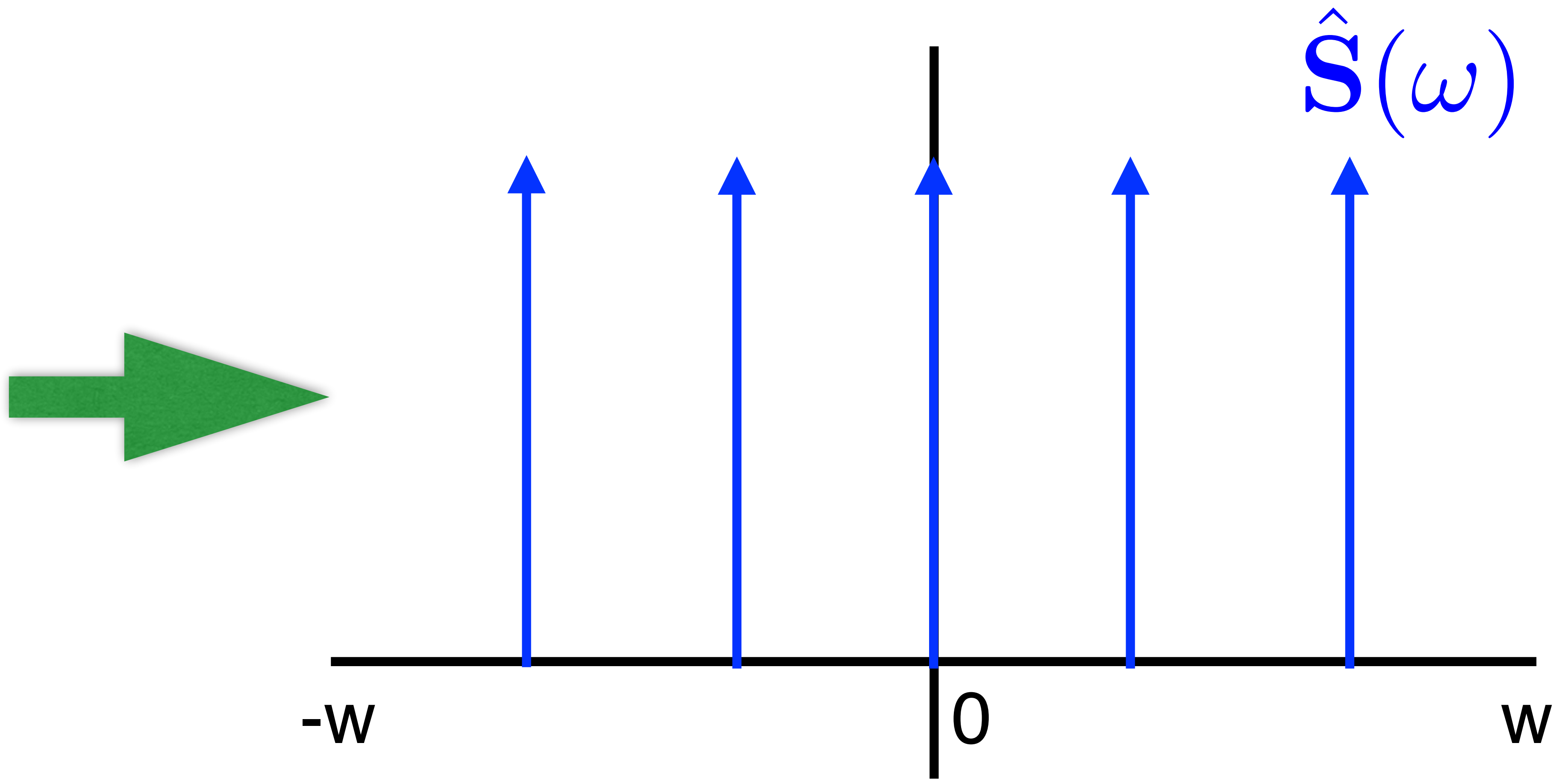


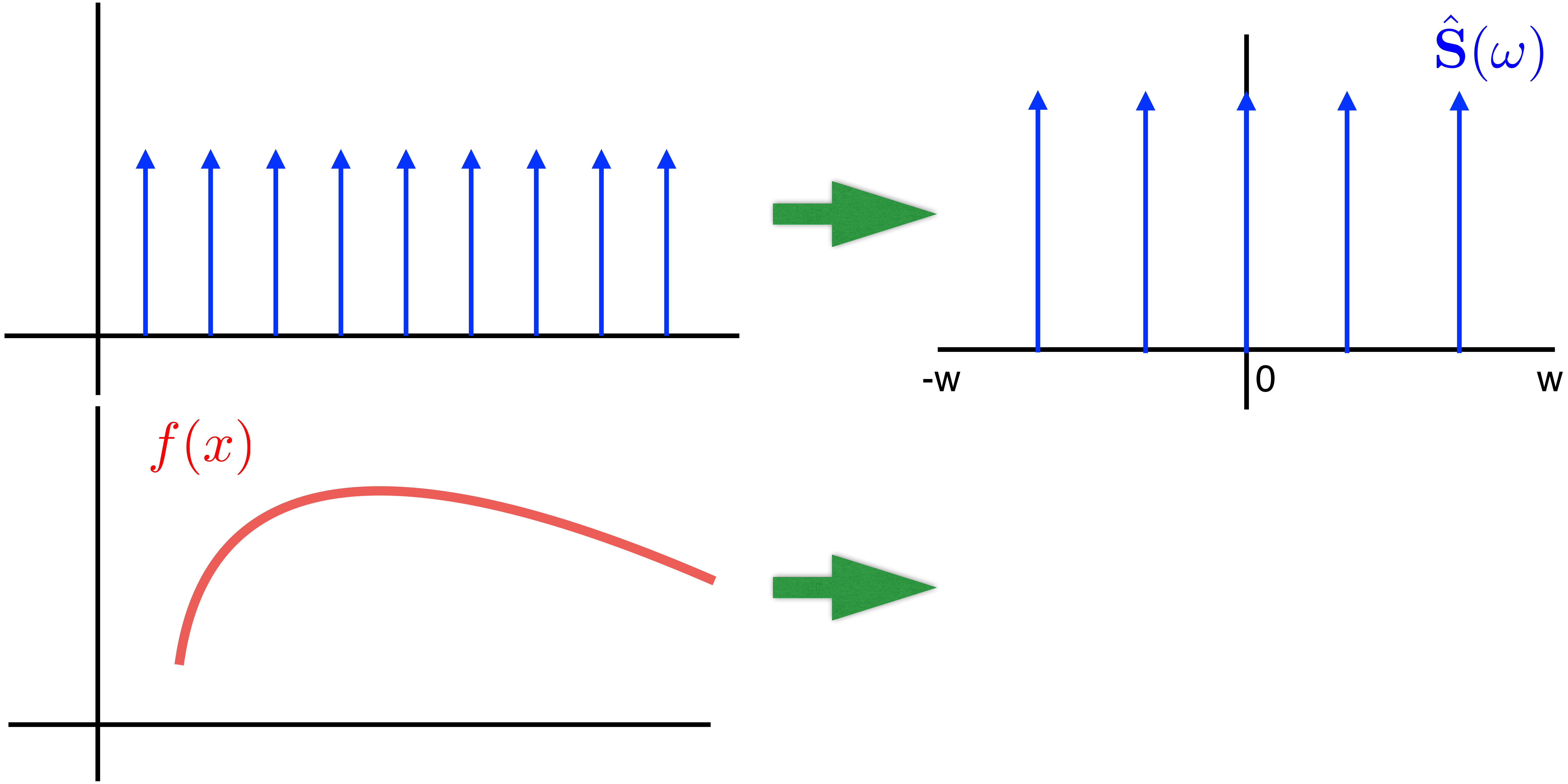
Fourier Domain



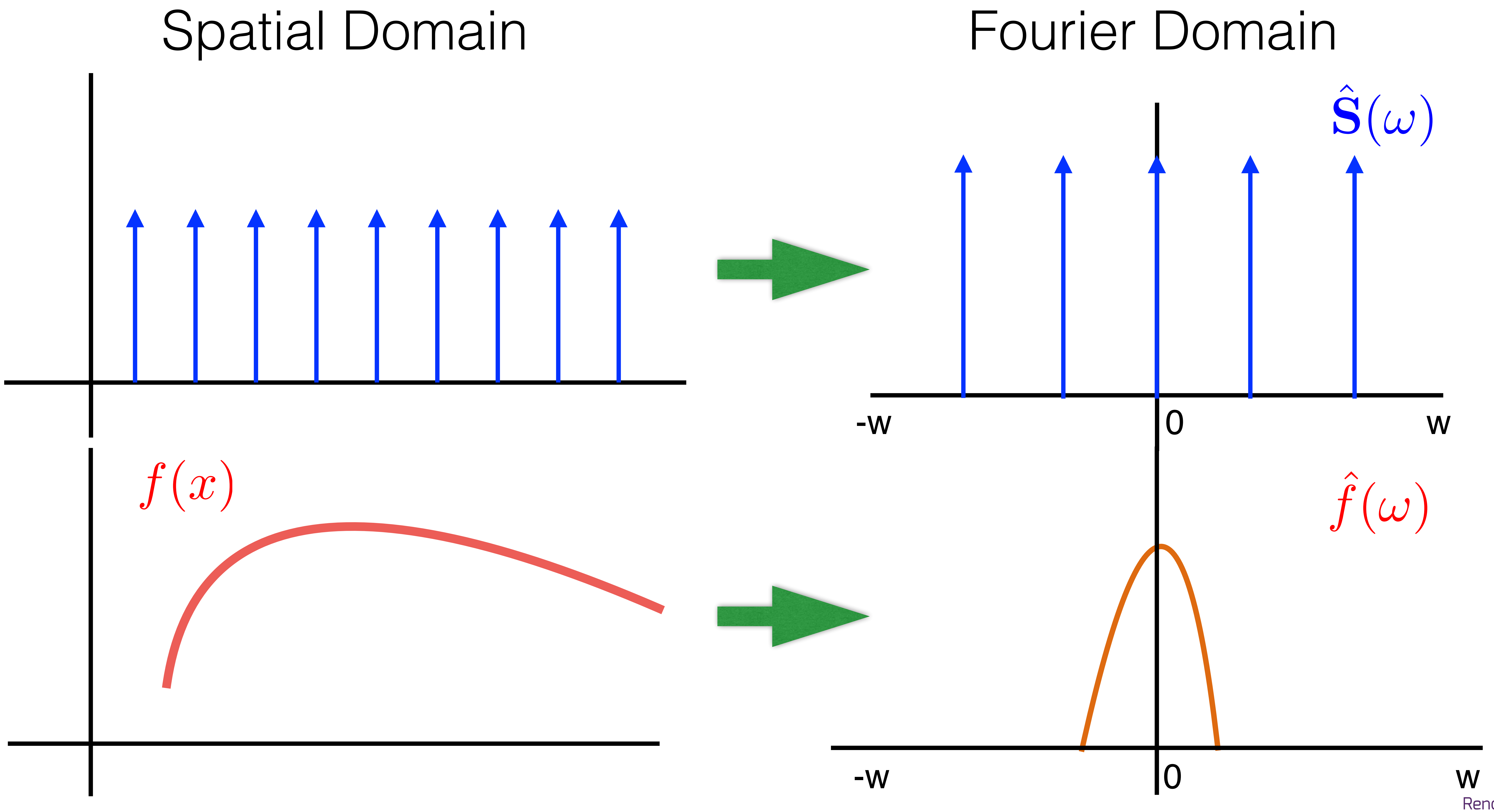
Samples and function in Fourier Domain

Spatial Domain

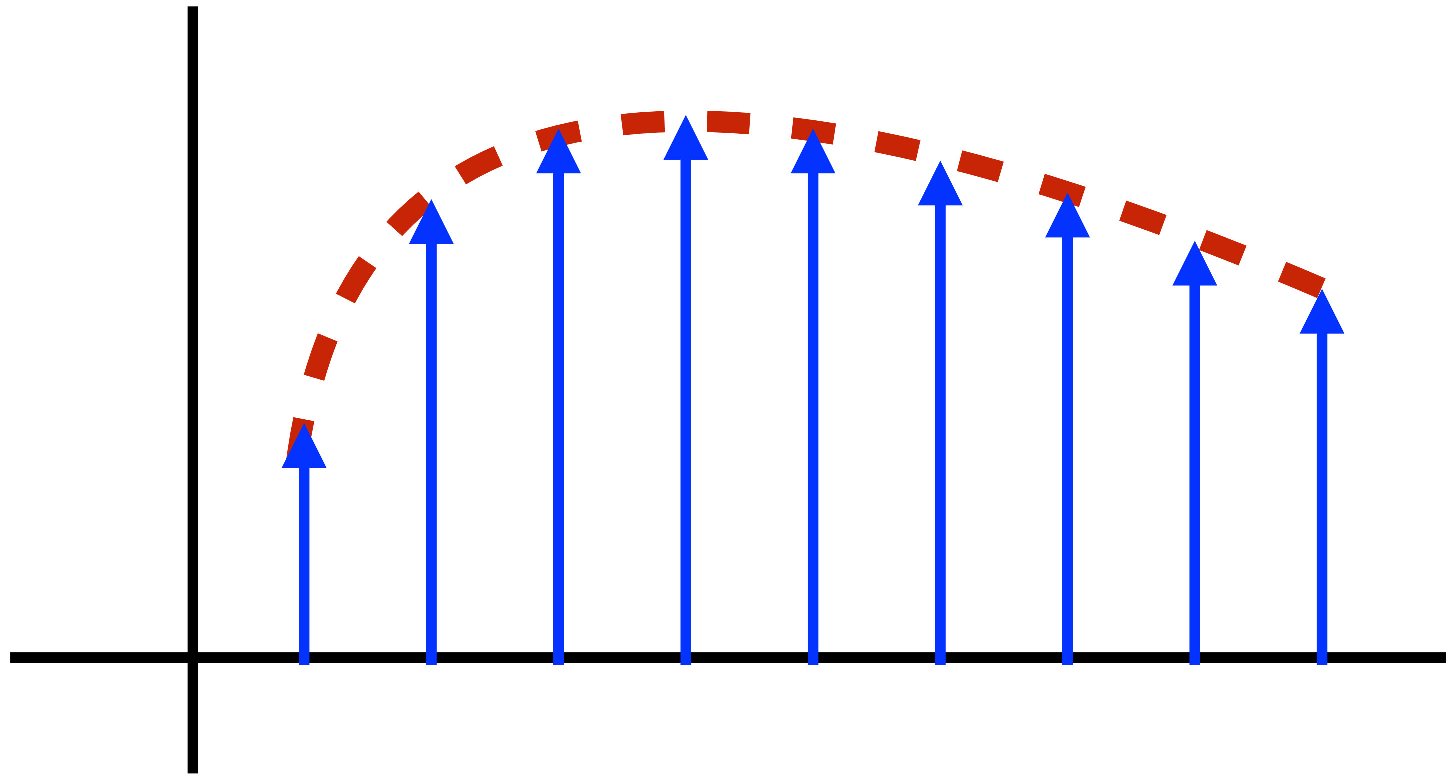
Fourier Domain



Samples and function in Fourier Domain

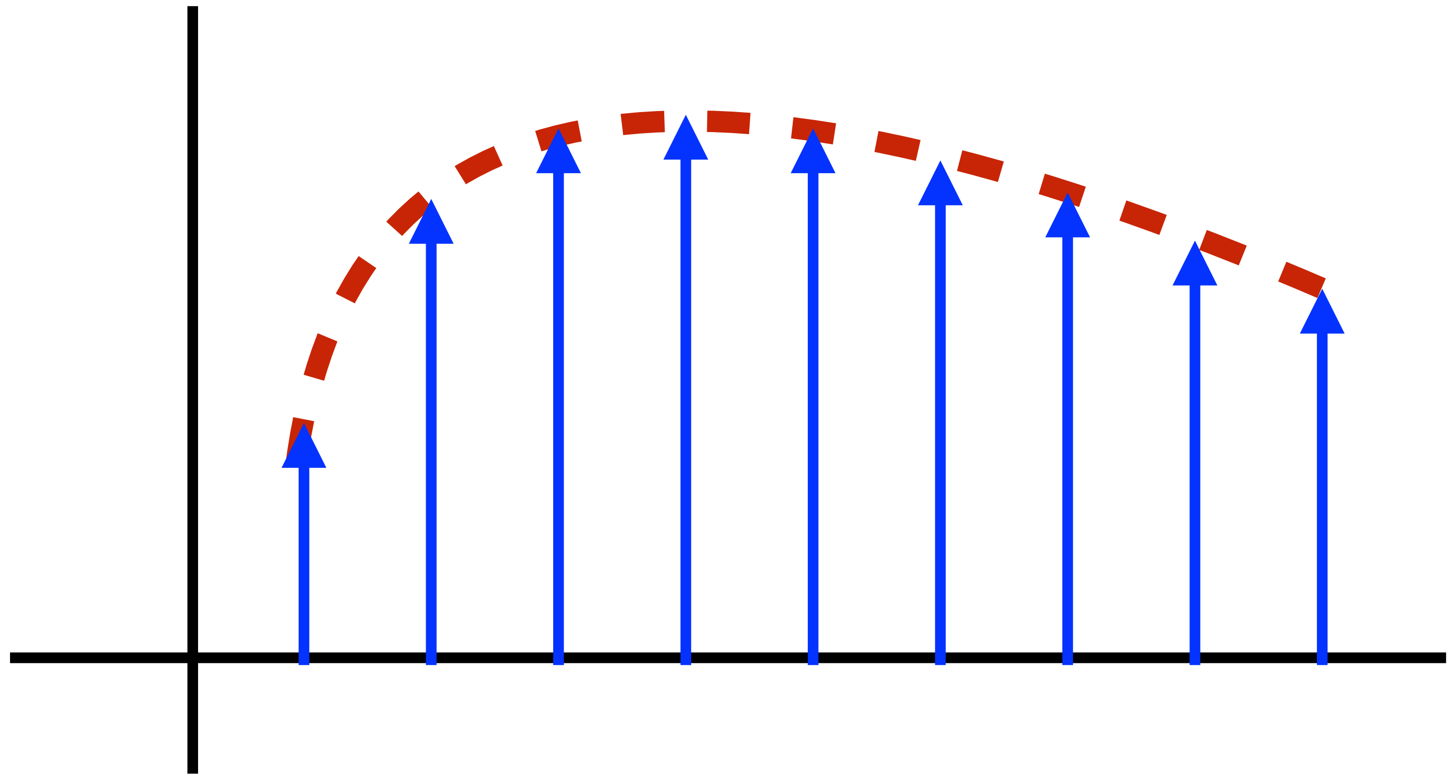


Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$

Sampling in Primal Domain is Convolution in Fourier Domain



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Fredo Durand [2011]

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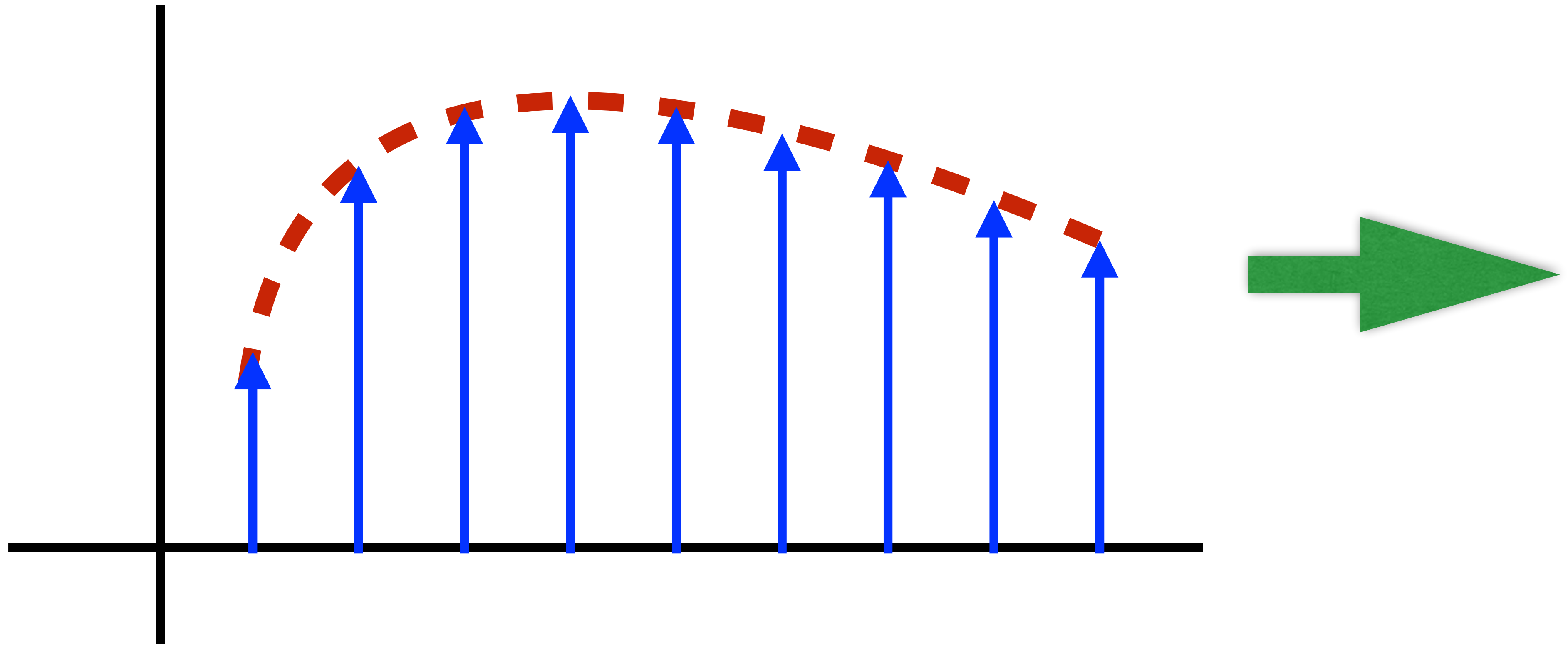

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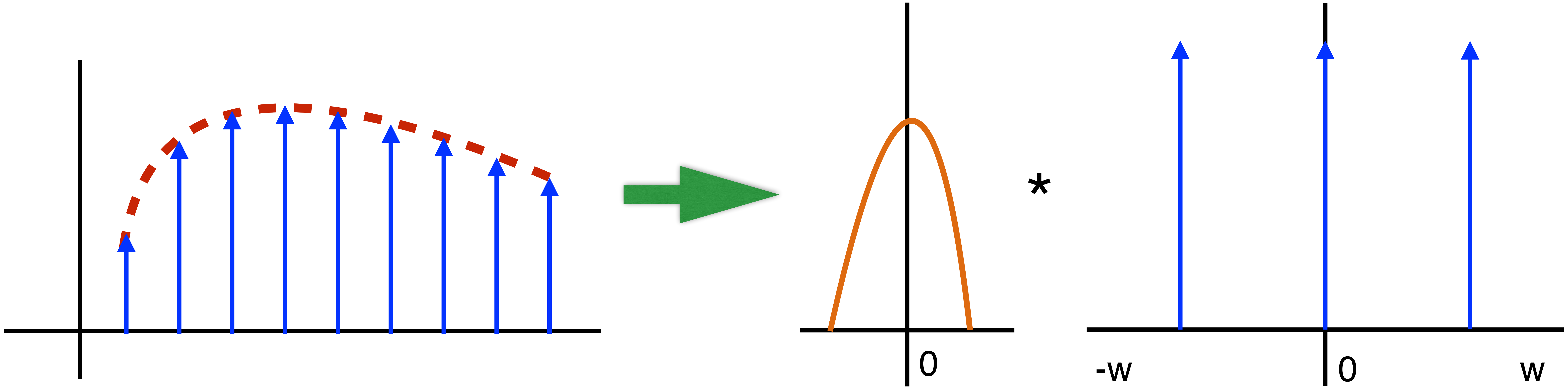
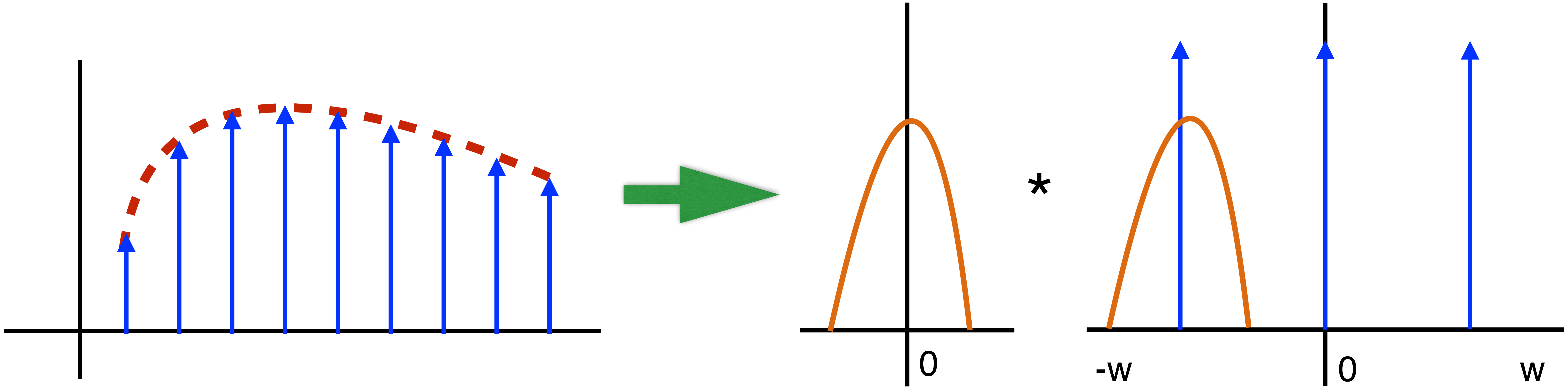


$f(x)\,\mathbf{S}(x)$ $\qquad\qquad$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

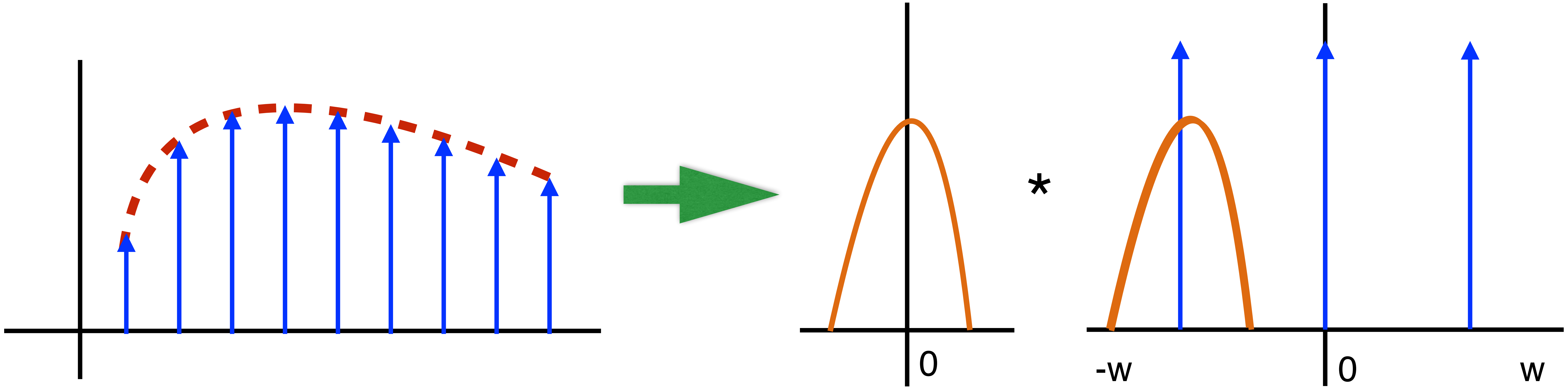
Sampling in Primal Domain is Convolution in Fourier Domain



$$f(x)\,\mathbf{S}(x) \qquad \hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]

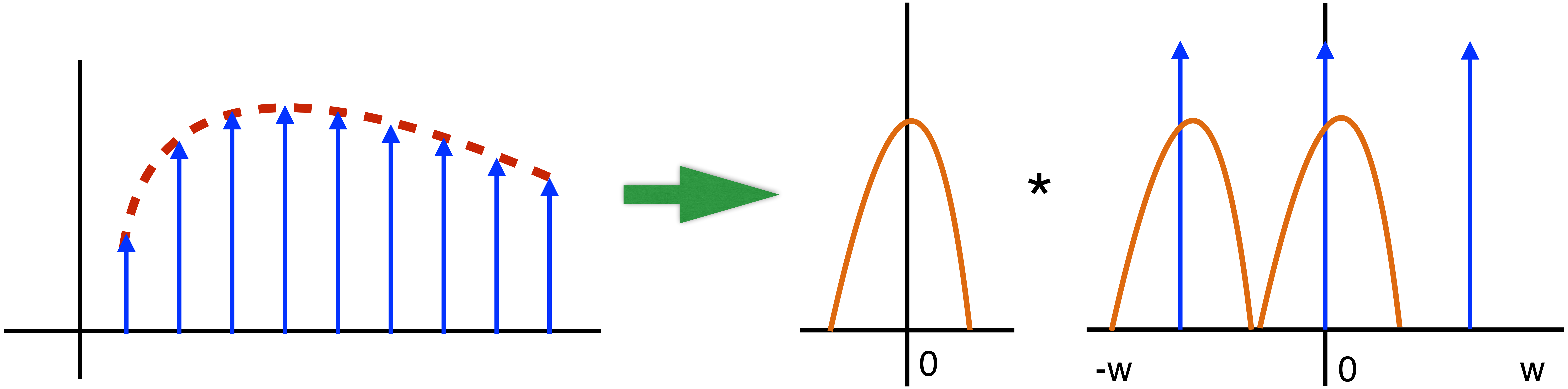
Sampling in Primal Domain is Convolution in Fourier Domain



$$f(x)\,\mathbf{S}(x) \qquad\qquad \hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain

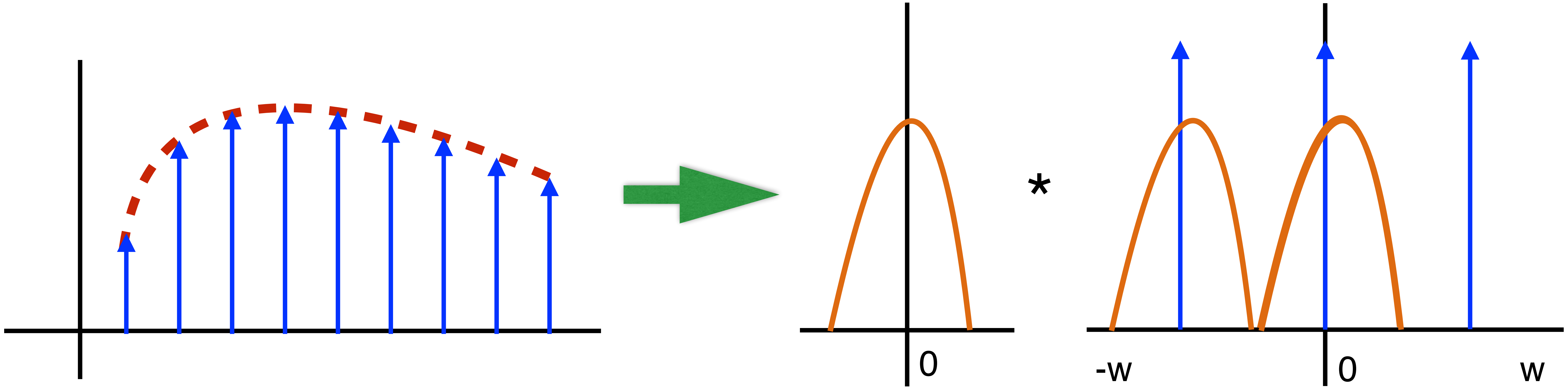


$$\color{red}{f(x)}\,\color{blue}{\mathbf{S}(x)}$$

$$\color{red}{\hat{f}(\omega)} \otimes \color{blue}{\hat{\mathbf{S}}(\omega)}$$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain



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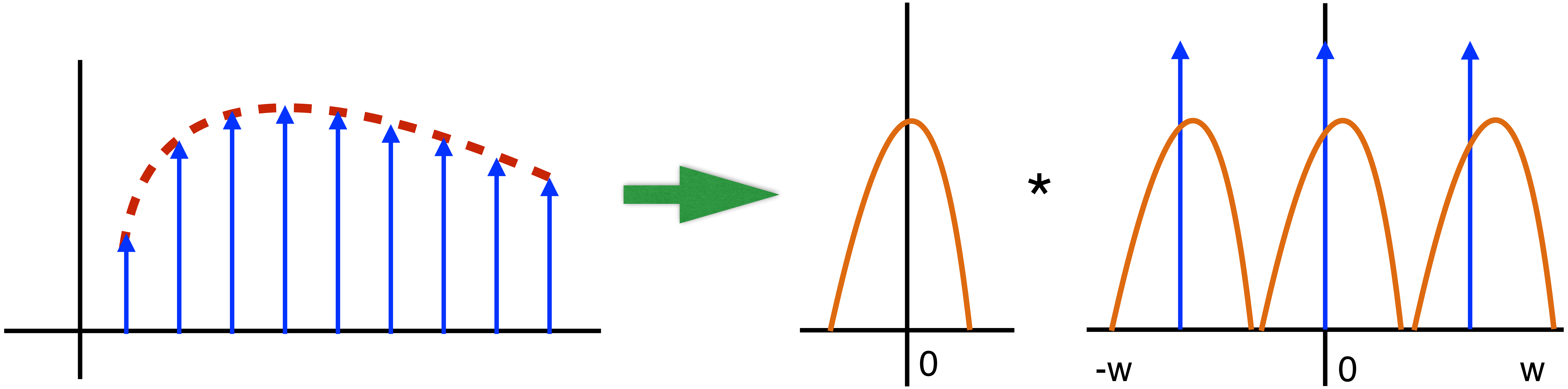
$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain



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Fredo Durand [2011]

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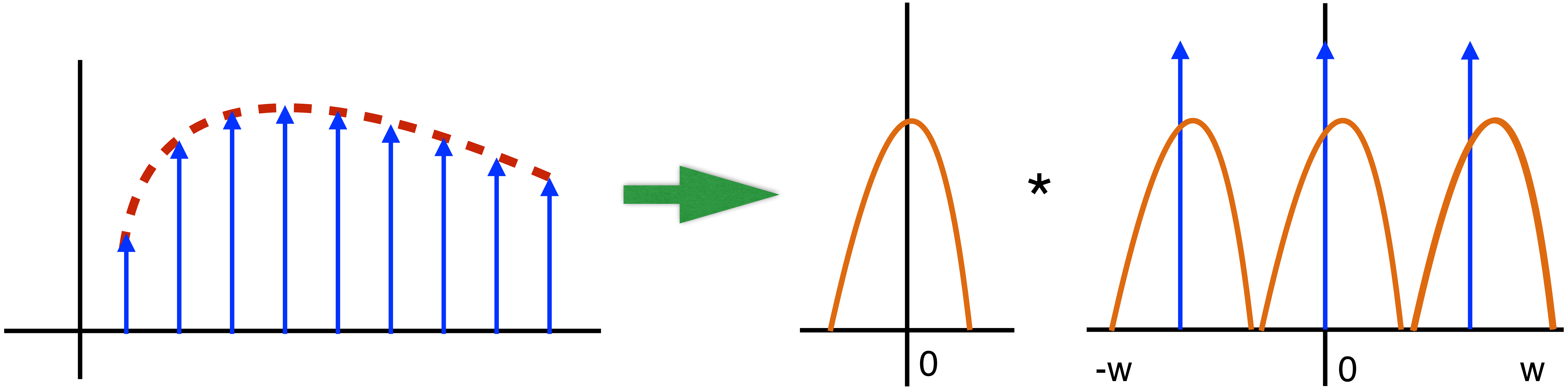
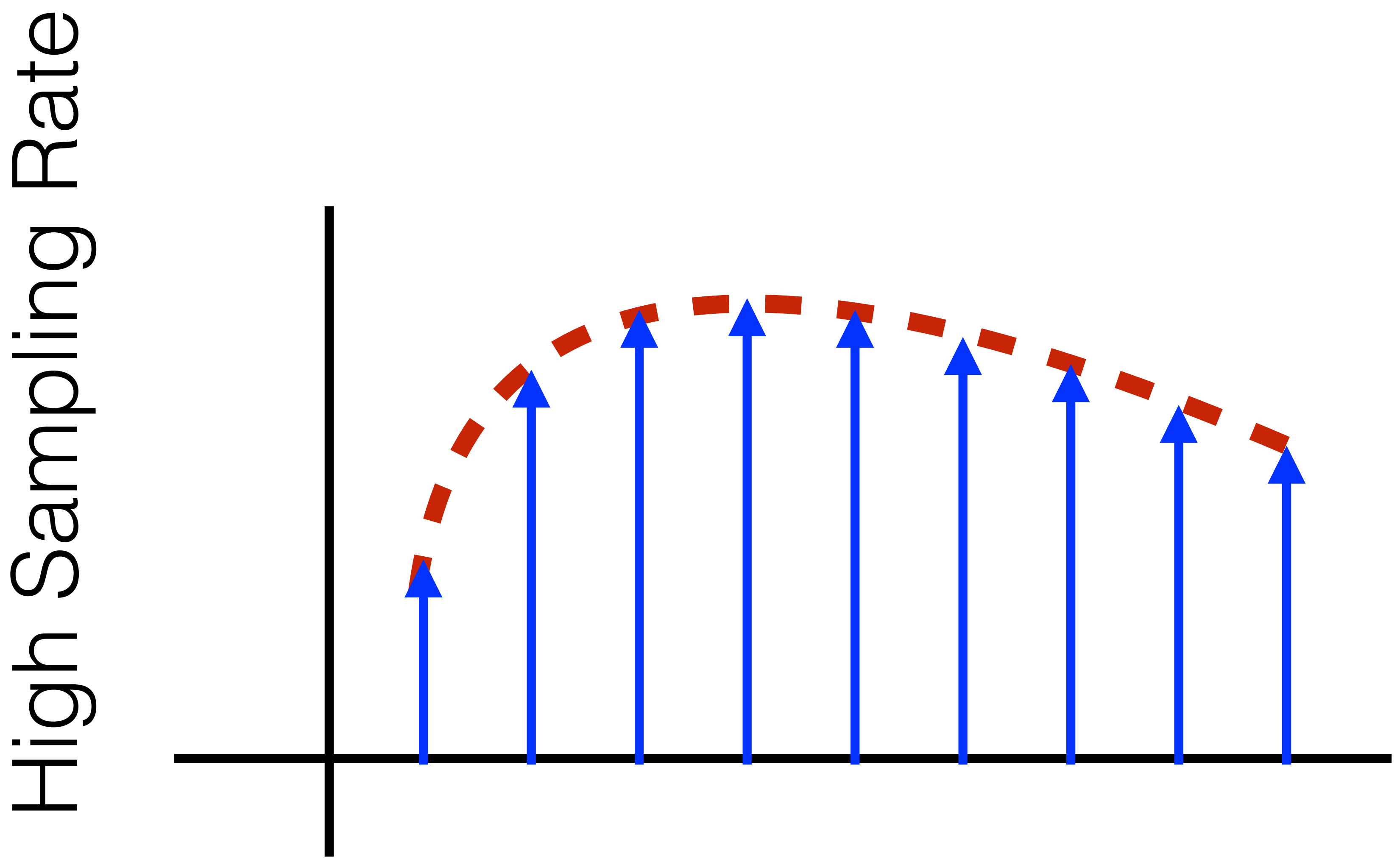


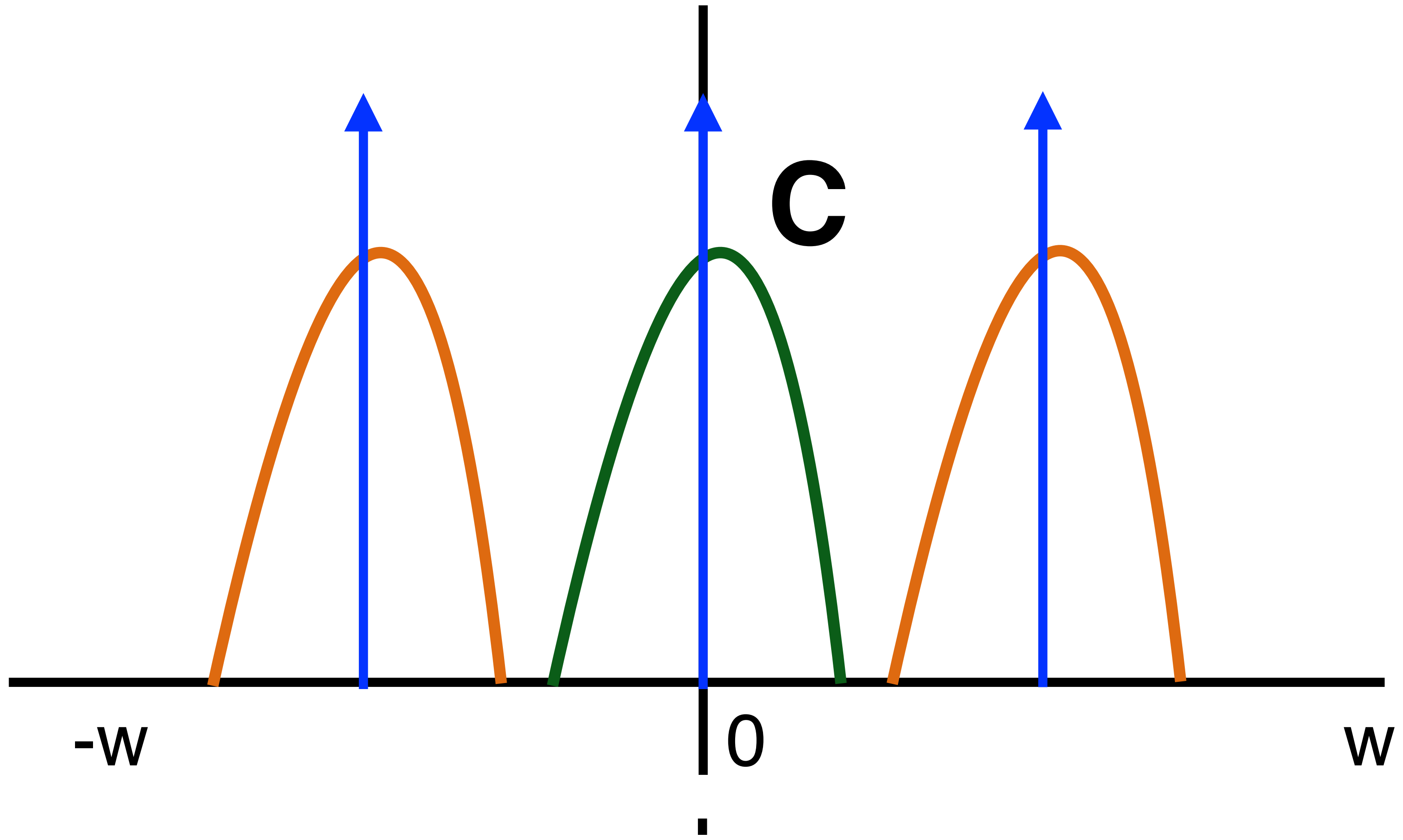
$f(x)\,\mathbf{S}(x)$

$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

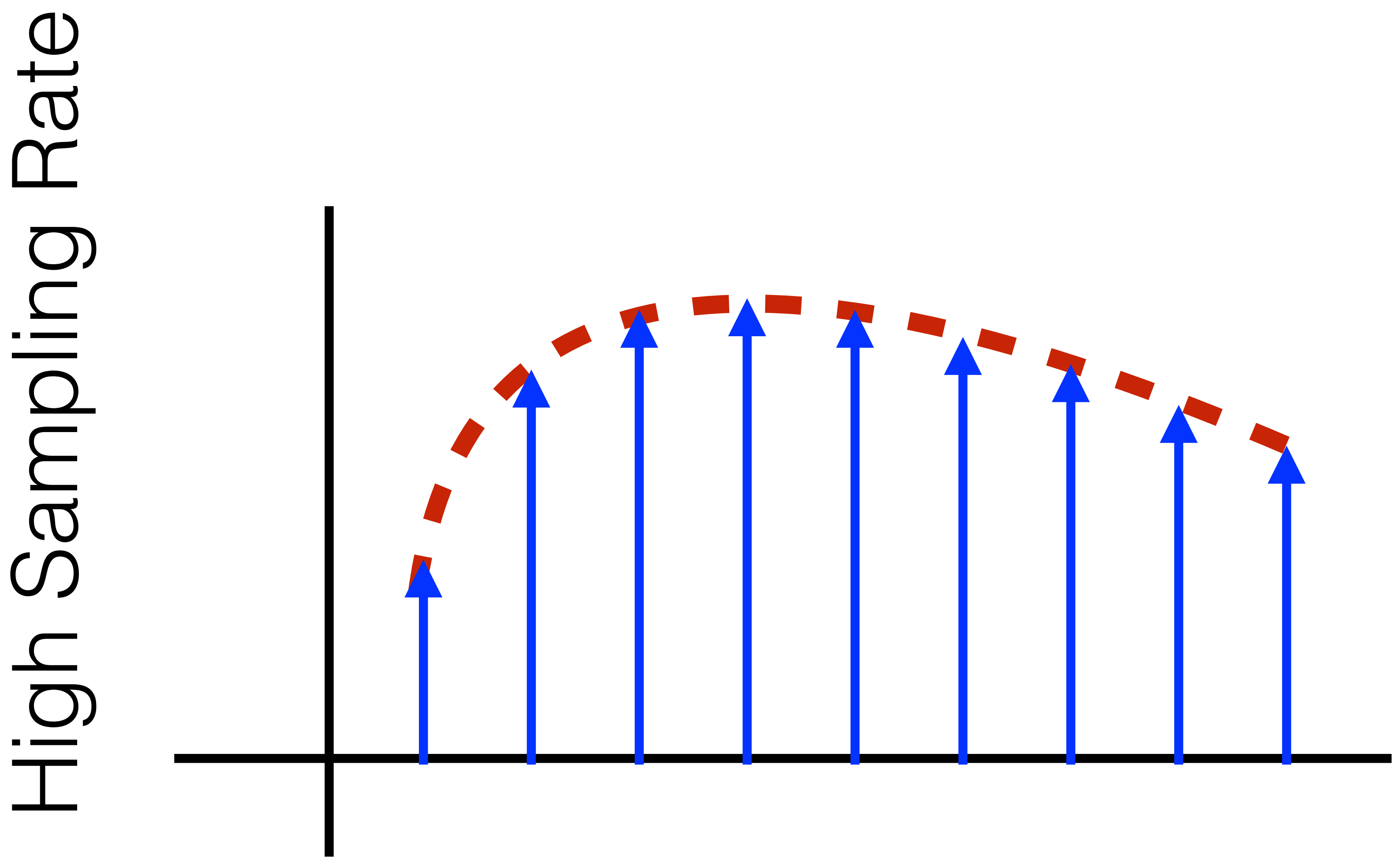
Fredo Durand [2011]

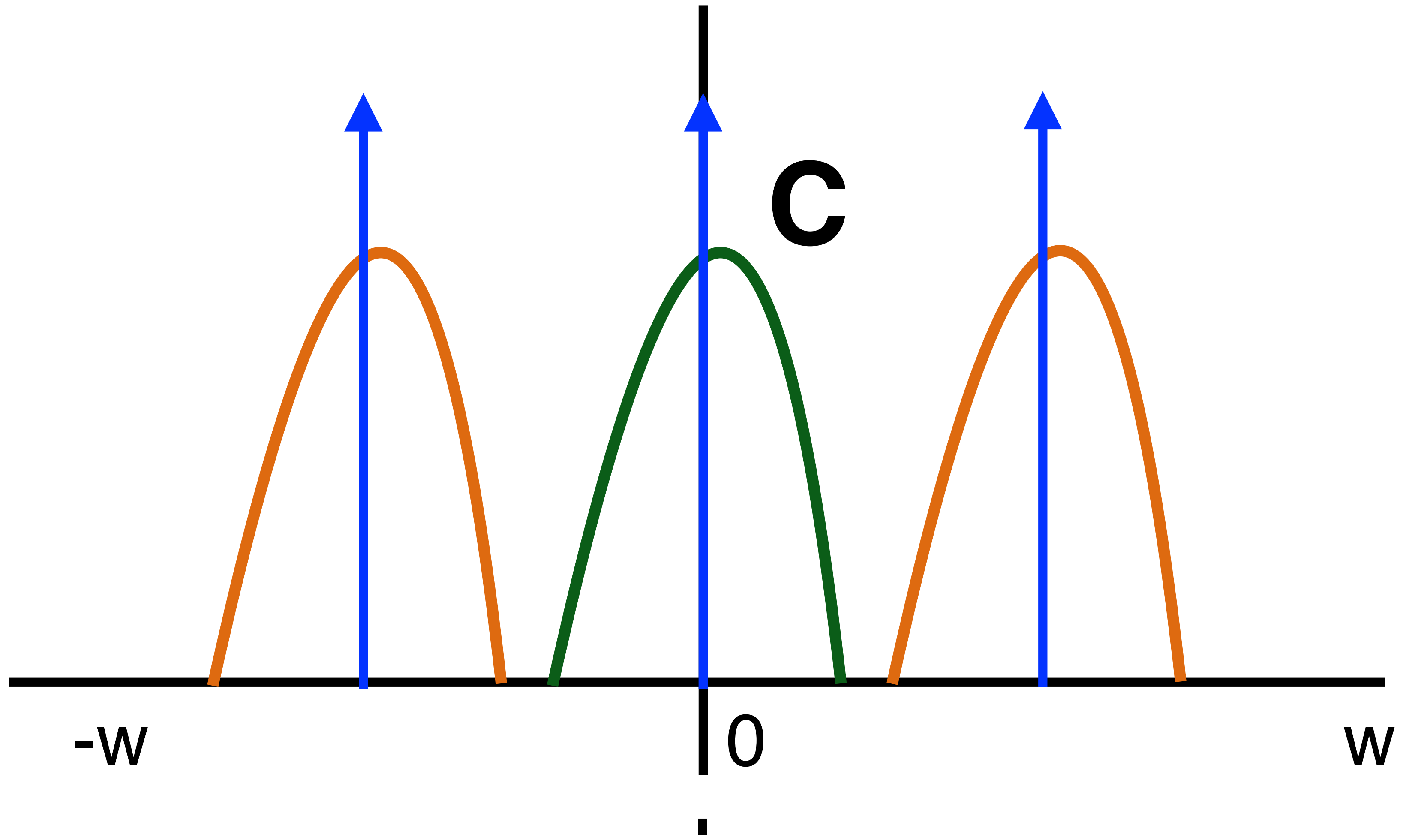
Aliasing in Reconstruction





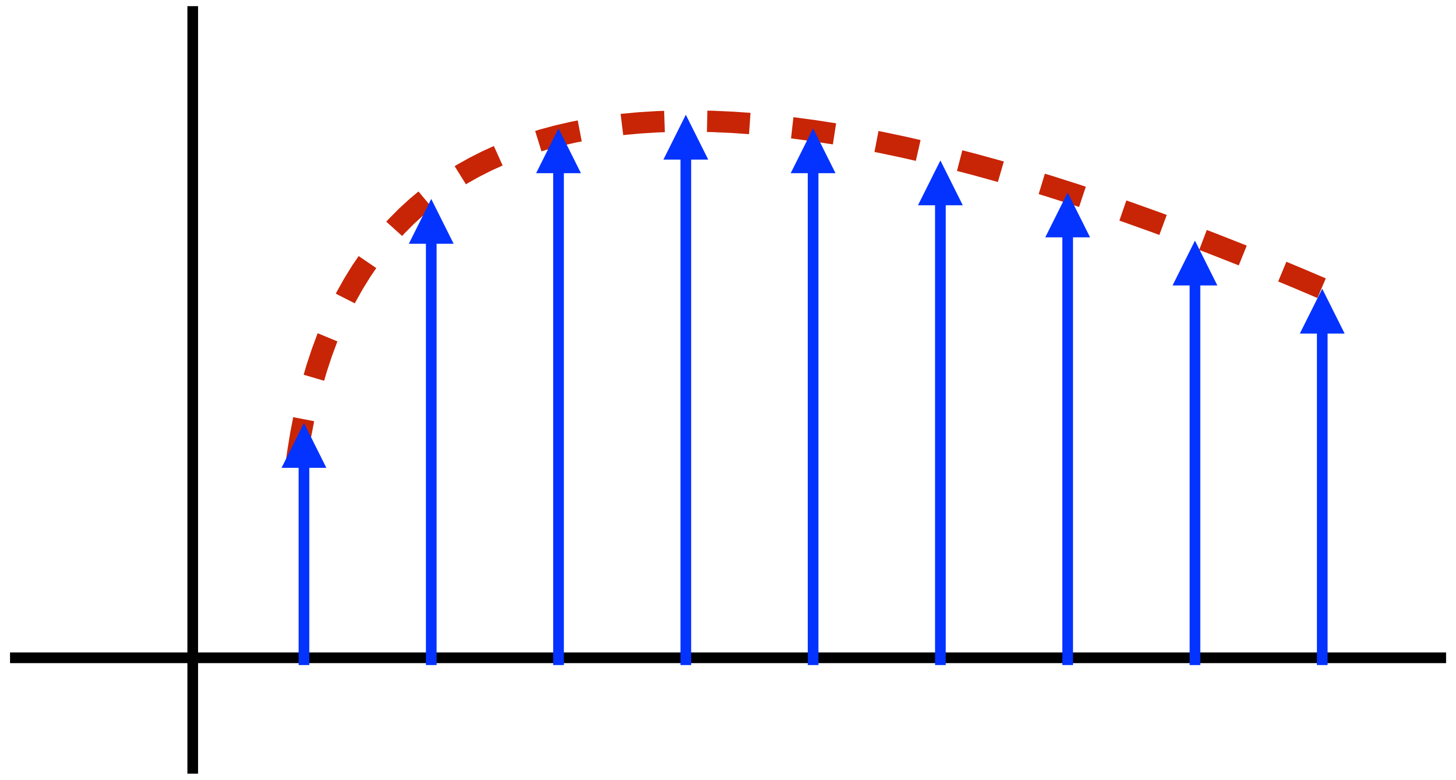
Aliasing in Reconstruction

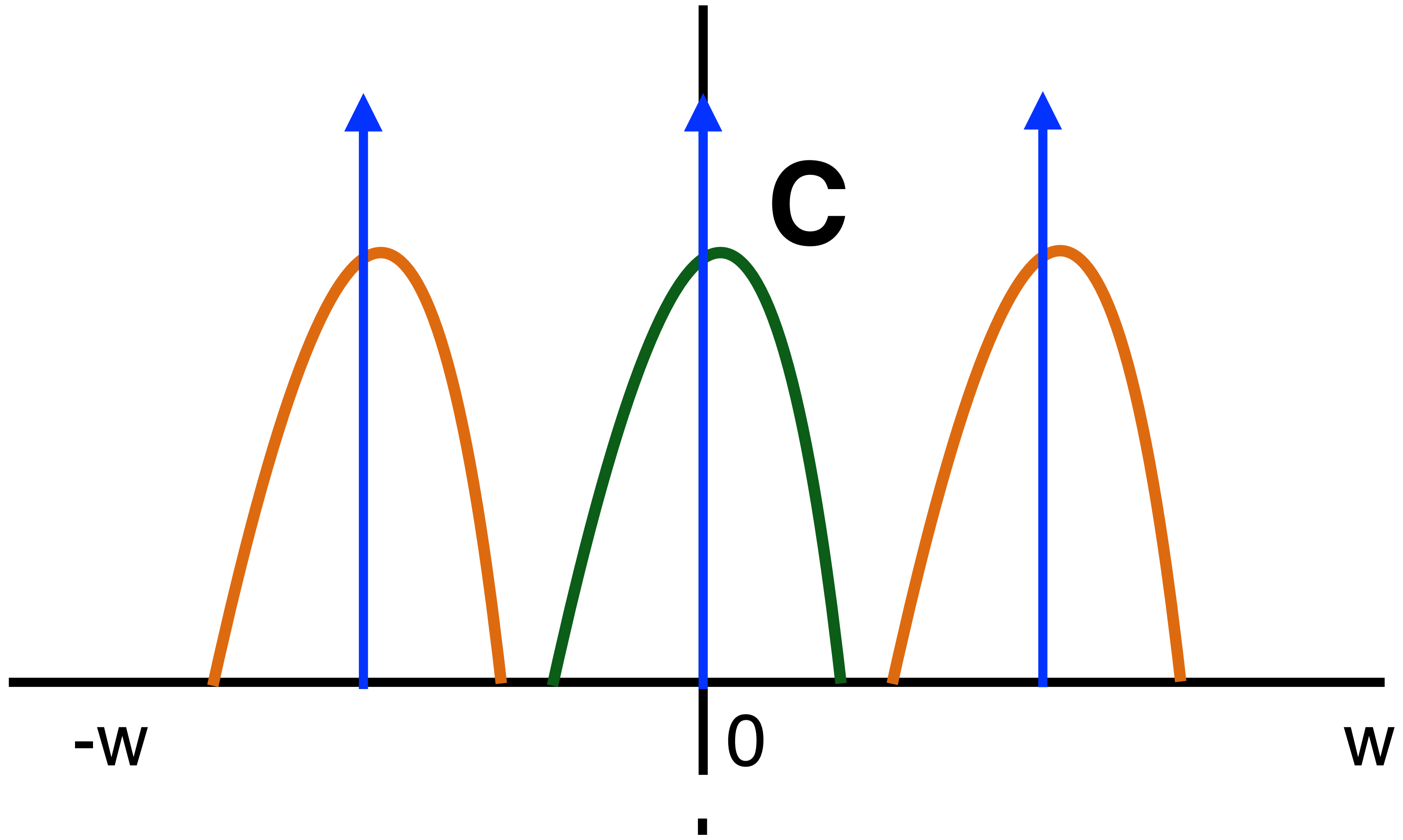




Aliasing in Reconstruction

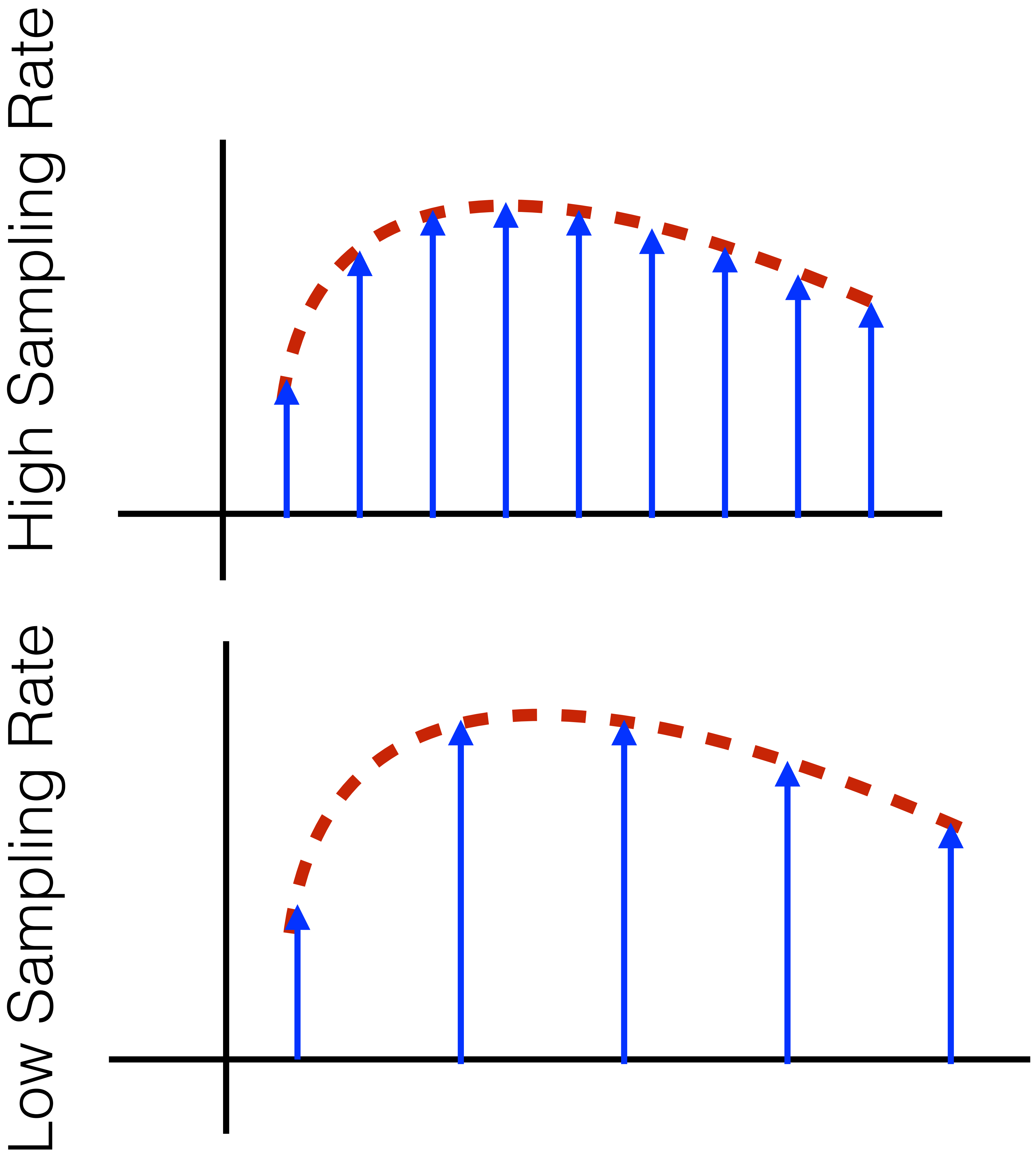
High Sampling Rate

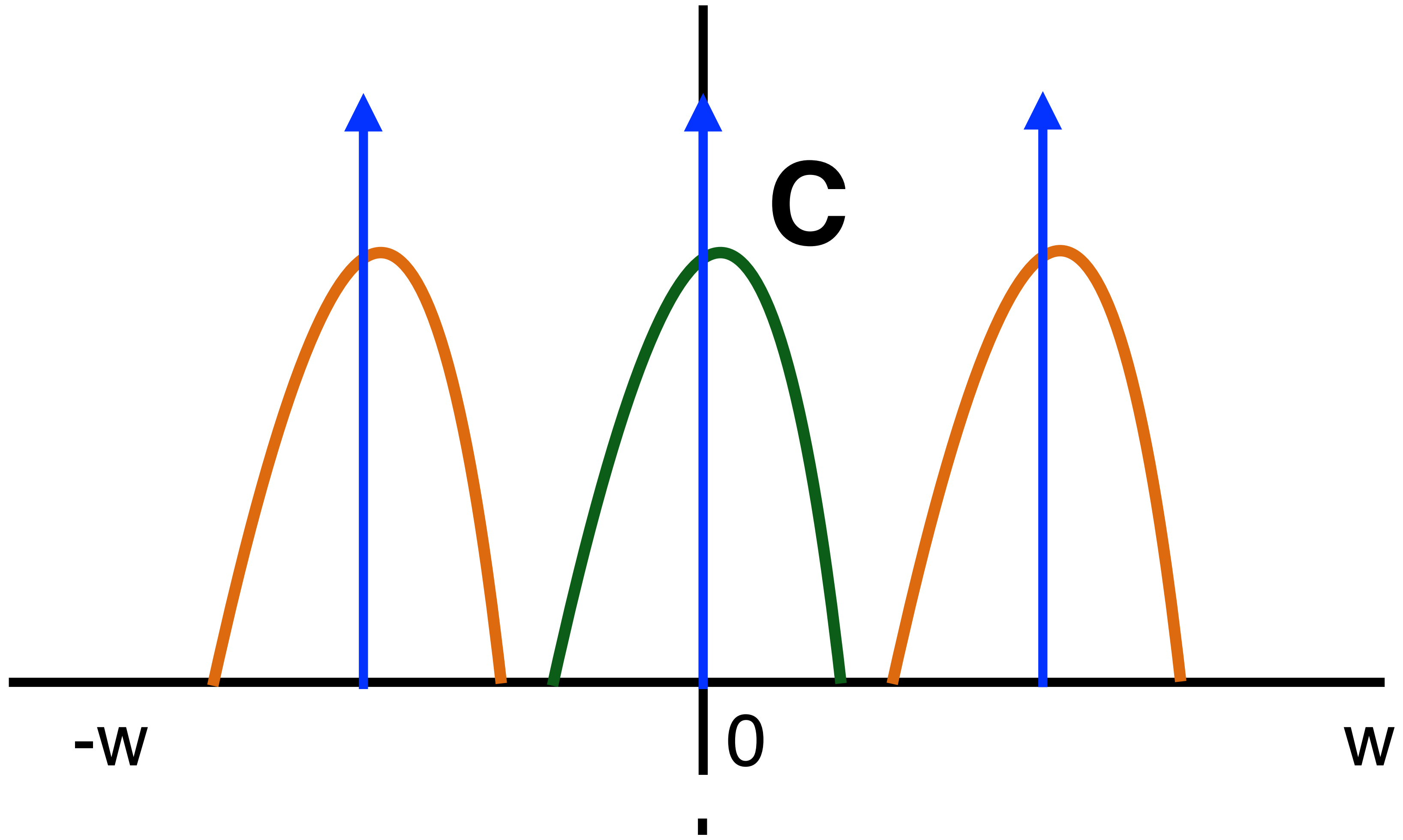




Low Sampling Rate

Aliasing in Reconstruction





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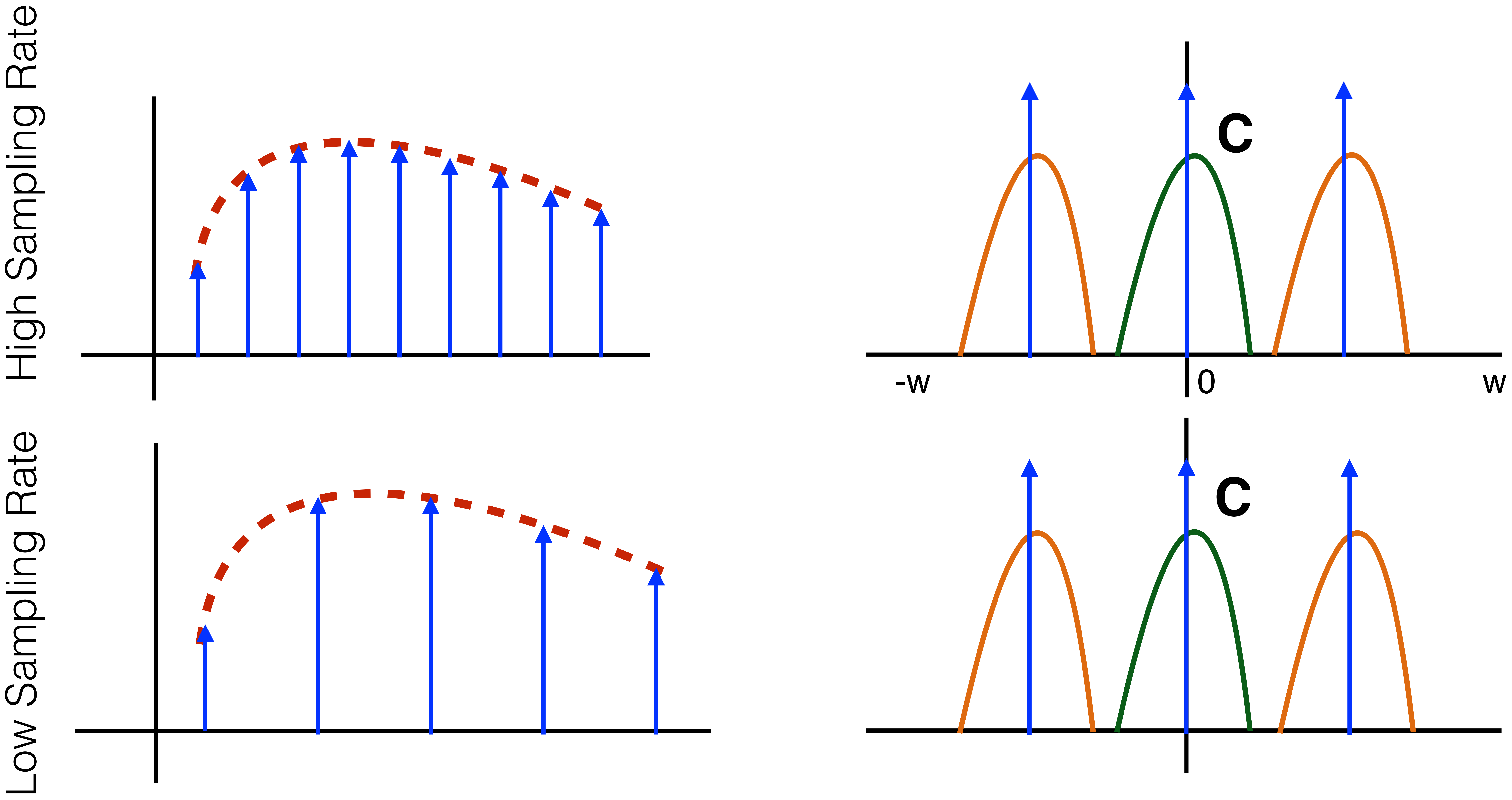

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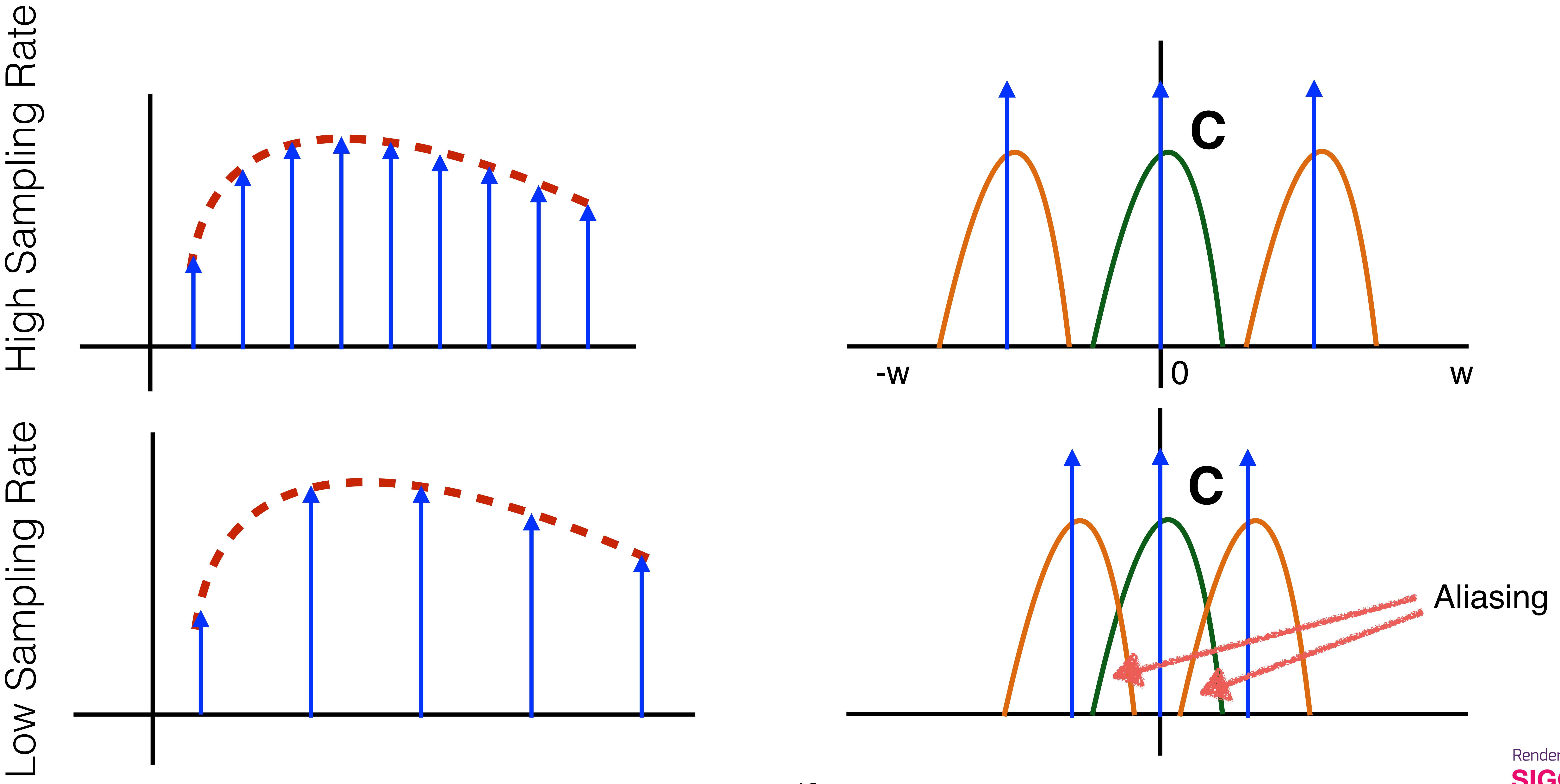
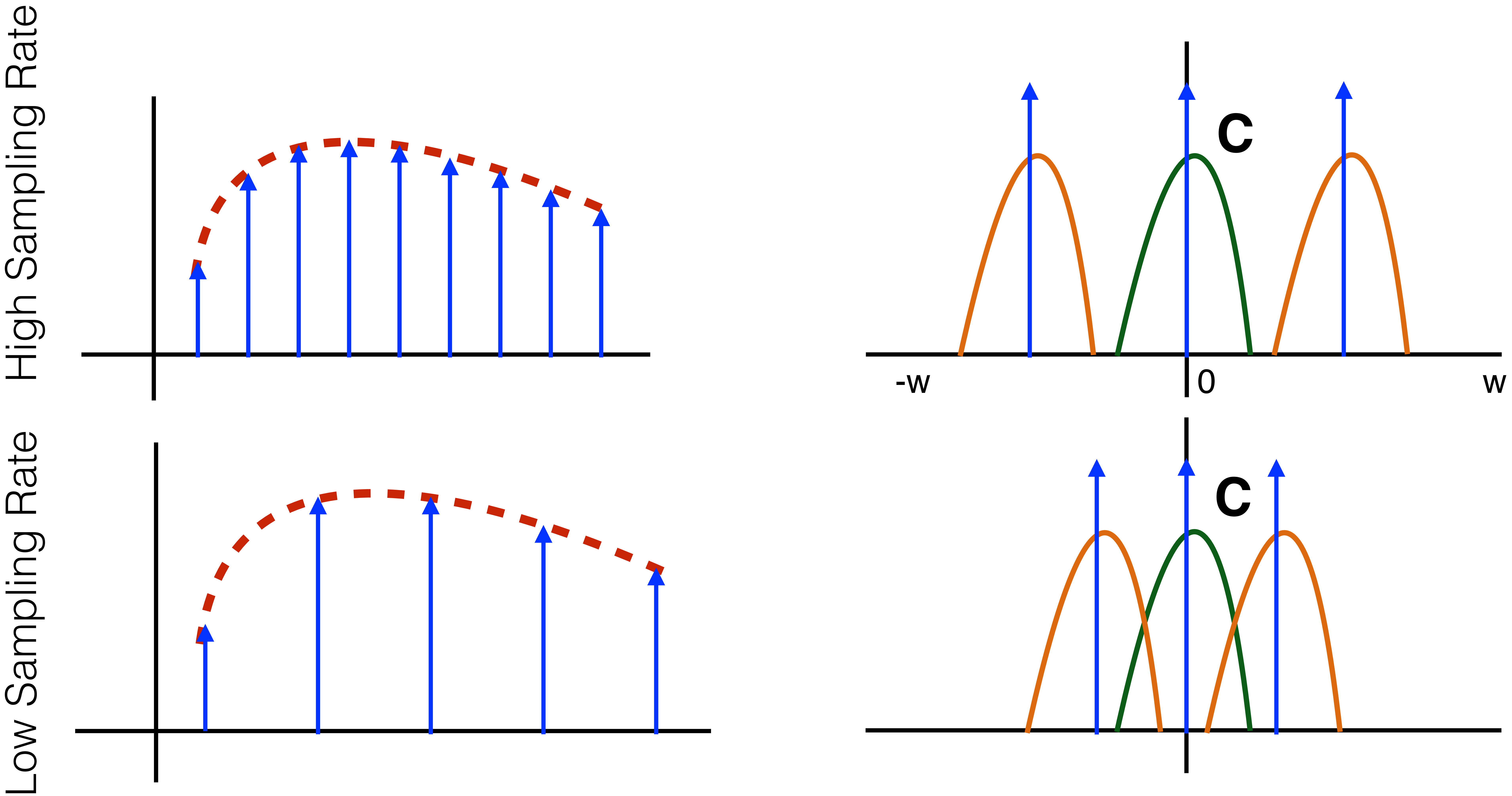
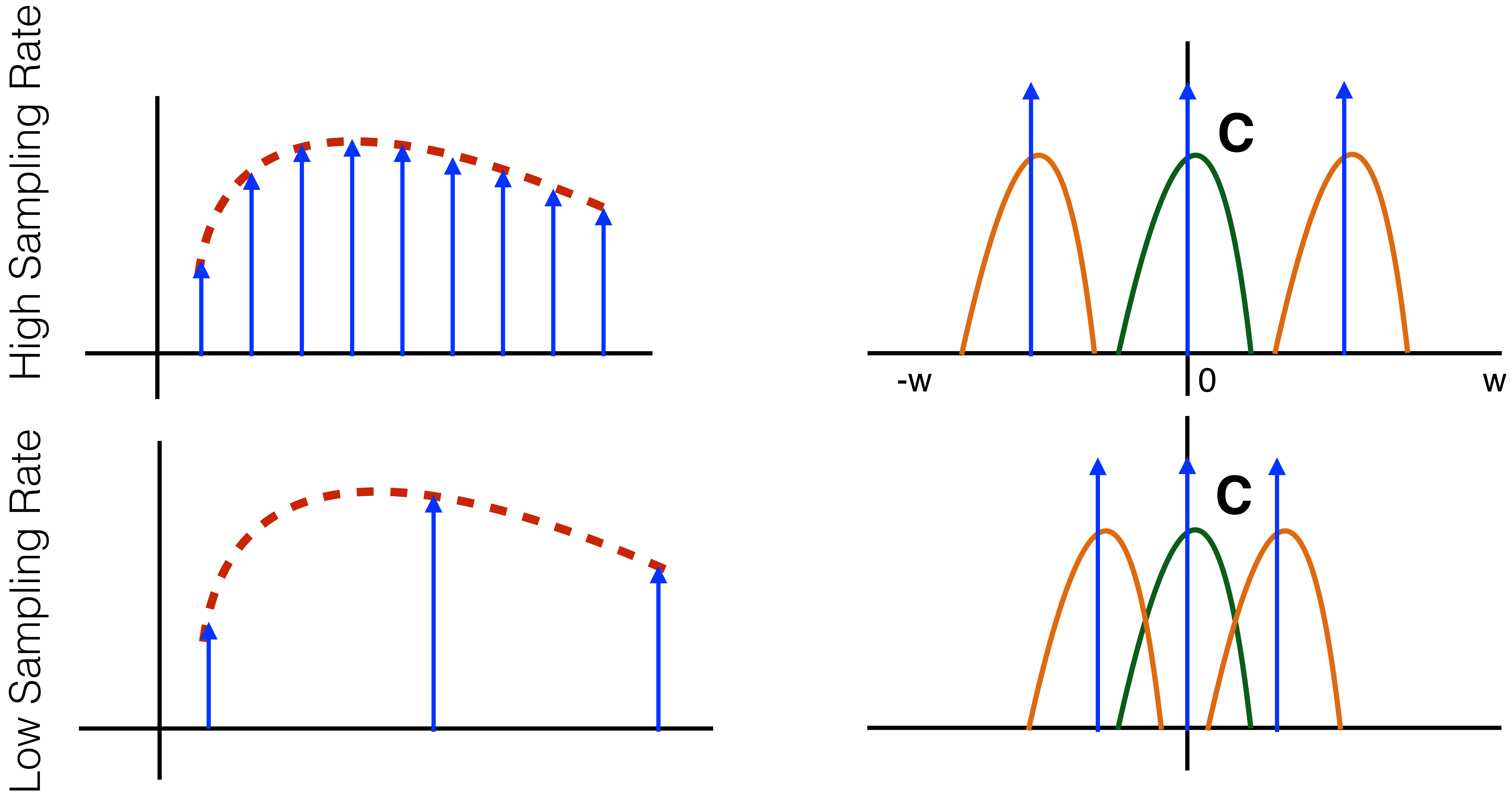


Aliasing in Reconstruction



Error in Monte Carlo Integration



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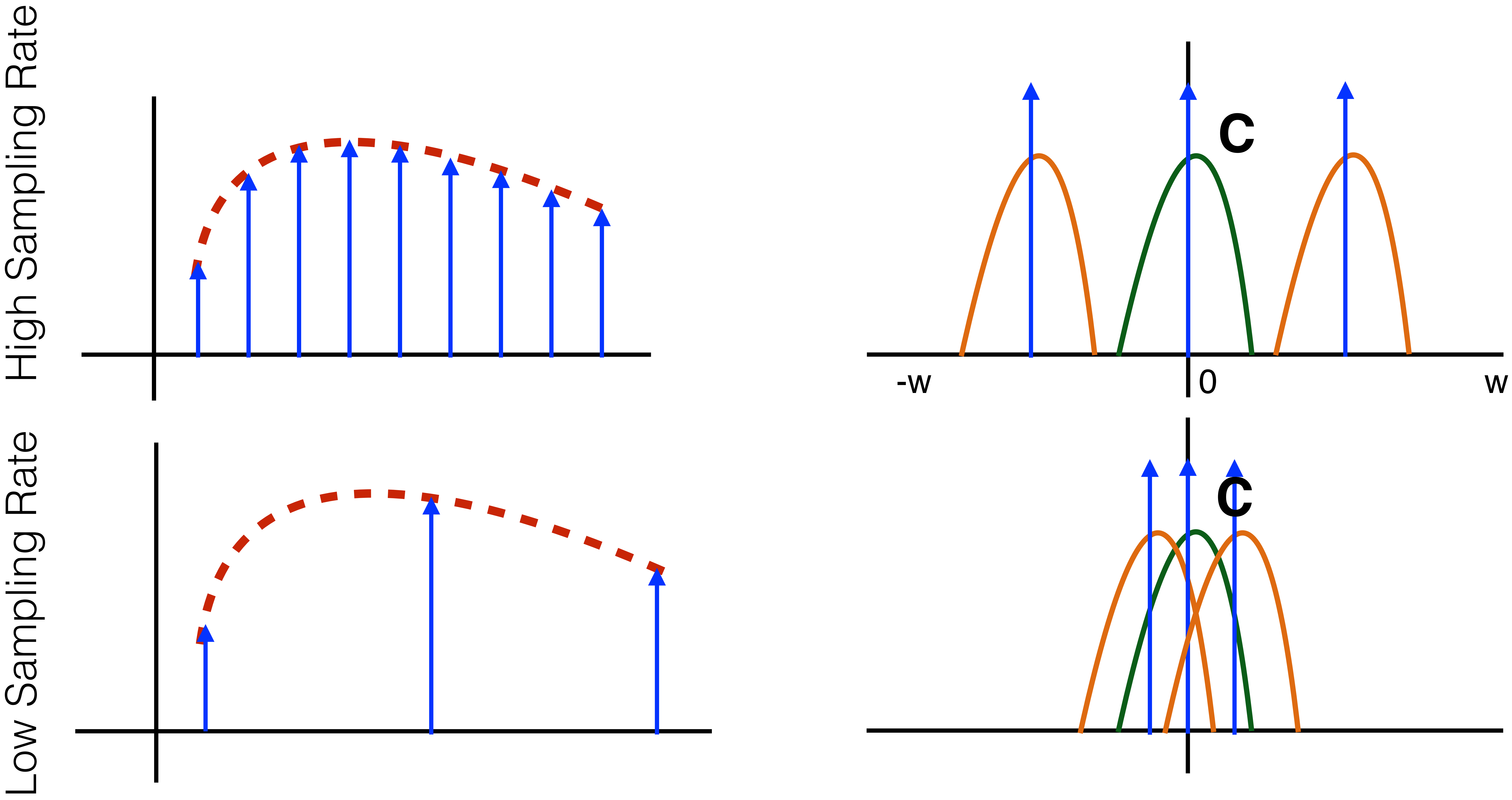

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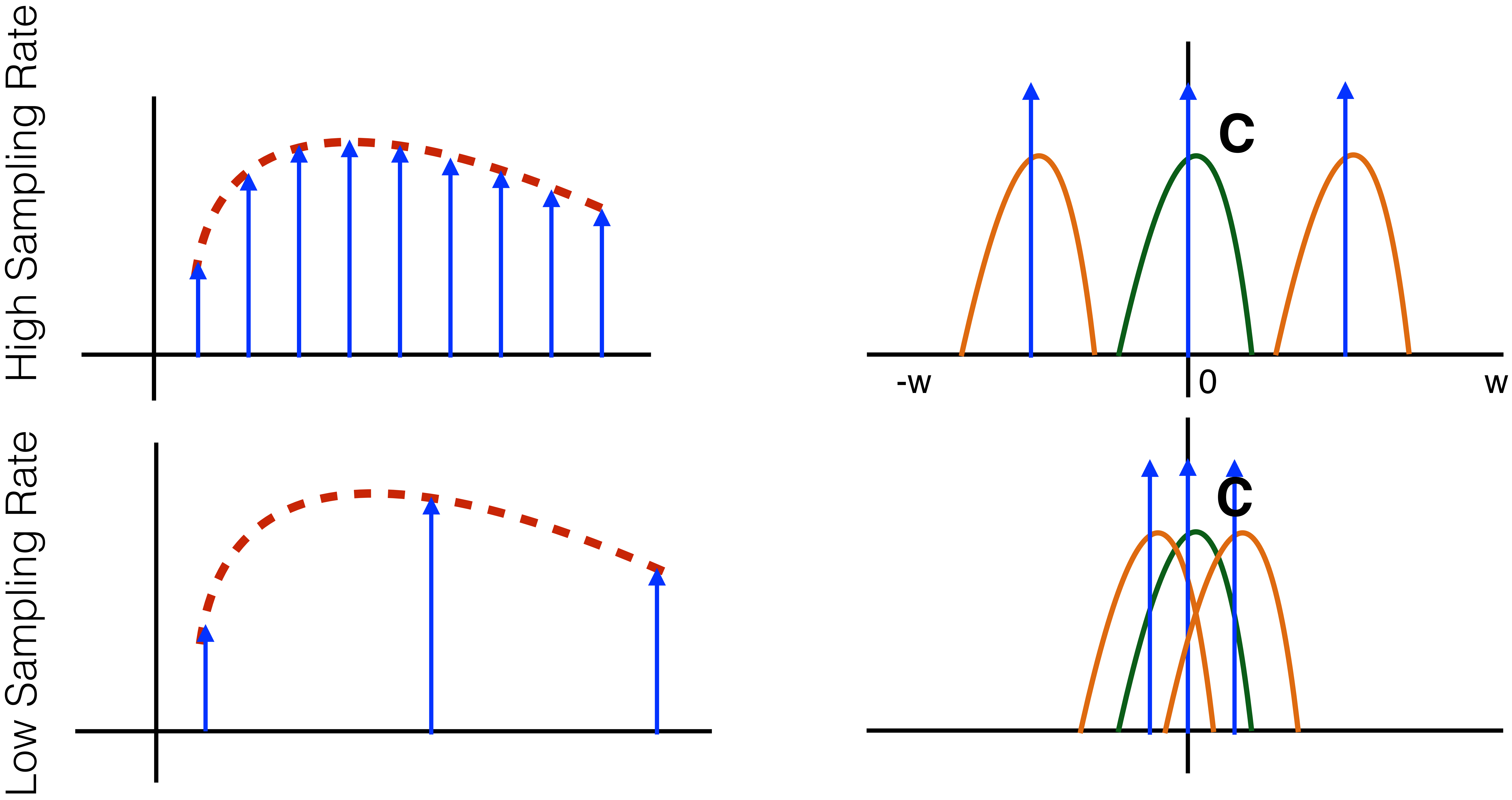

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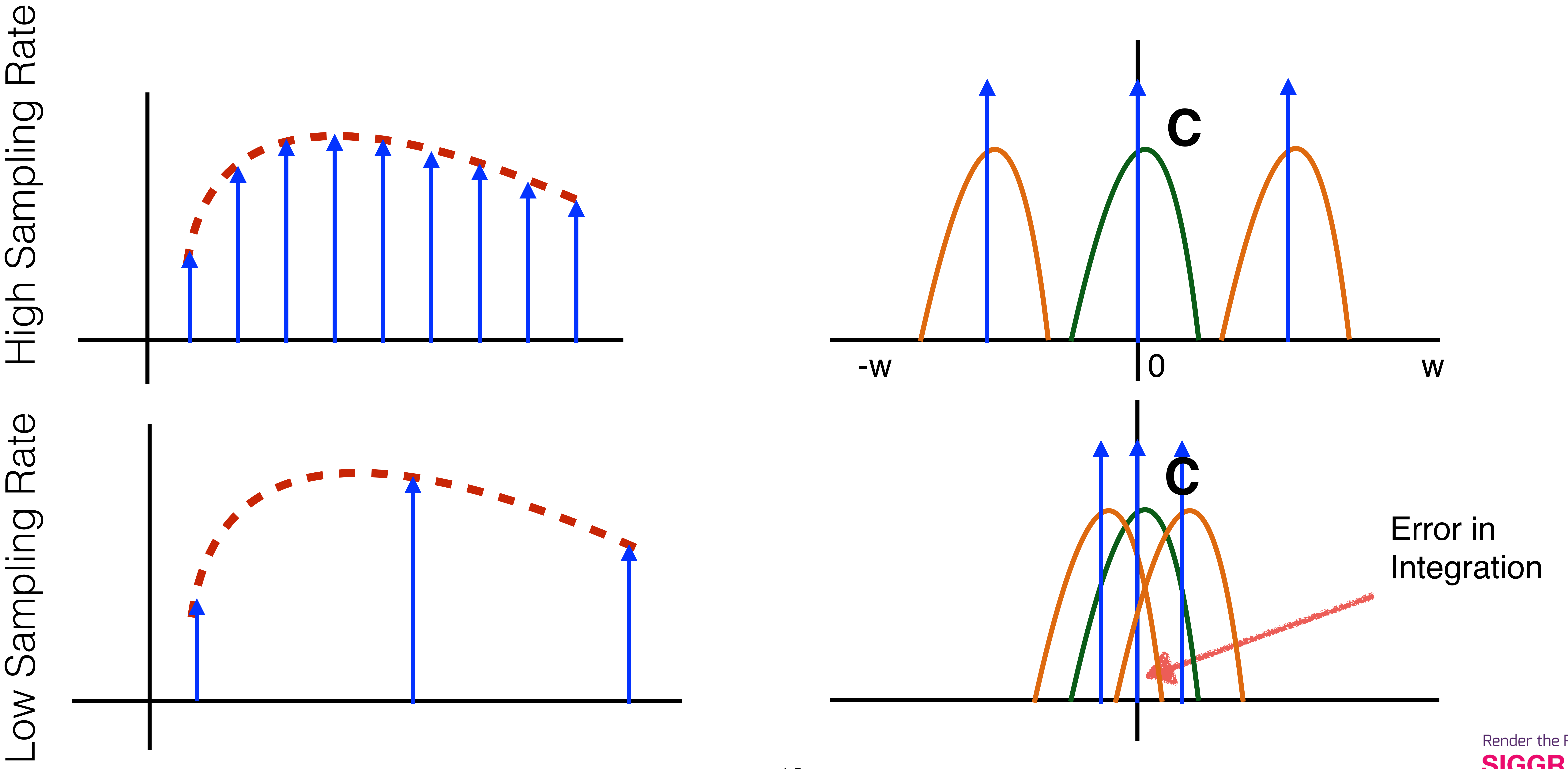
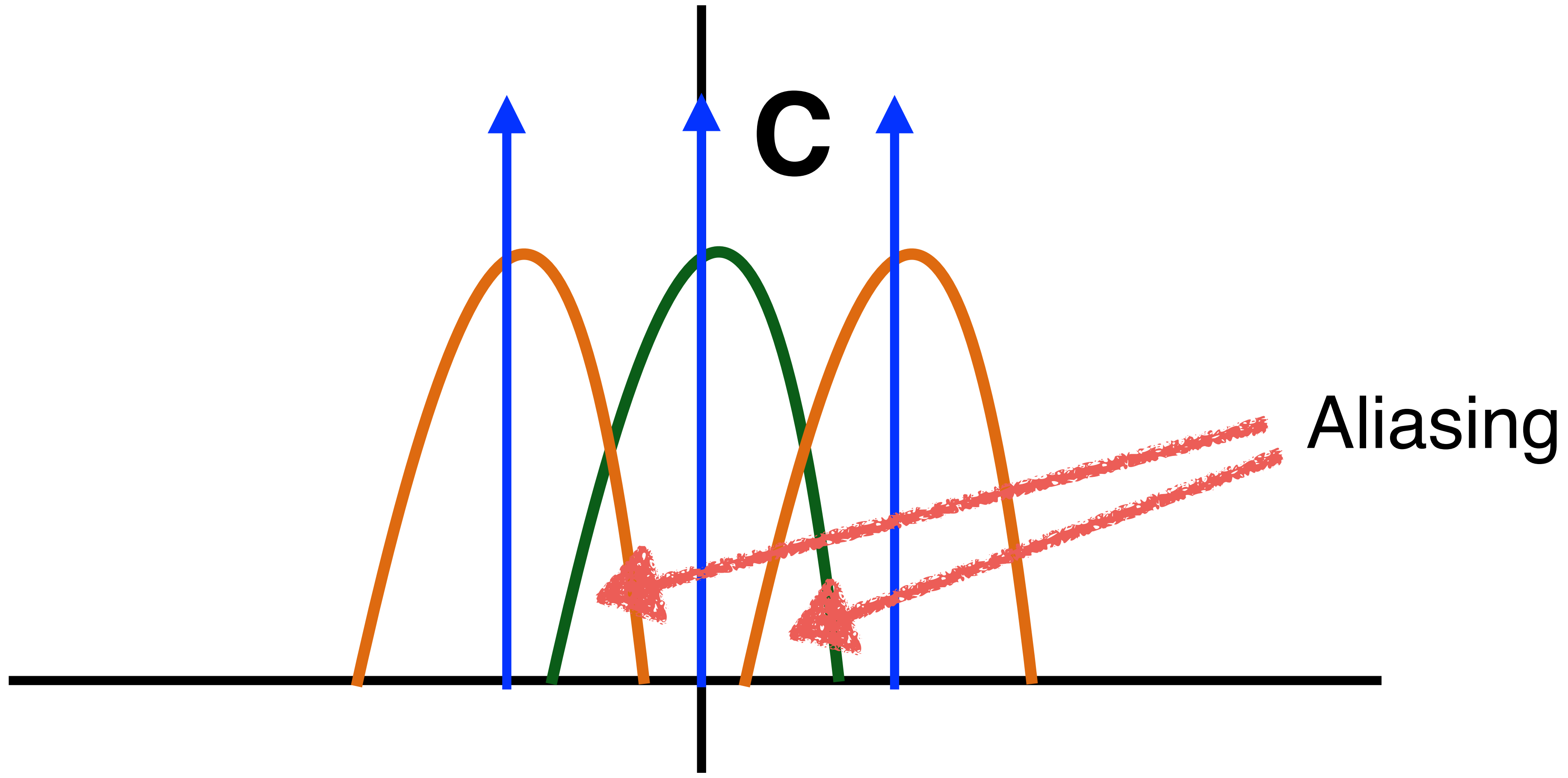
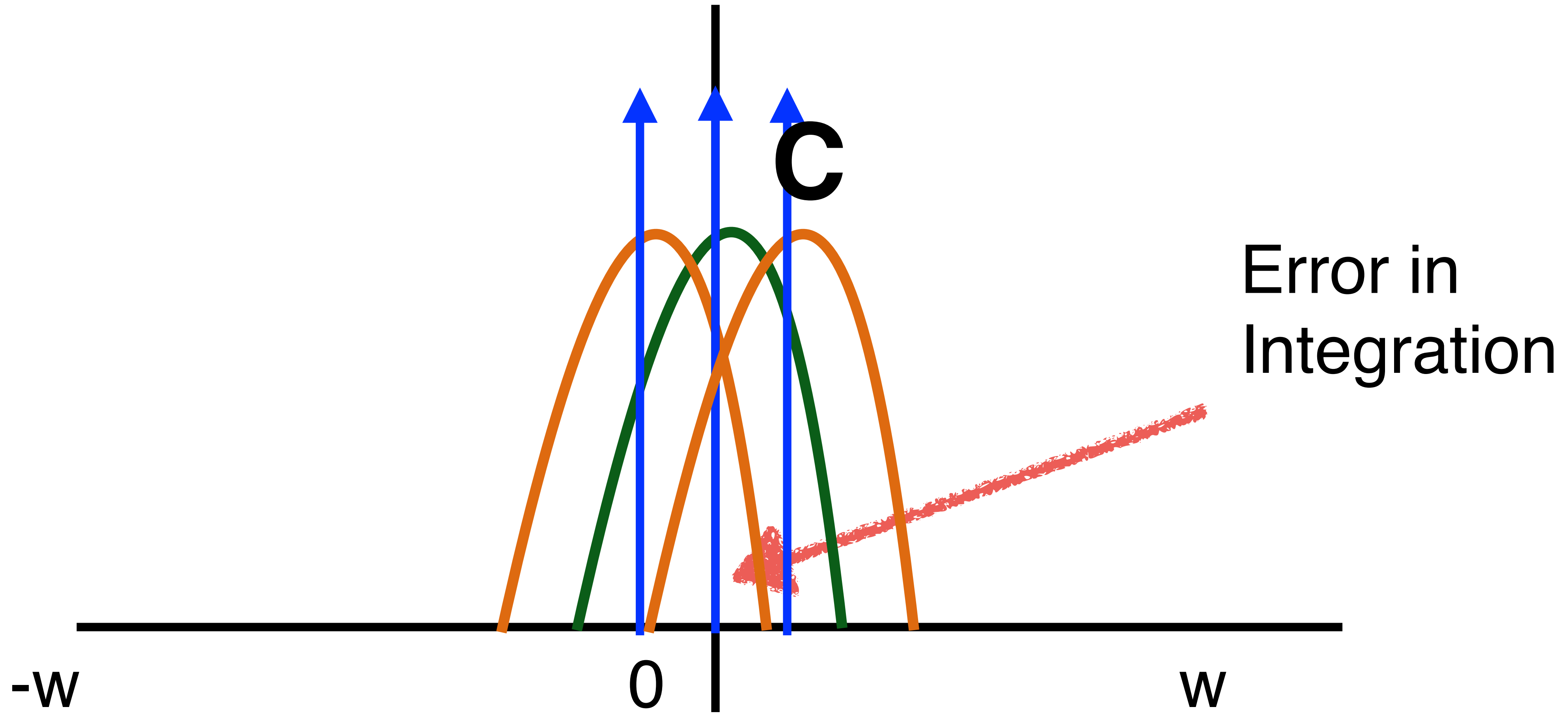


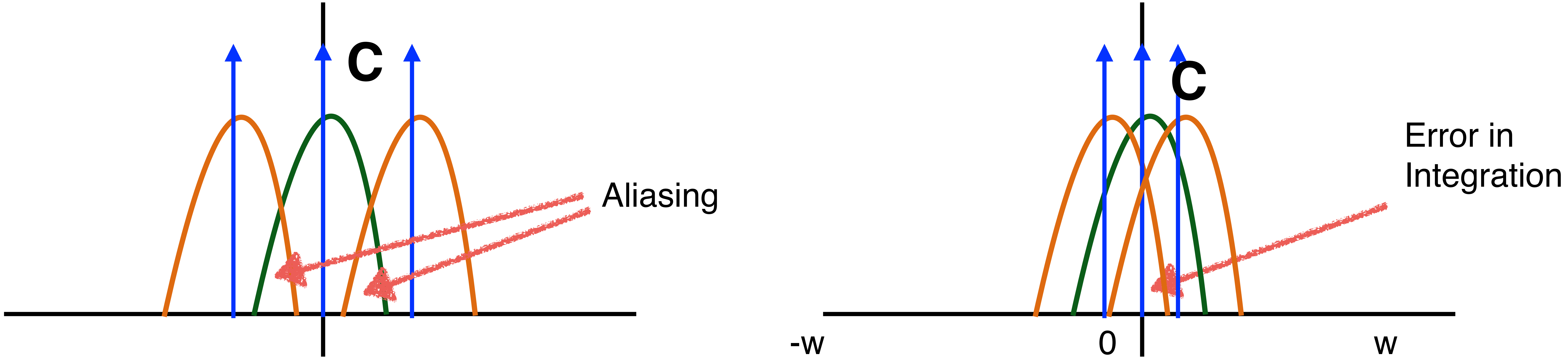
Aliasing (Reconstruction) vs. Error (Integration)





Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]



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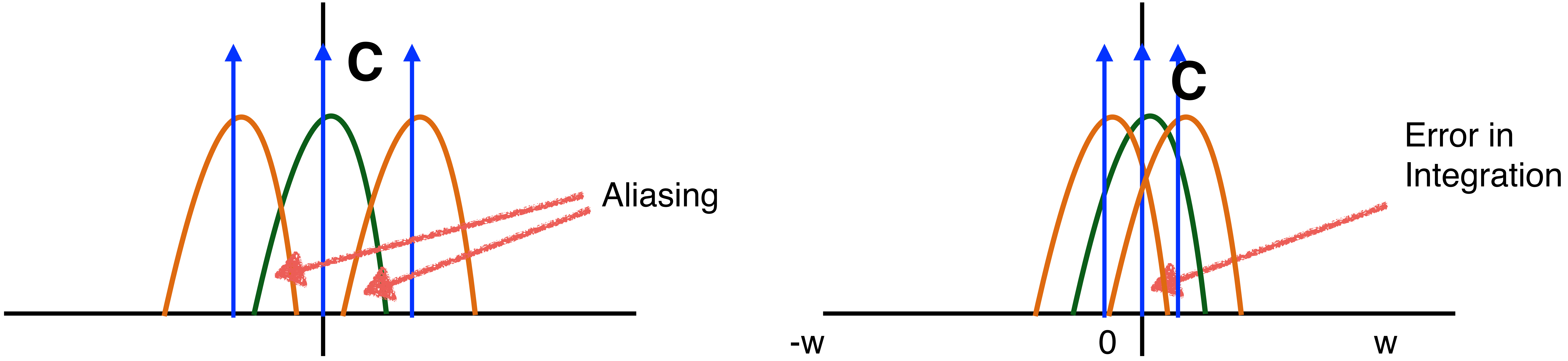

Integration in the Fourier Domain

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Integration is the DC term in the Fourier Domain

Spatial Domain:

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Fourier Domain:

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Fourier Domain:

$$\hat{f}(0)$$

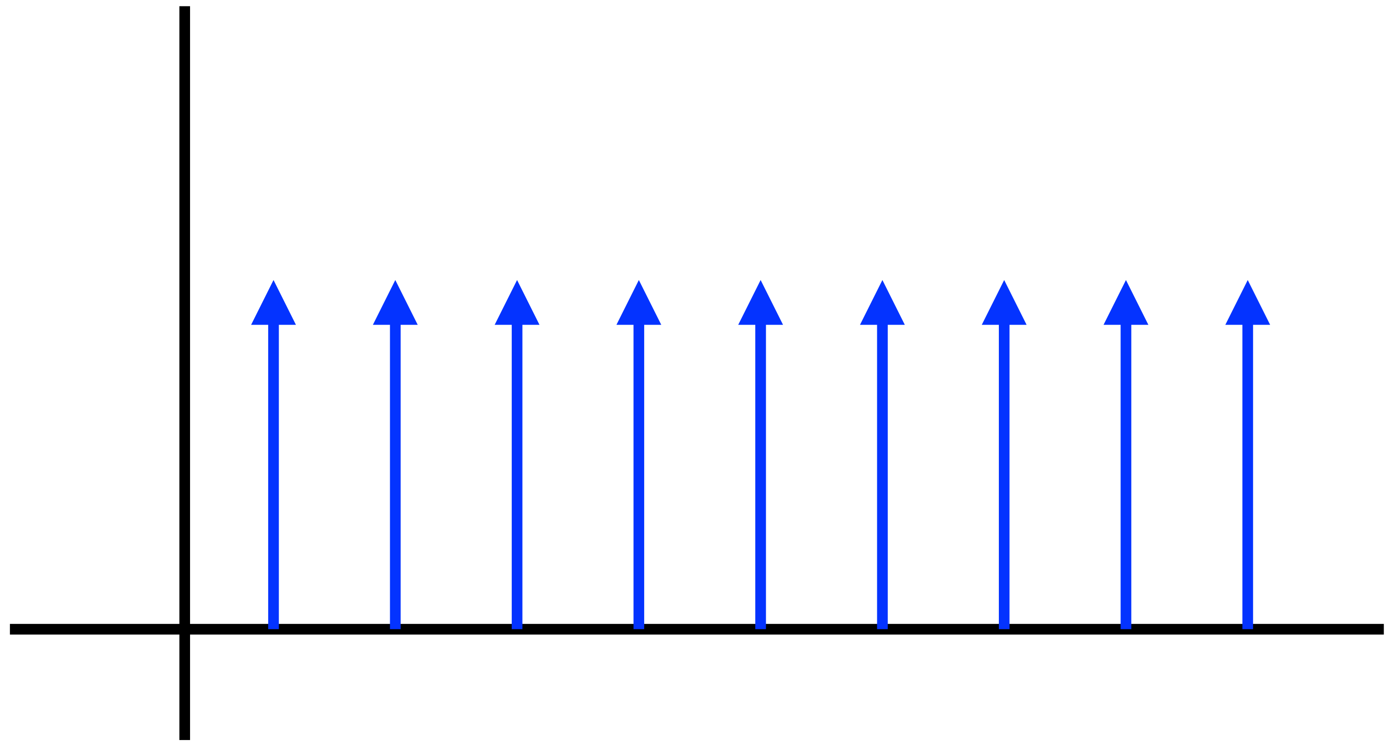
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



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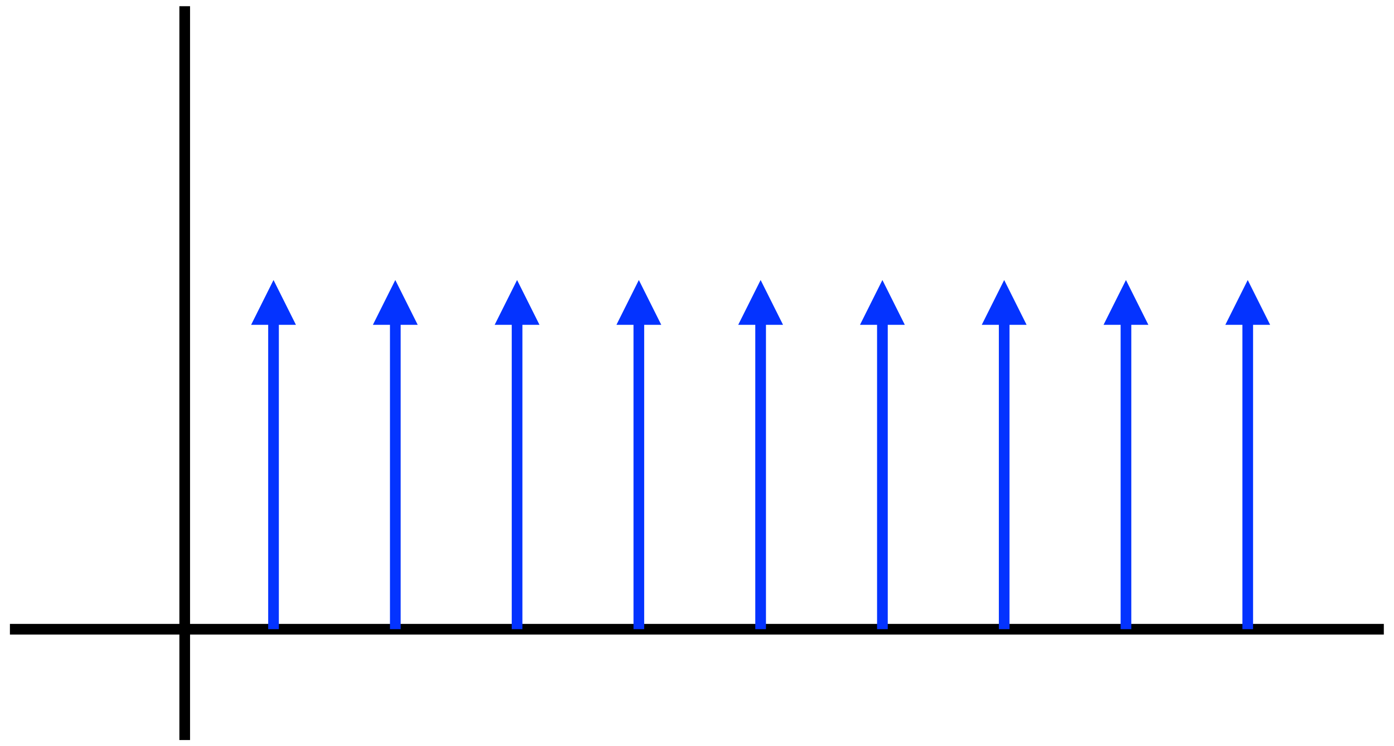

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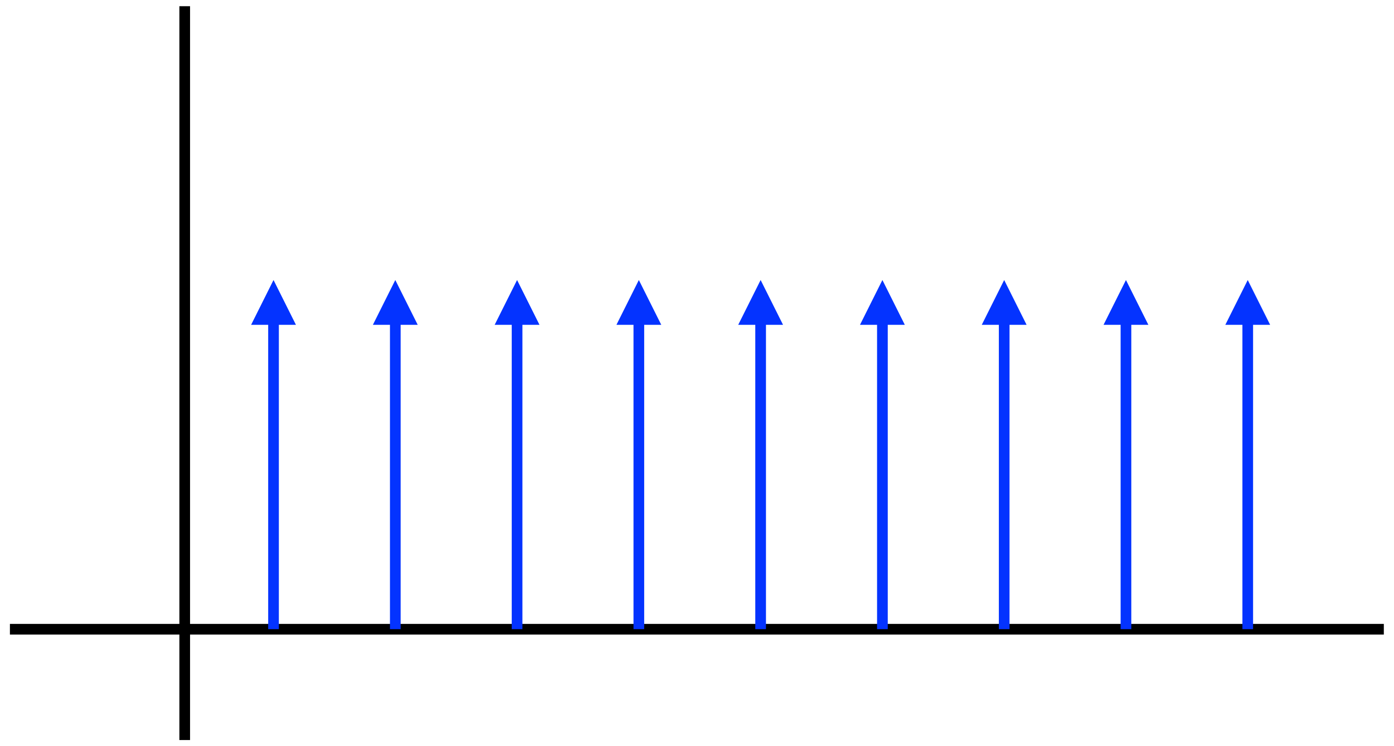

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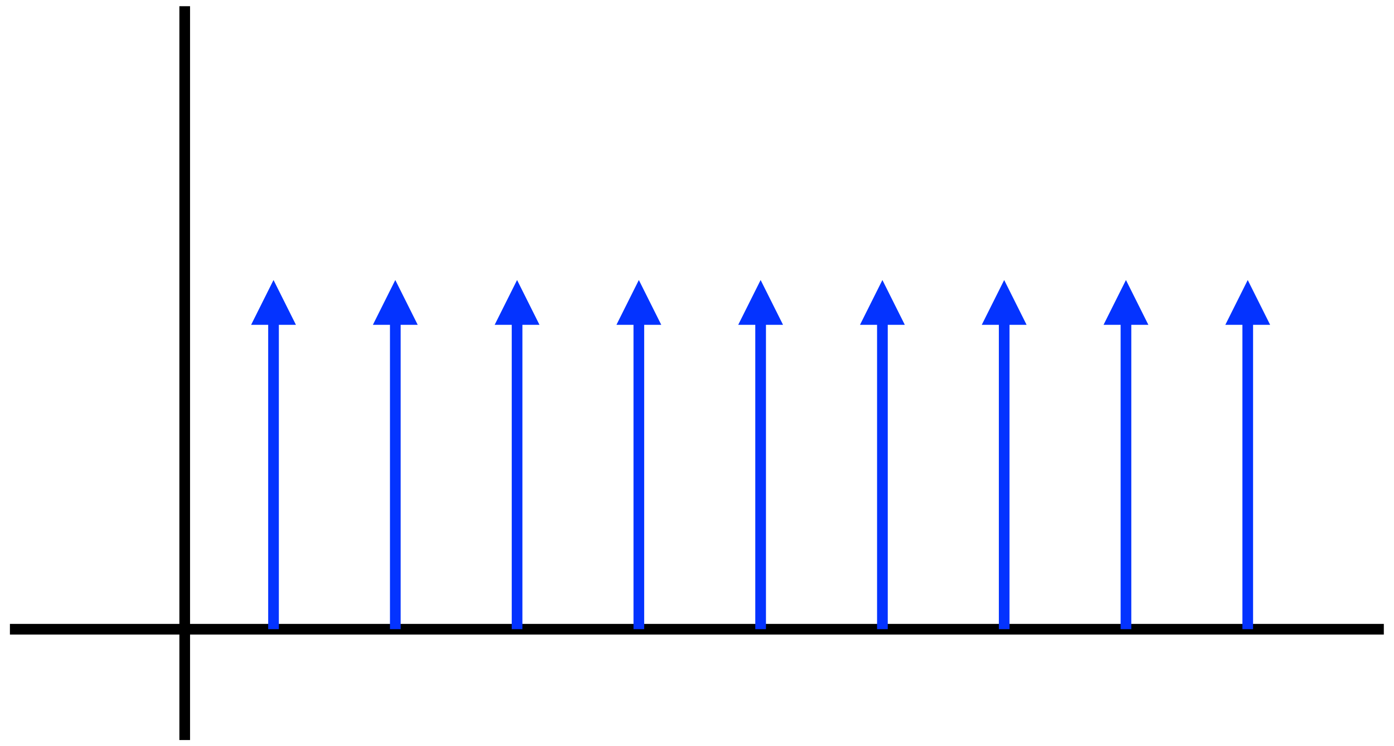

Monte Carlo Estimator in Spatial Domain

$$\boxed{\tilde{\mu}_N} = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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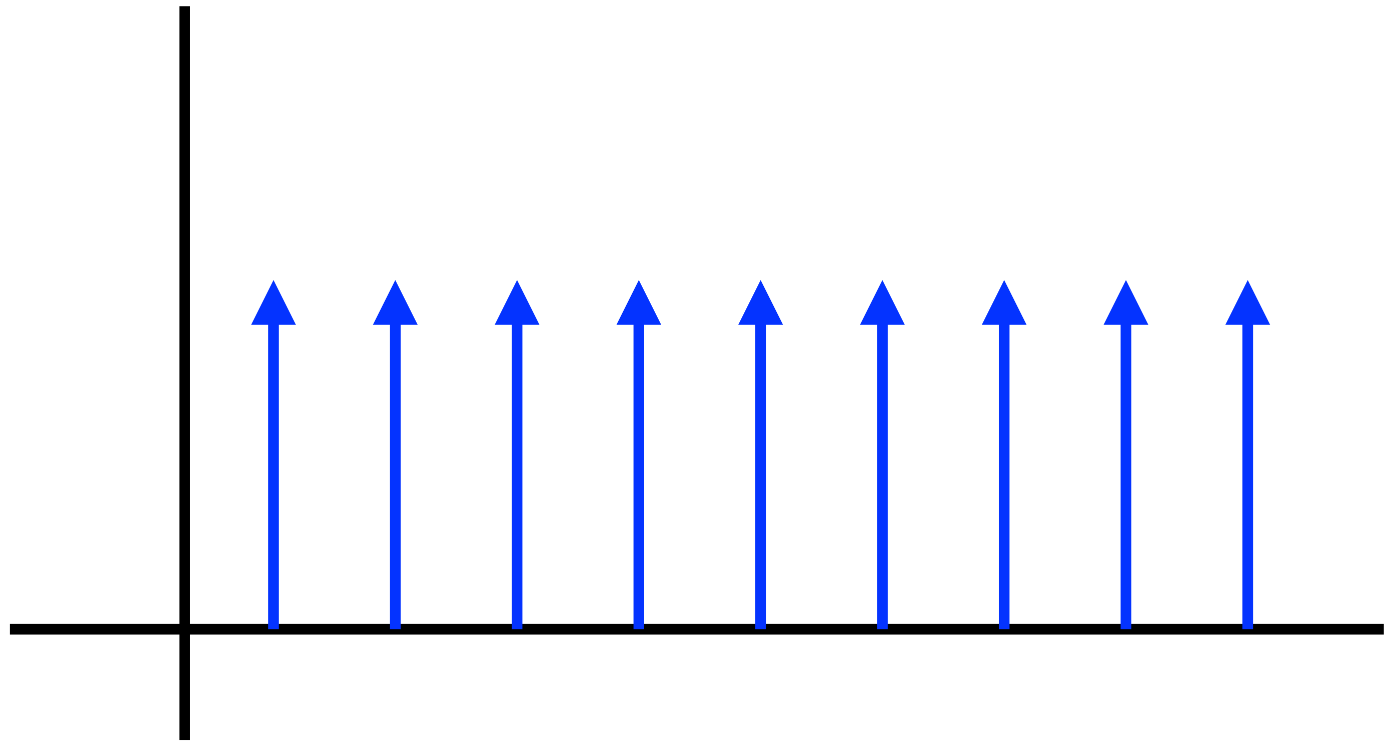

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Monte Carlo Estimator in Fourier Domain

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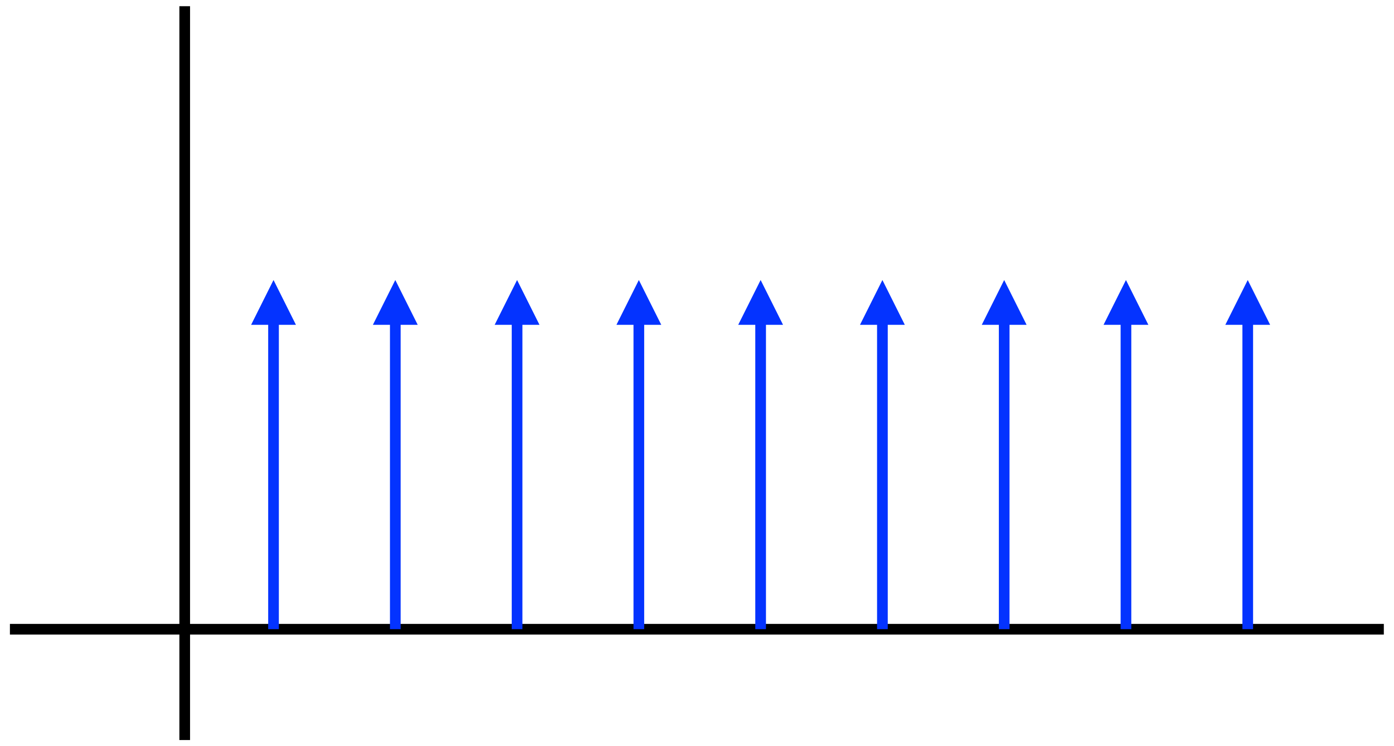

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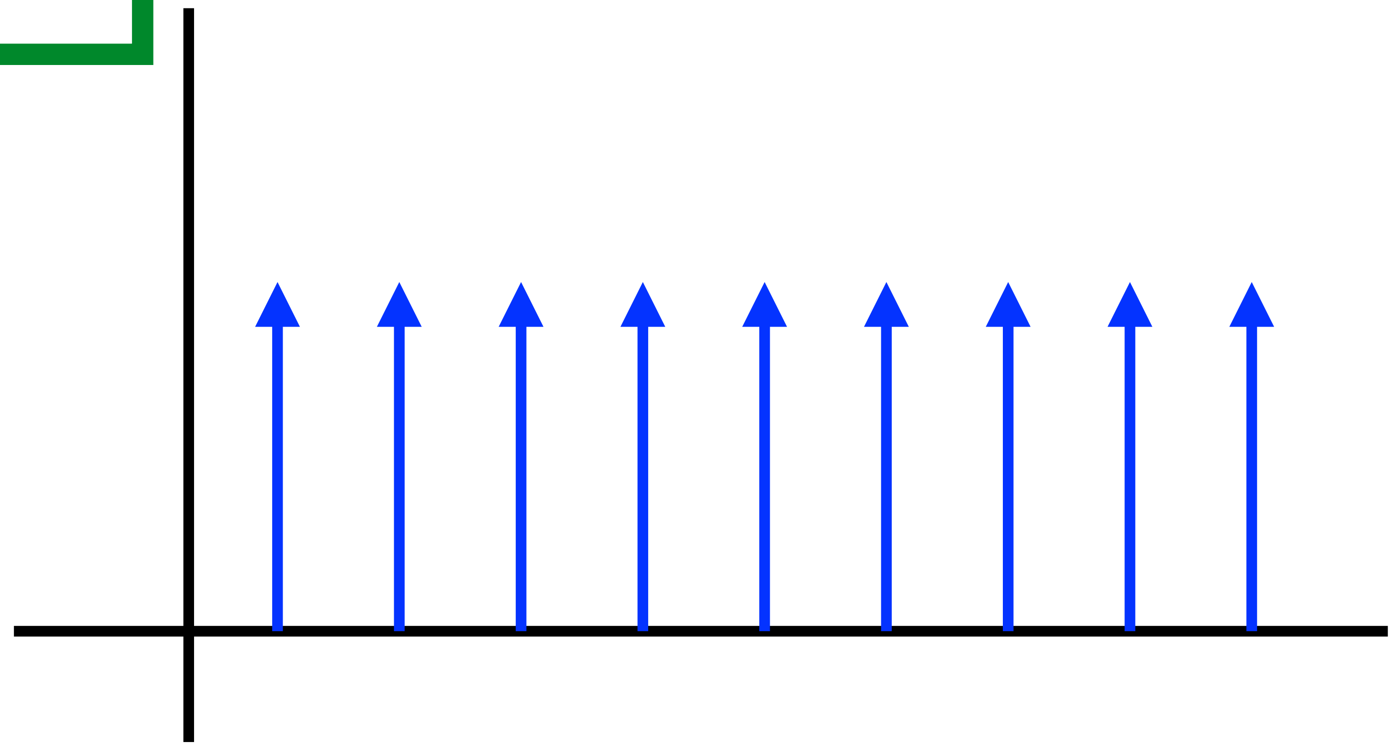

Monte Carlo Estimator in Fourier Domain

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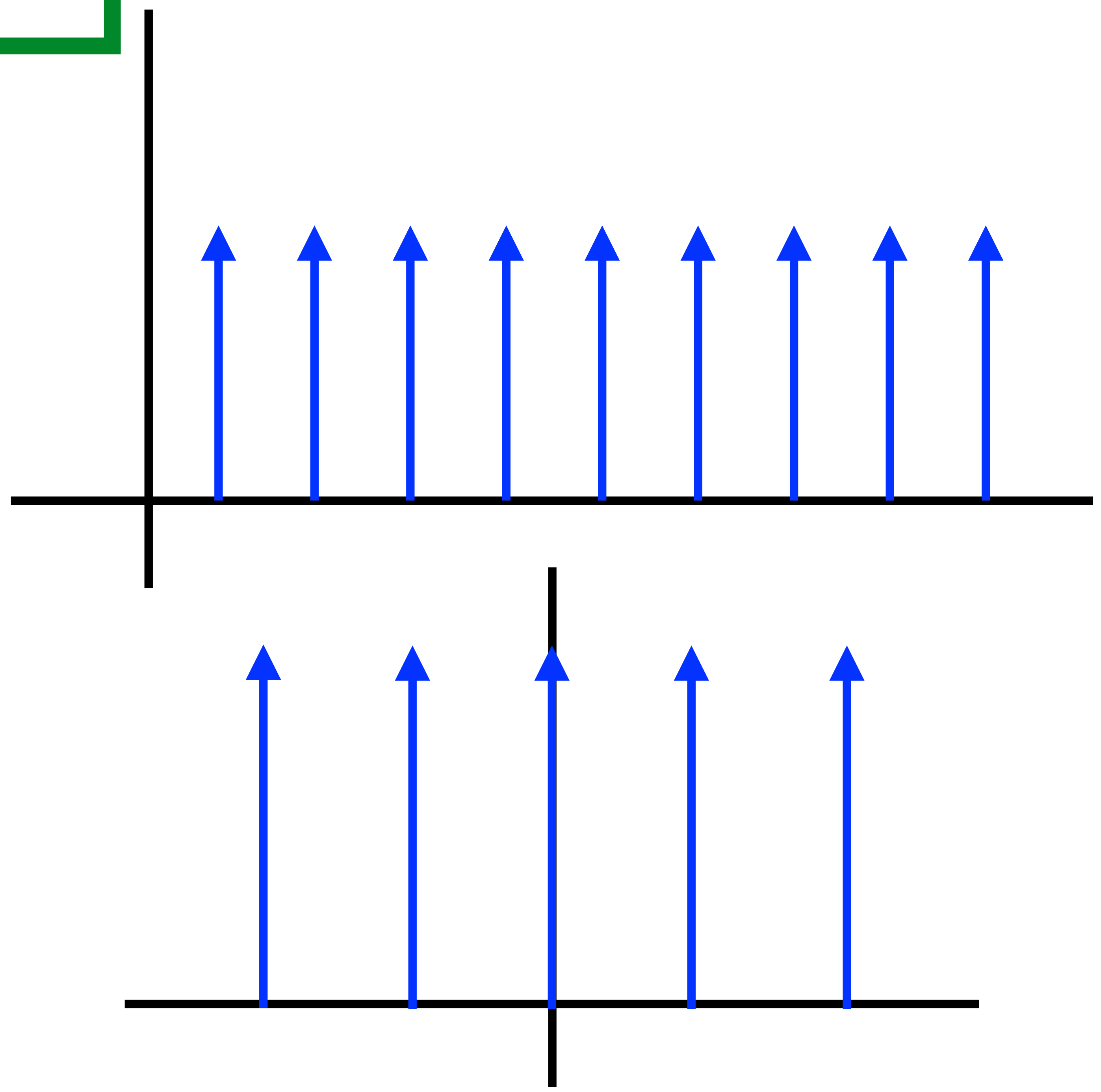
$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$

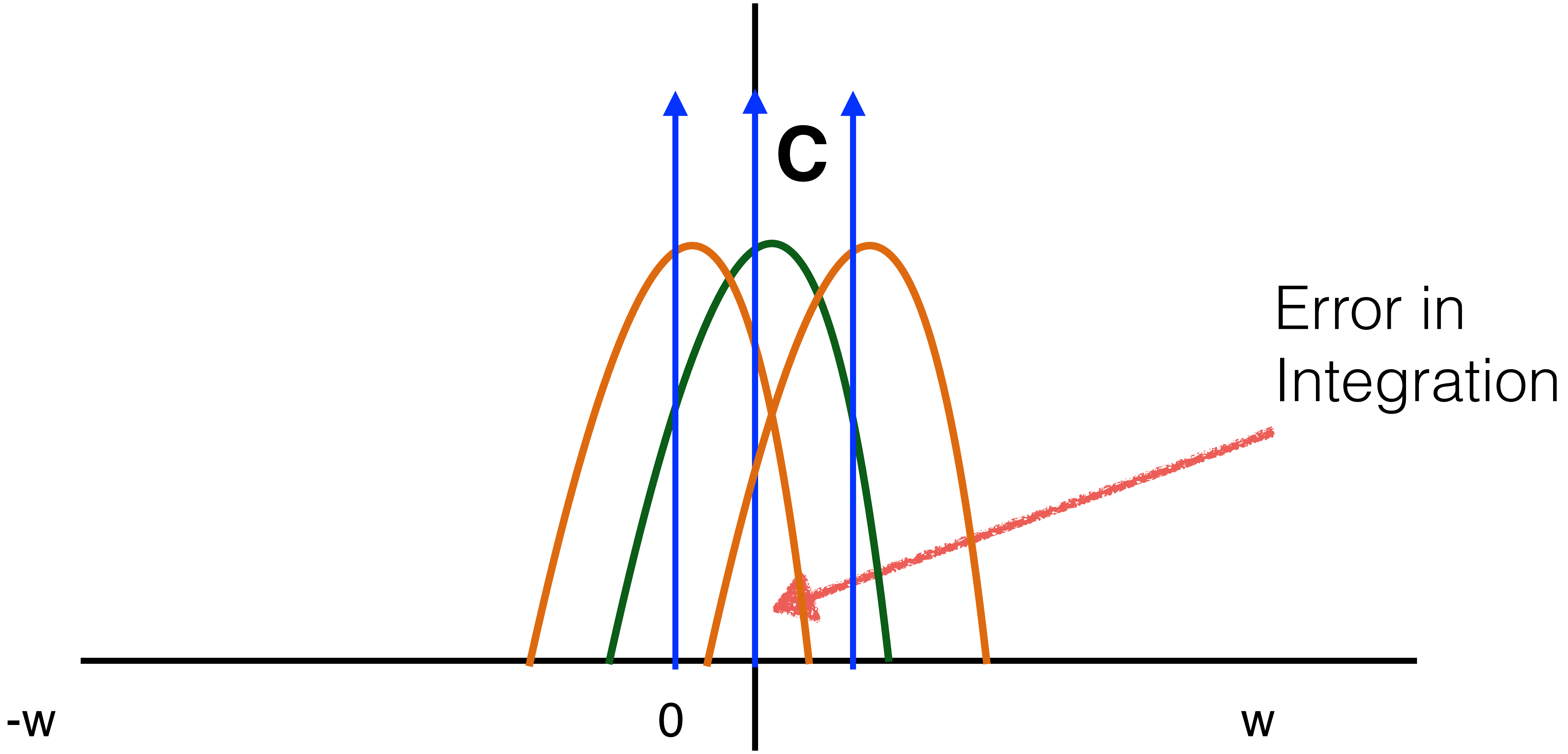
$$\hat{\mathbf{S}}(\omega) = \frac{1}{N}\sum_{k=1}^{N}e^{-i2\pi\omega x_k}$$



How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



Fredo Durand [2011]

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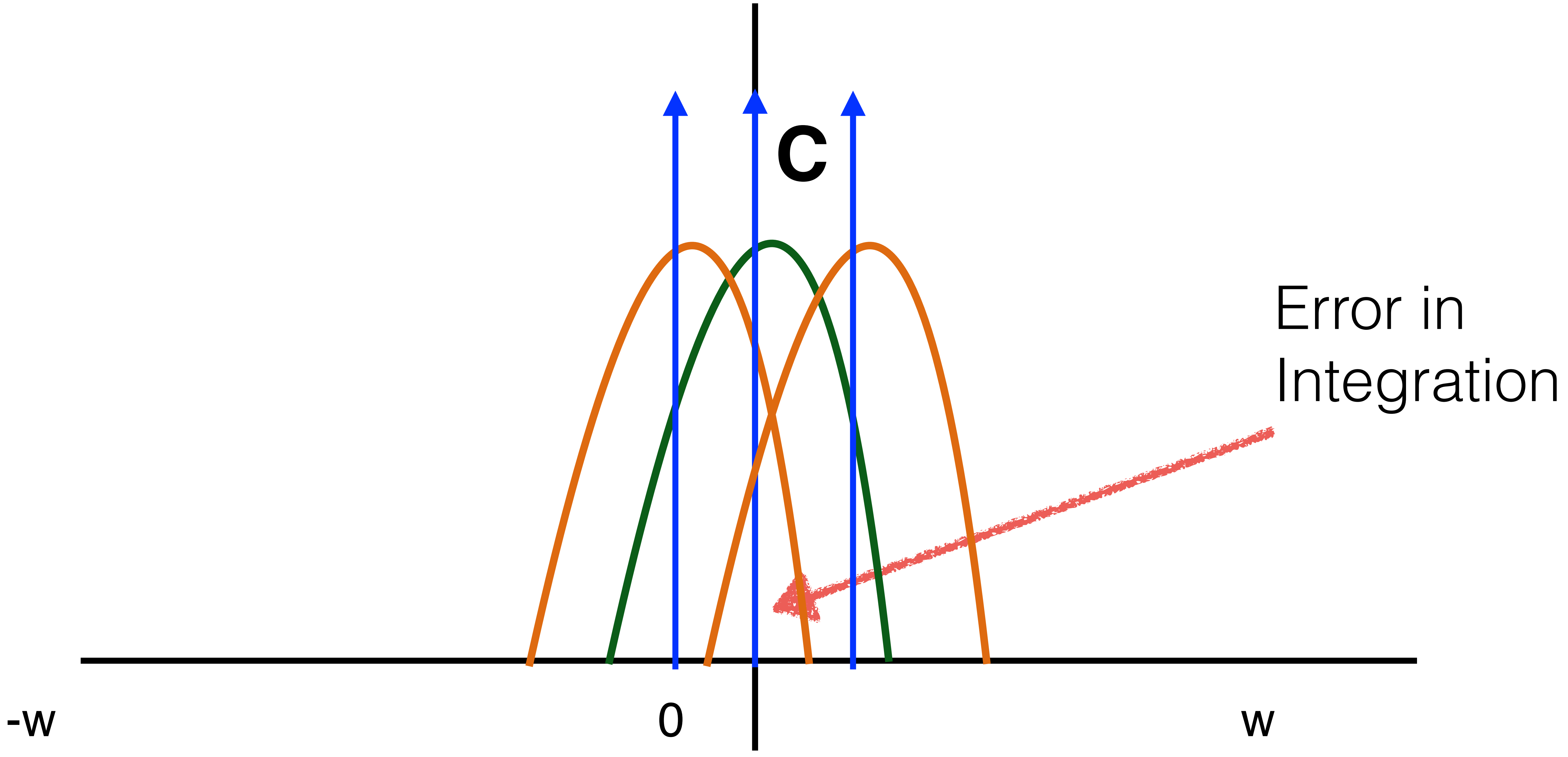

Fredo Durand [2011]

Error in Spatial Domain

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$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

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Monte Carlo Estimator

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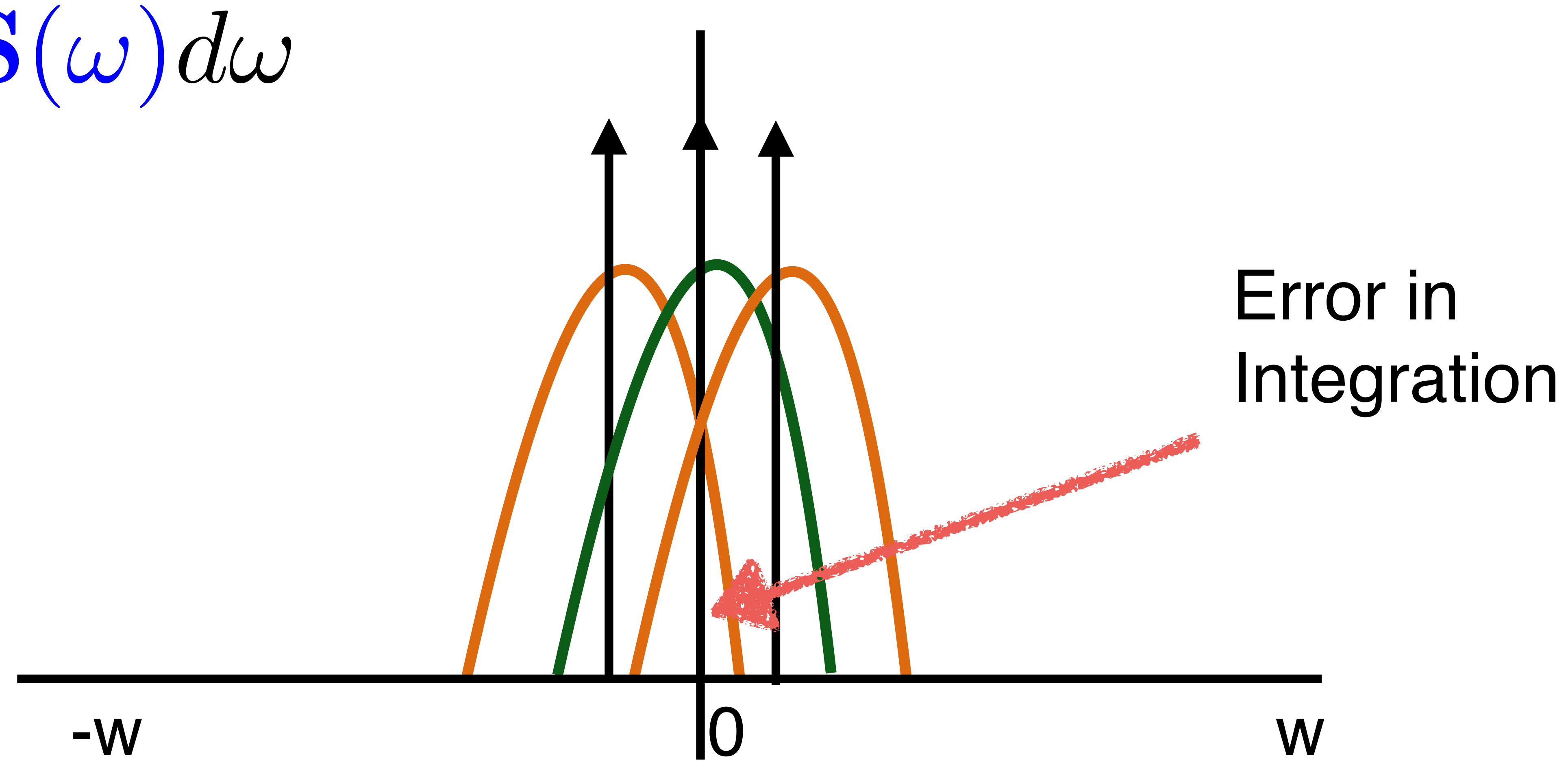

Fredo Durand [2011]

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Properties of Error

- Bias
- Variance

Properties of Error

- Bias: Expected value of the Error
- Variance

Properties of Error

- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
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Properties of Error

- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
- Variance: $\mathrm{Var}(I - \mu_N)$

Subr and Kautz [2013]

Bias in the Monte Carlo Estimator

Bias in Fourier Domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Bias in Fourier Domain

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Bias: $$\langle I - \tilde{\mu}_N \rangle$$

Bias in Fourier Domain

Bias: $$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_\Omega \hat{f}^*(\omega) \hat{\mathbf{S}}(\omega) d\omega \right\rangle$$

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Bias in Fourier Domain

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To obtain an unbiased estimator:

Subr and Kautz [2013]

Bias in Fourier Domain

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To obtain an unbiased estimator:

Subr and Kautz [2013]

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = |\langle \hat{\mathbf{S}}(\omega) \rangle|\, e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

Complex form in Amplitude and Phase

Amplitude

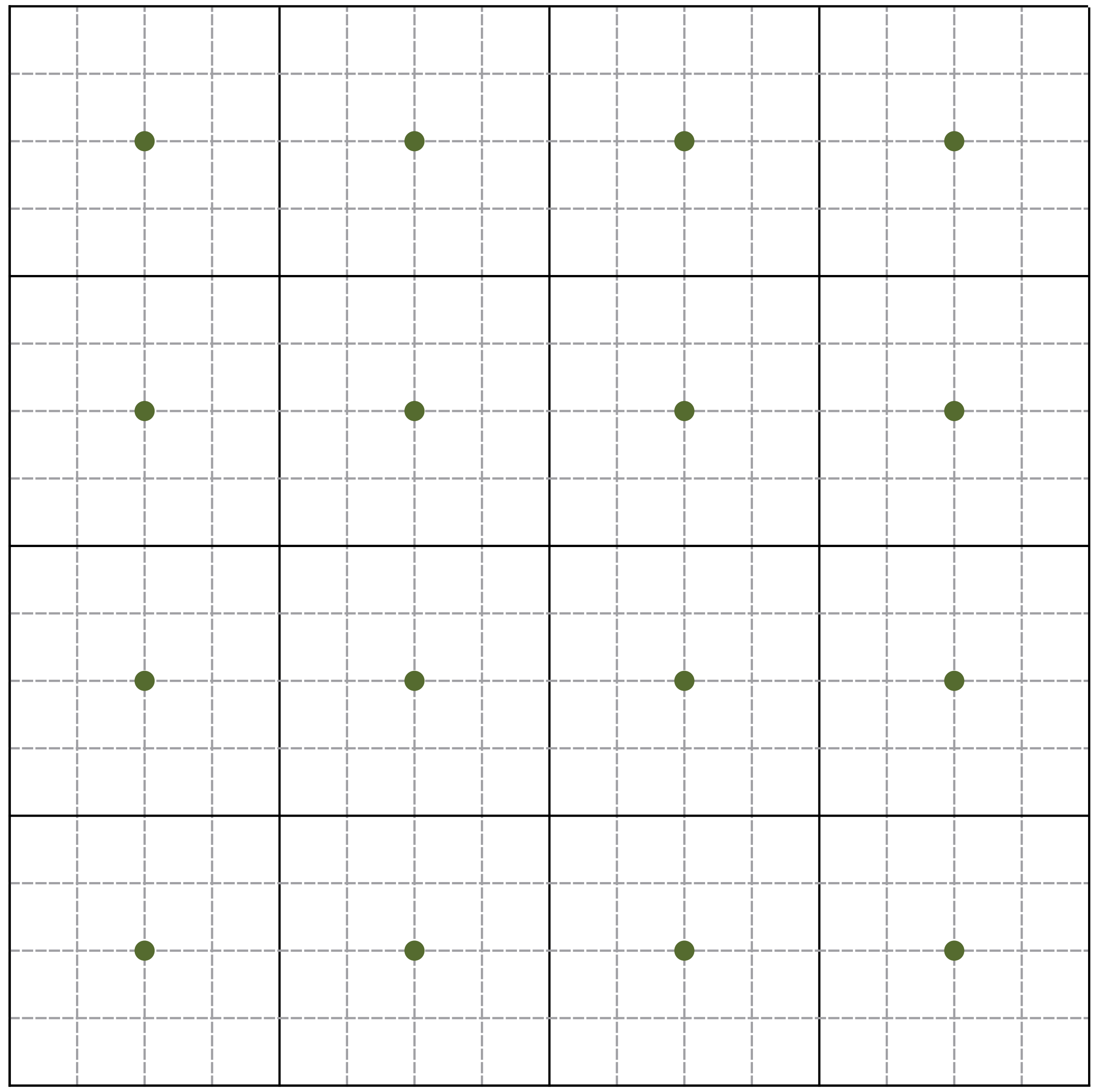
$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

Complex form in Amplitude and Phase

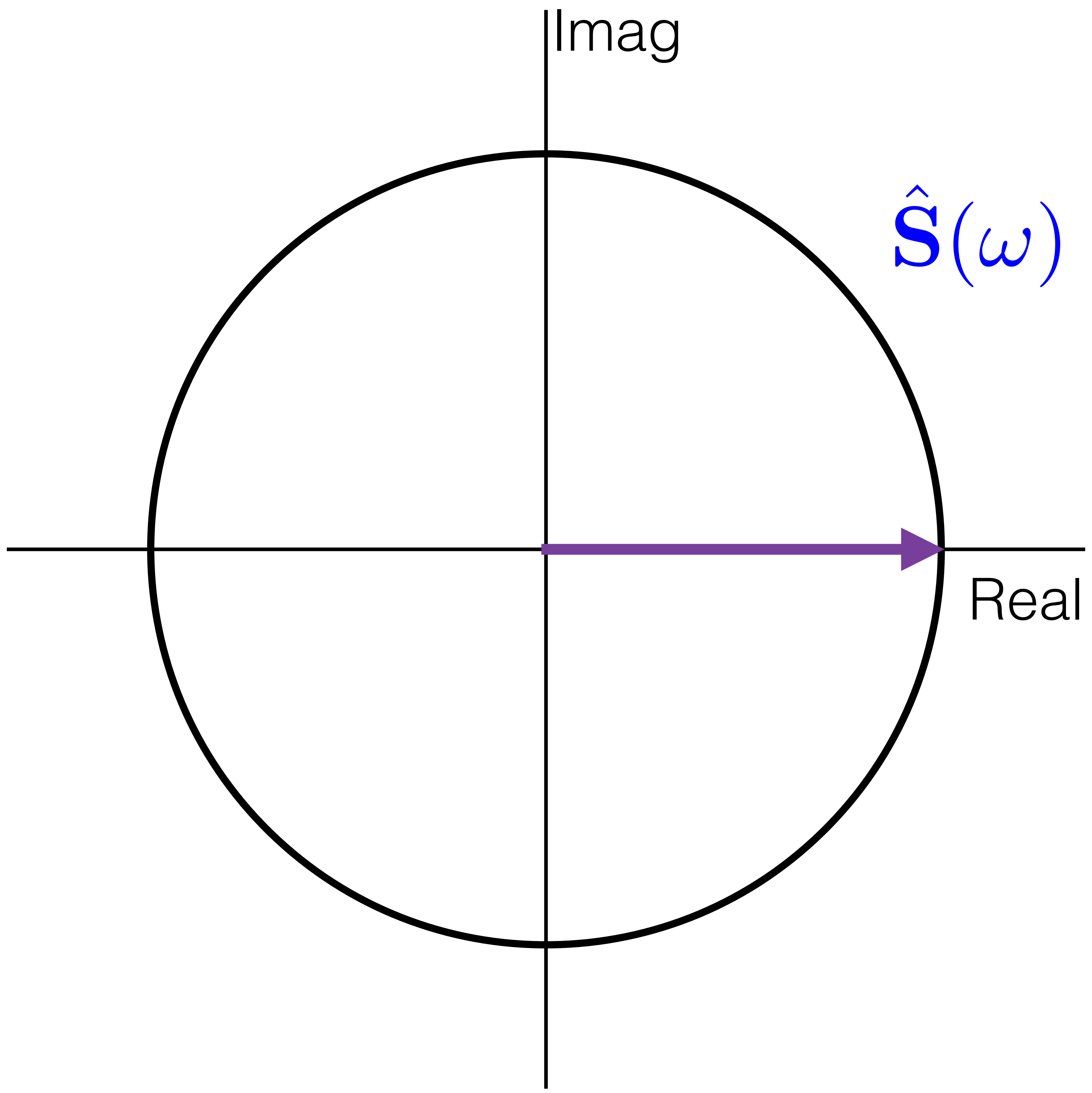
Amplitude Phase

$$\langle\hat{\mathbf{S}}(\omega)\rangle = \underbrace{\boxed{|\langle\hat{\mathbf{S}}(\omega)\rangle|}}_{} e^{-\boxed{\Phi(\langle\hat{\mathbf{S}}(\omega)\rangle)}}$$

Phase change due to Random Shift



For a given frequency ω



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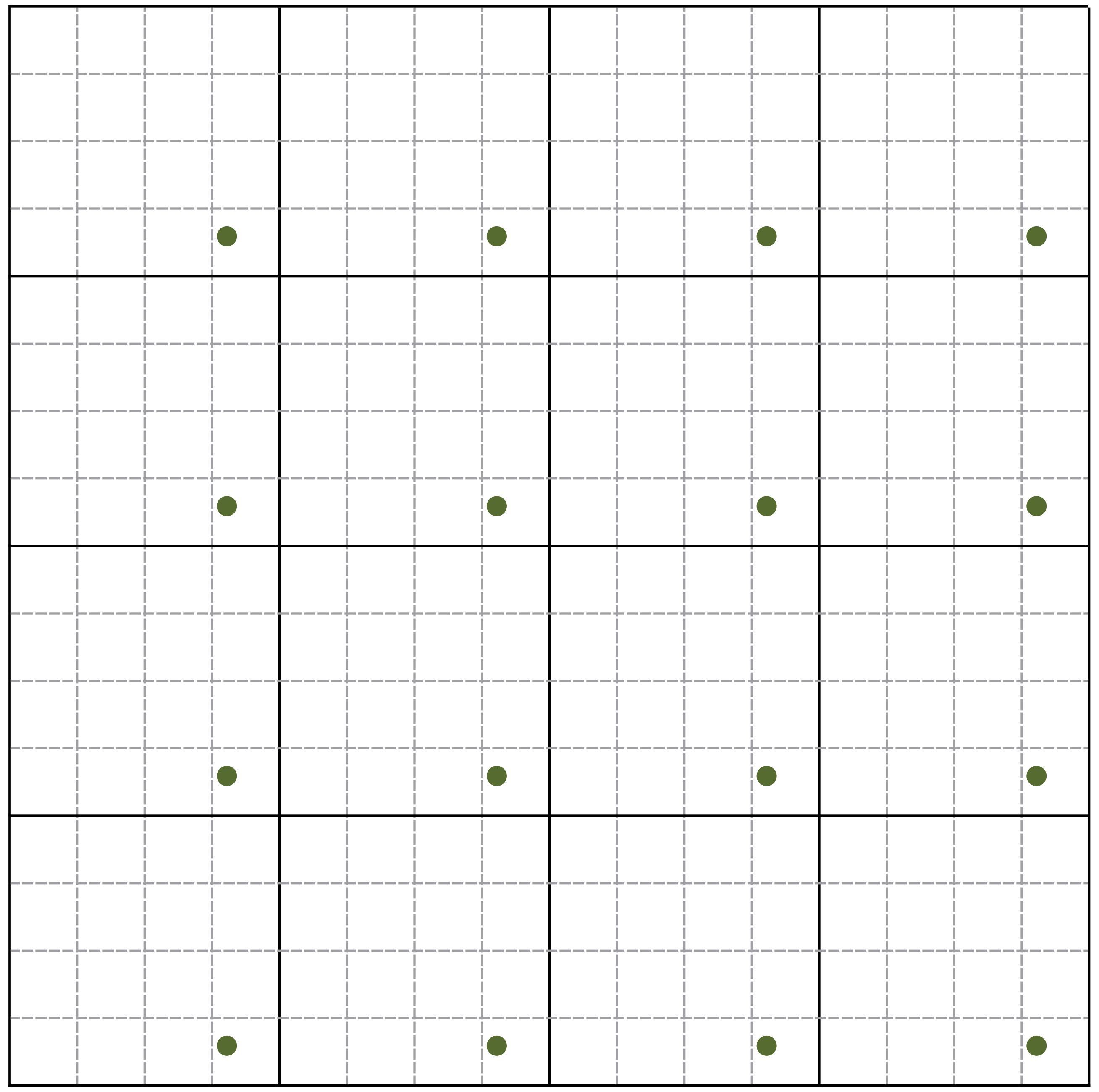

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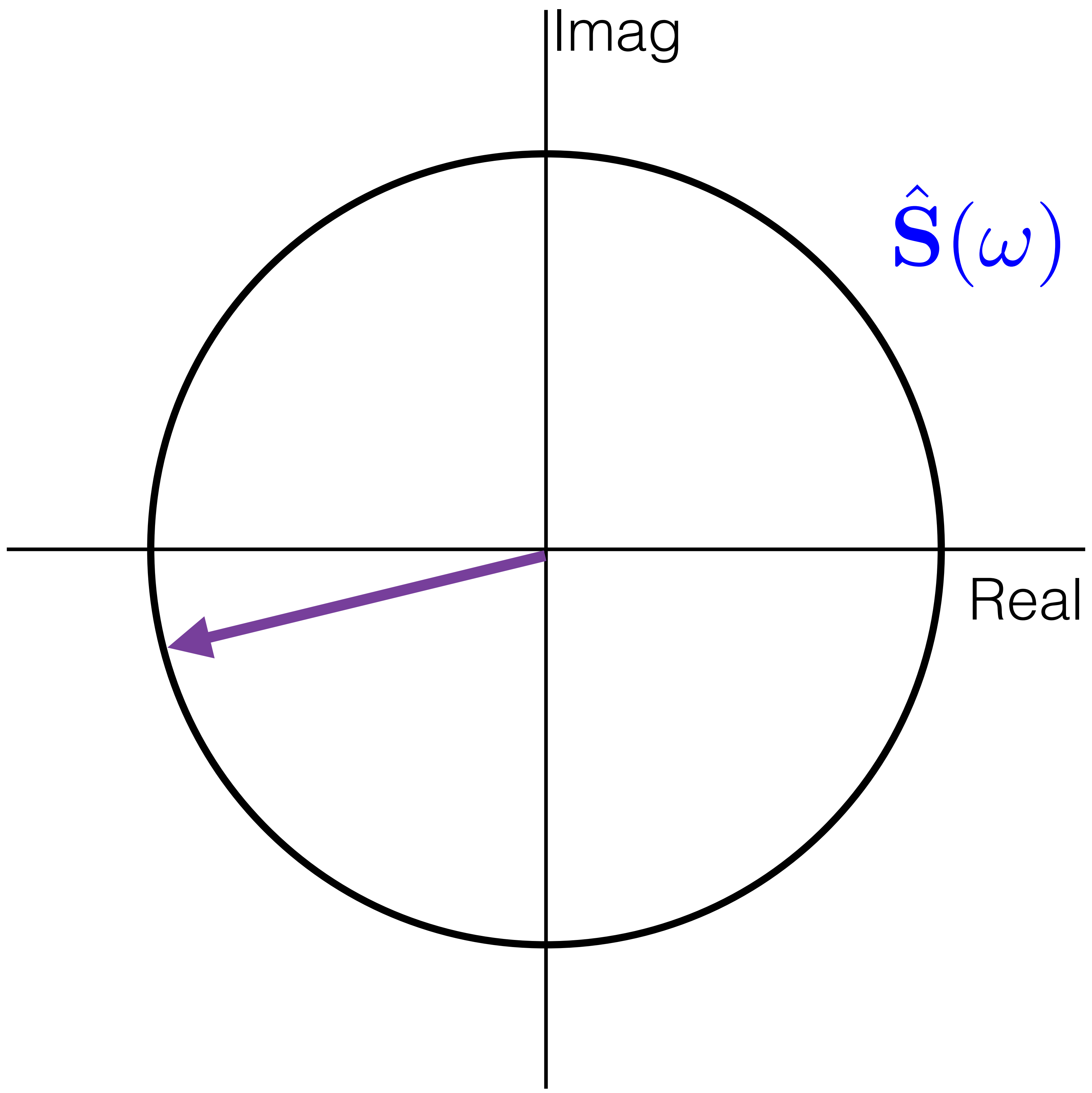

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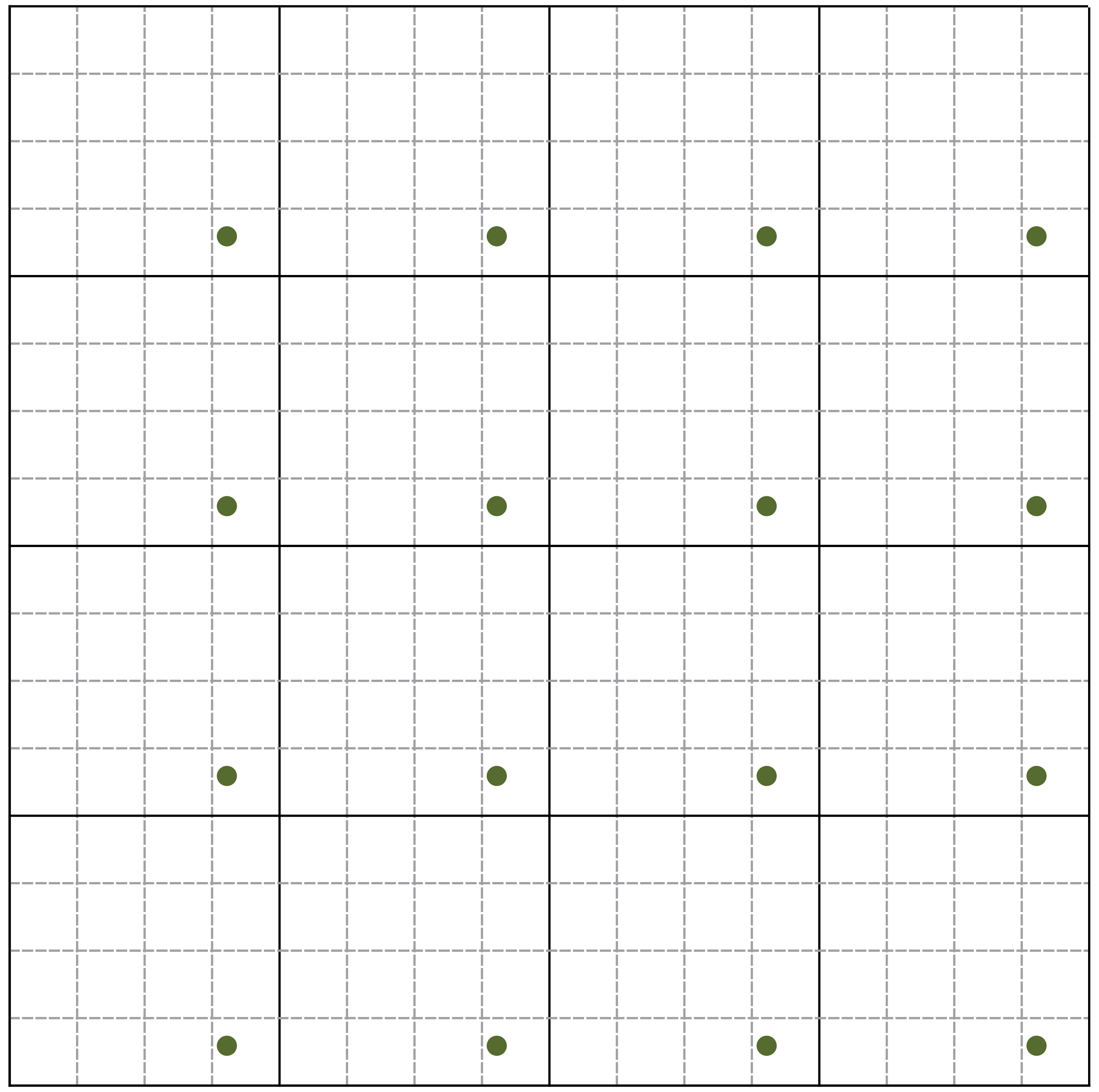
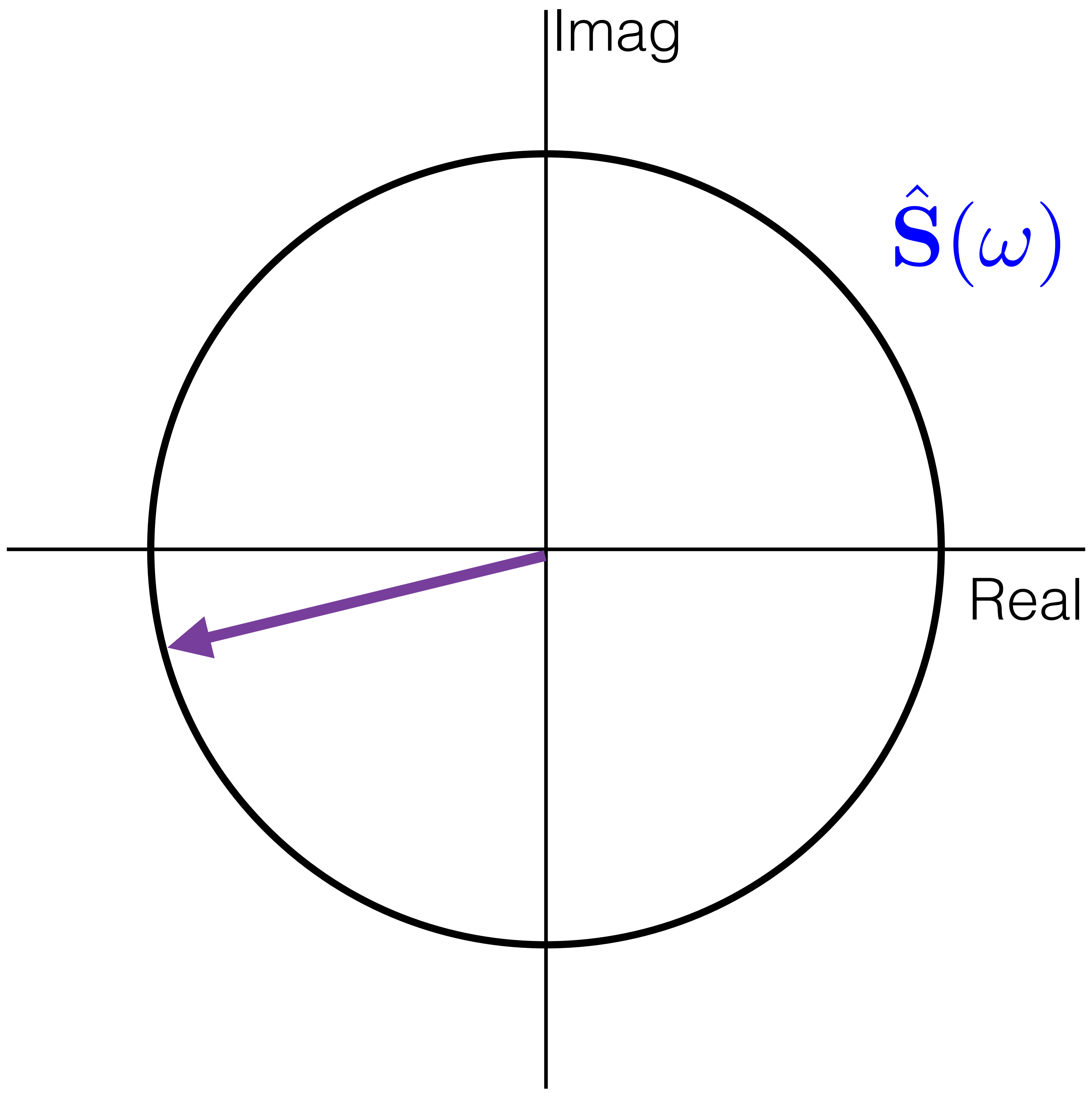


Pauly et al. [2000]
Ramamoorthi et al. [2012]

For a given frequency ω



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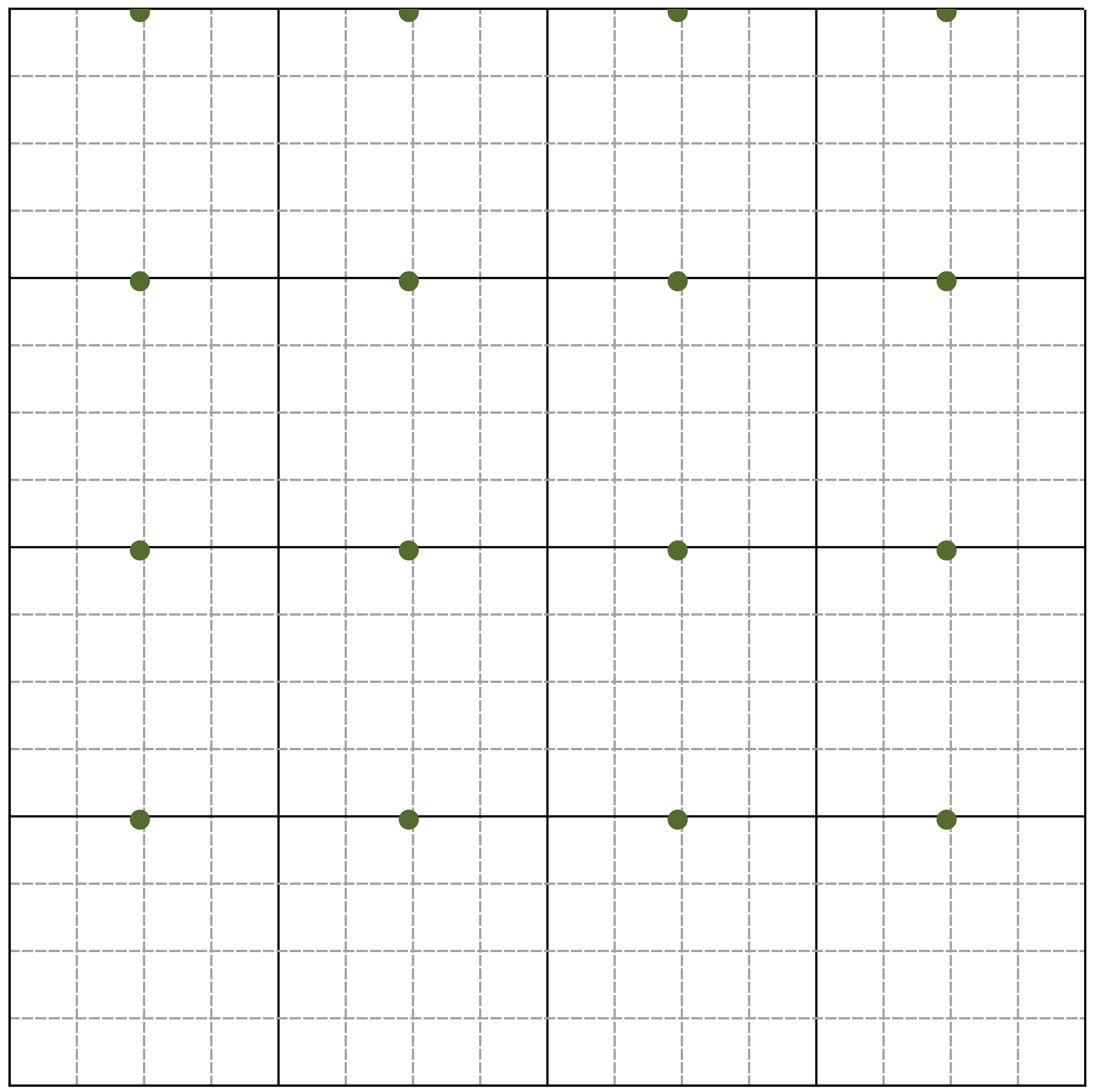
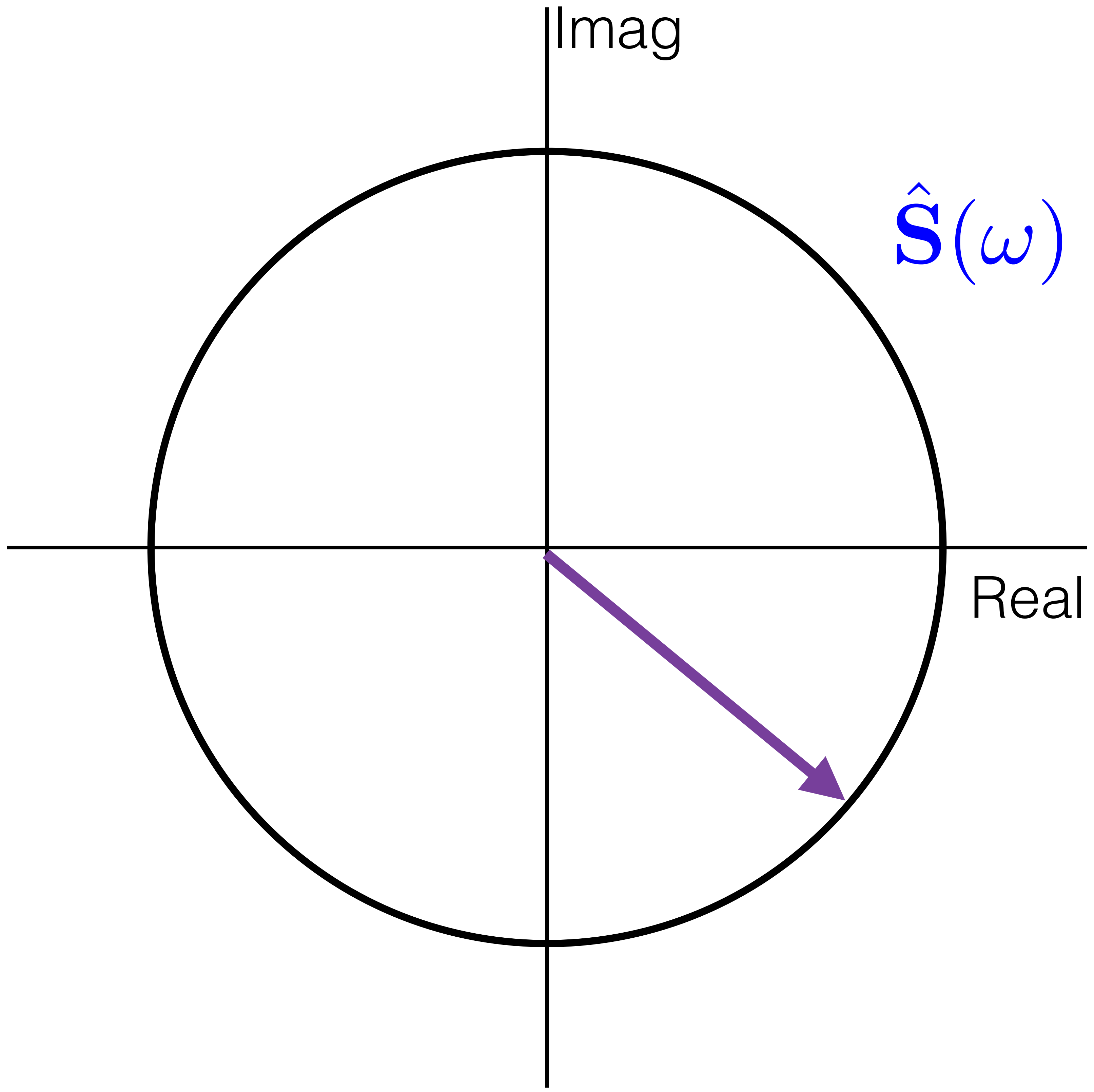
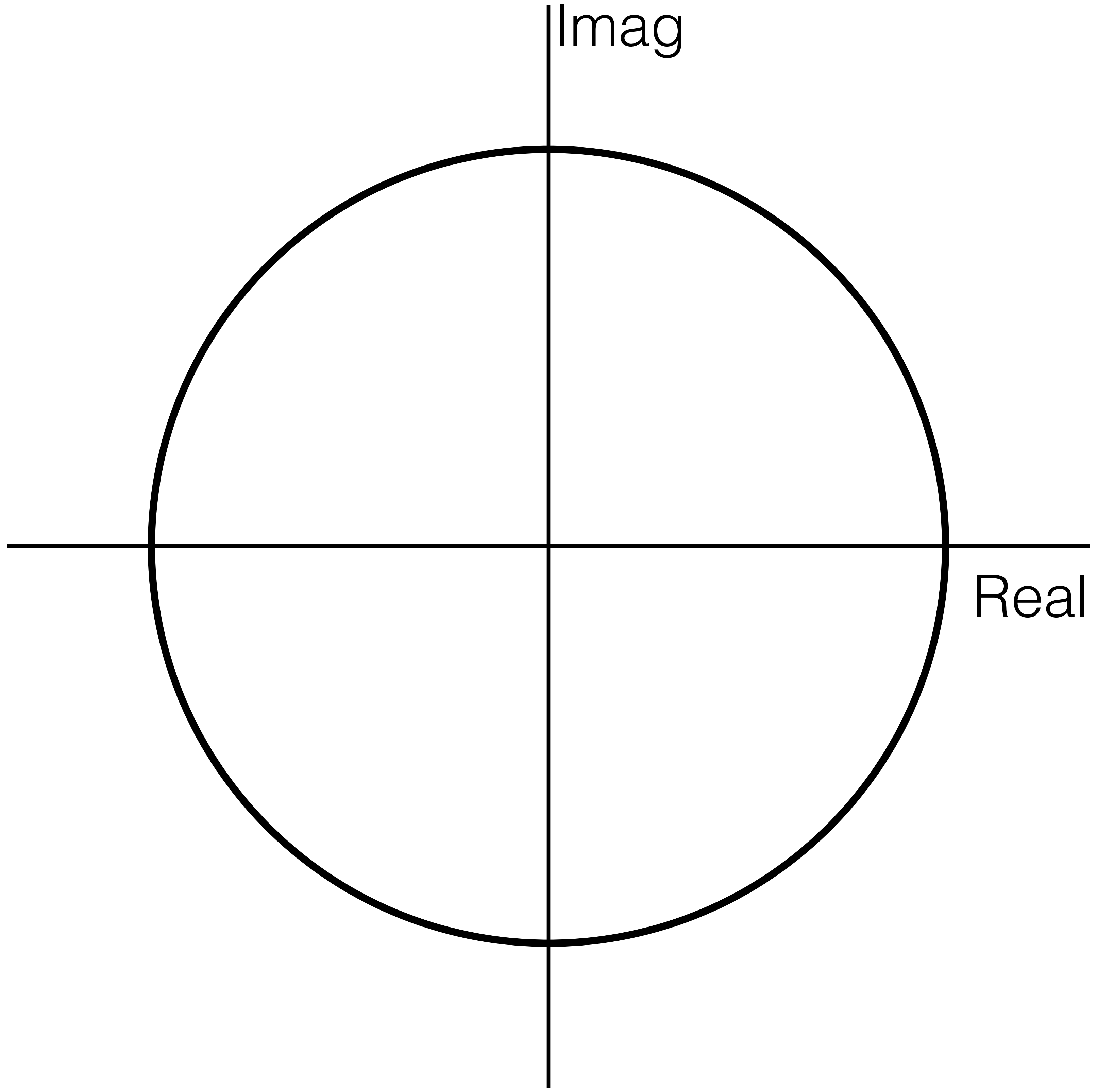




Phase change due to Random Shift

Multiple realizations

For a given frequency ω



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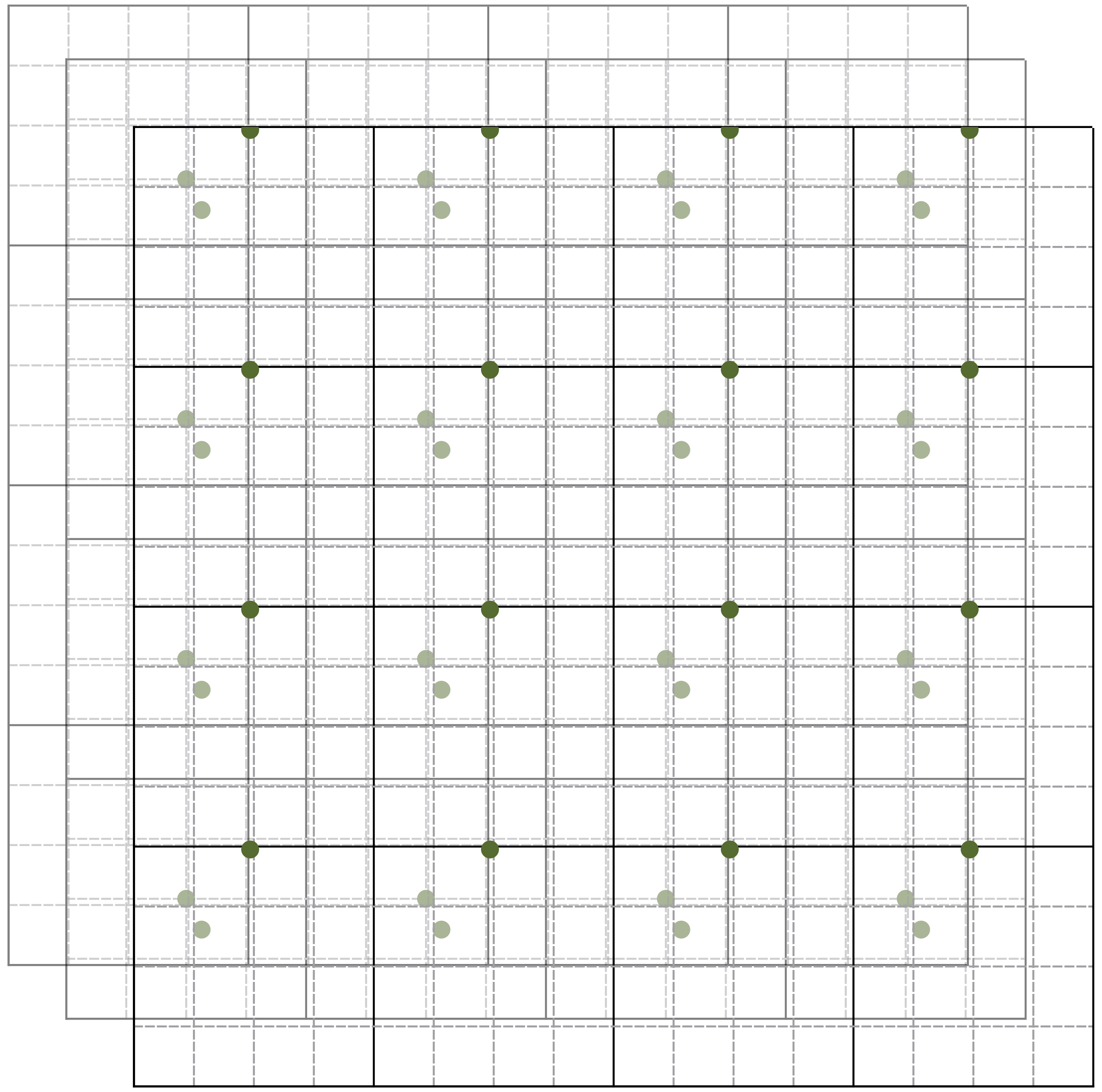

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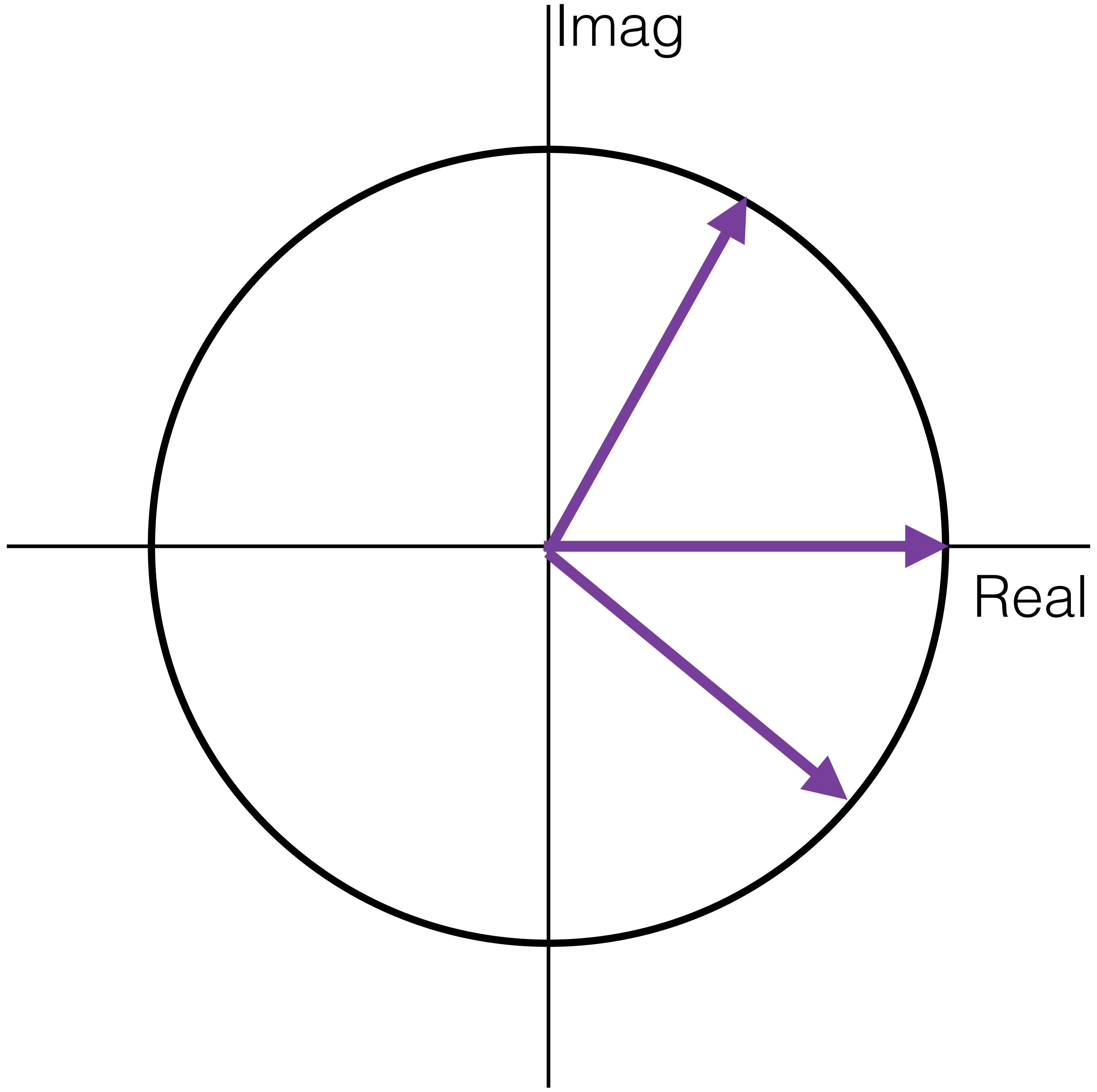

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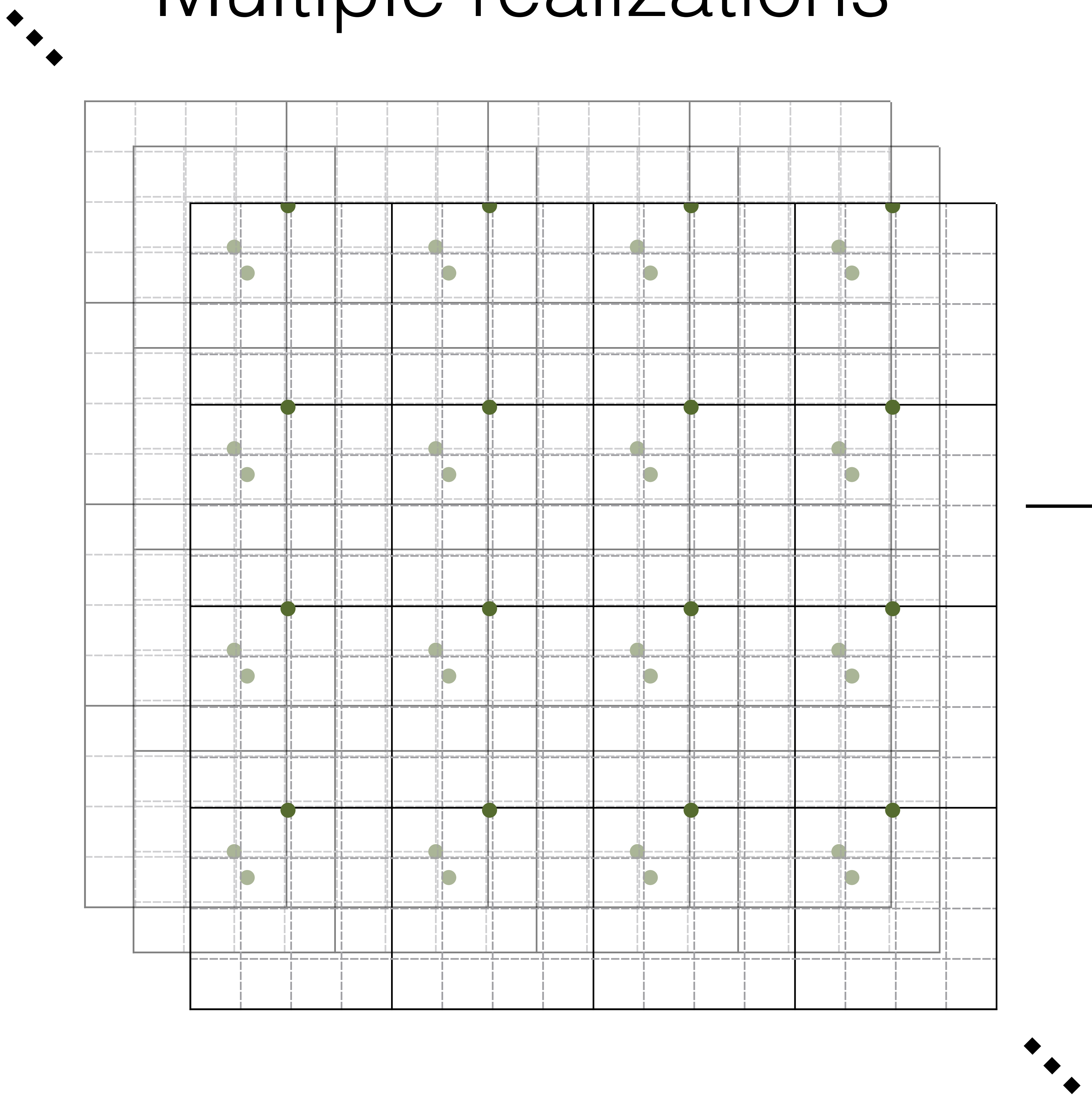

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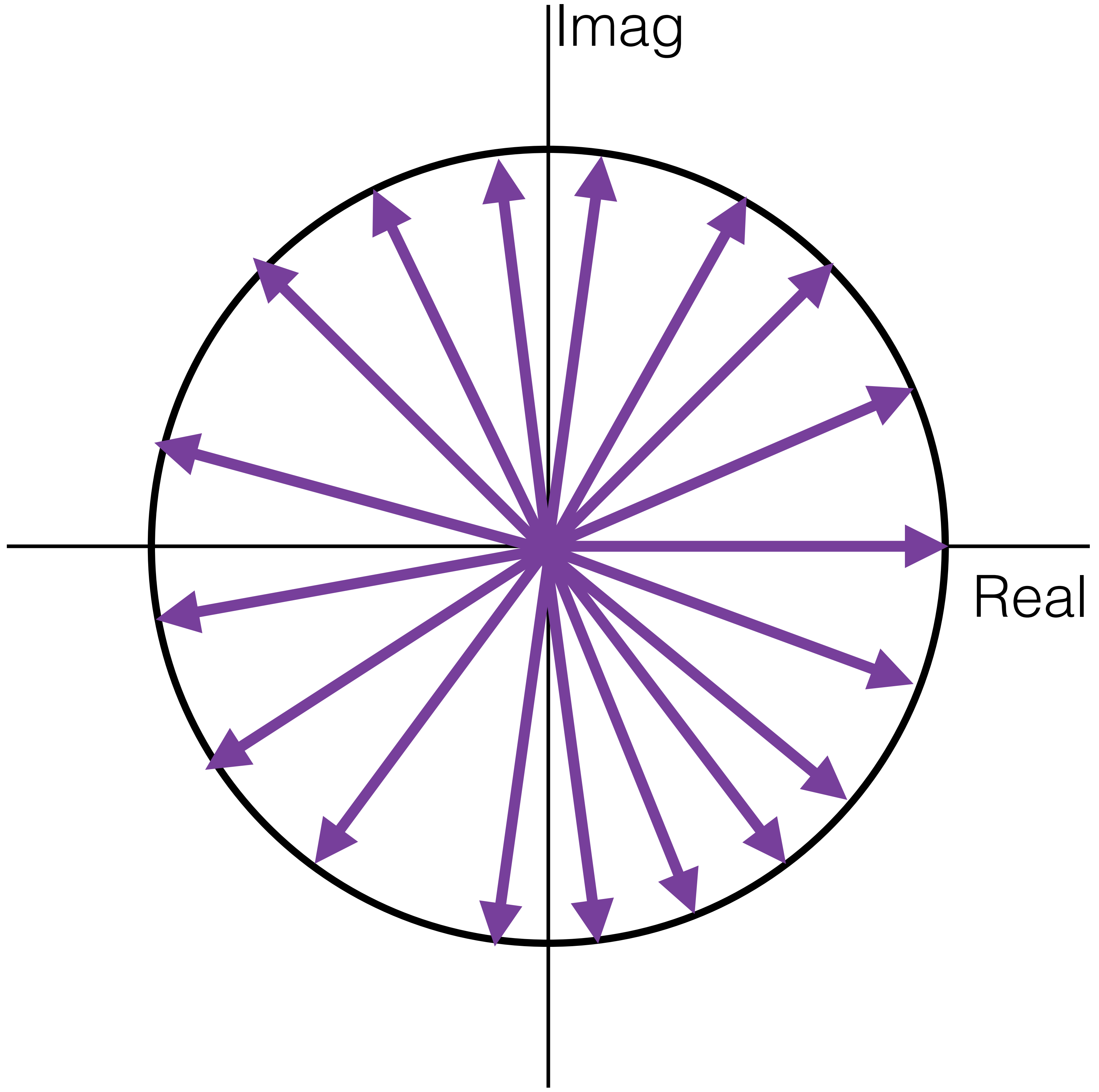

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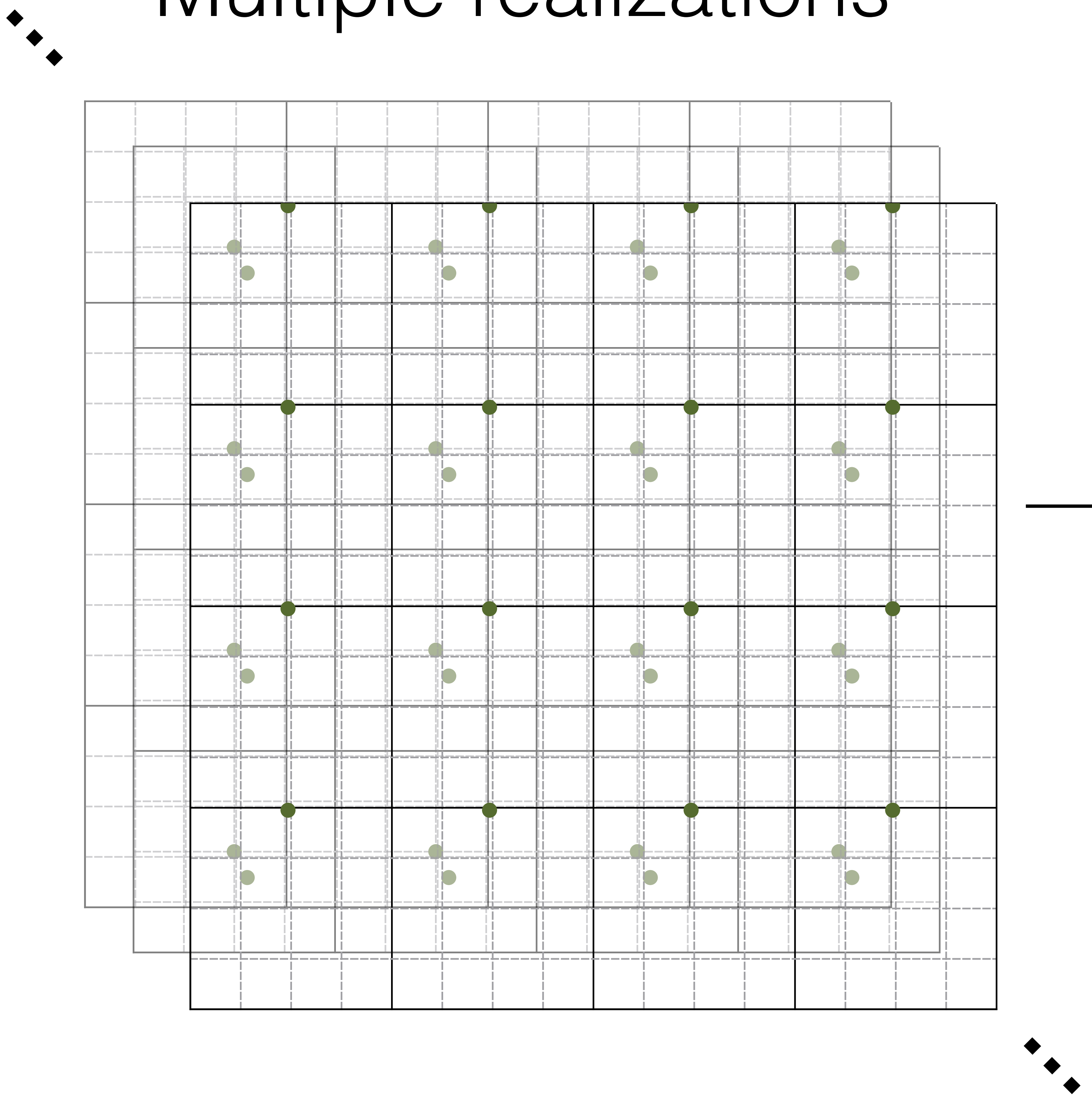

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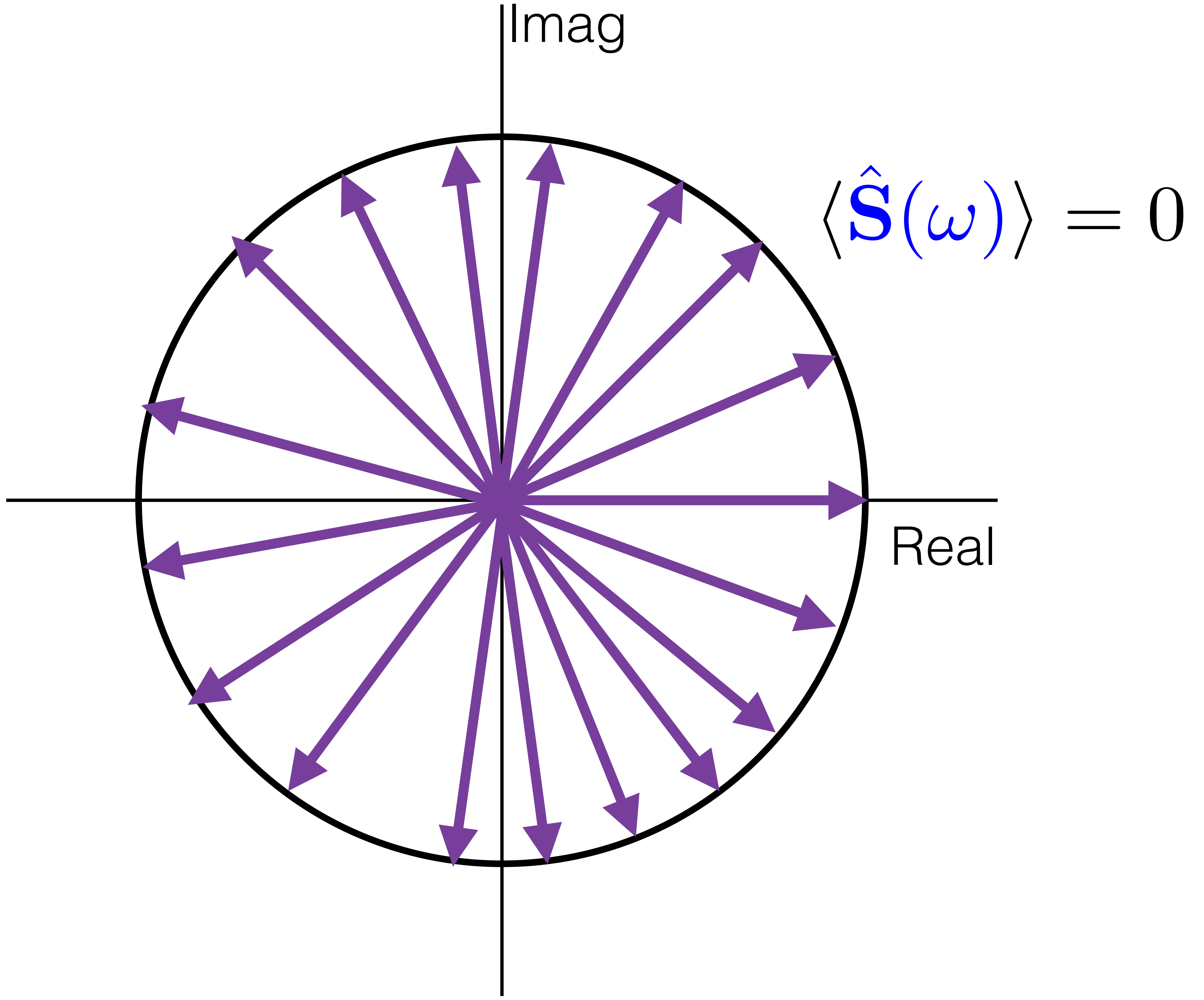

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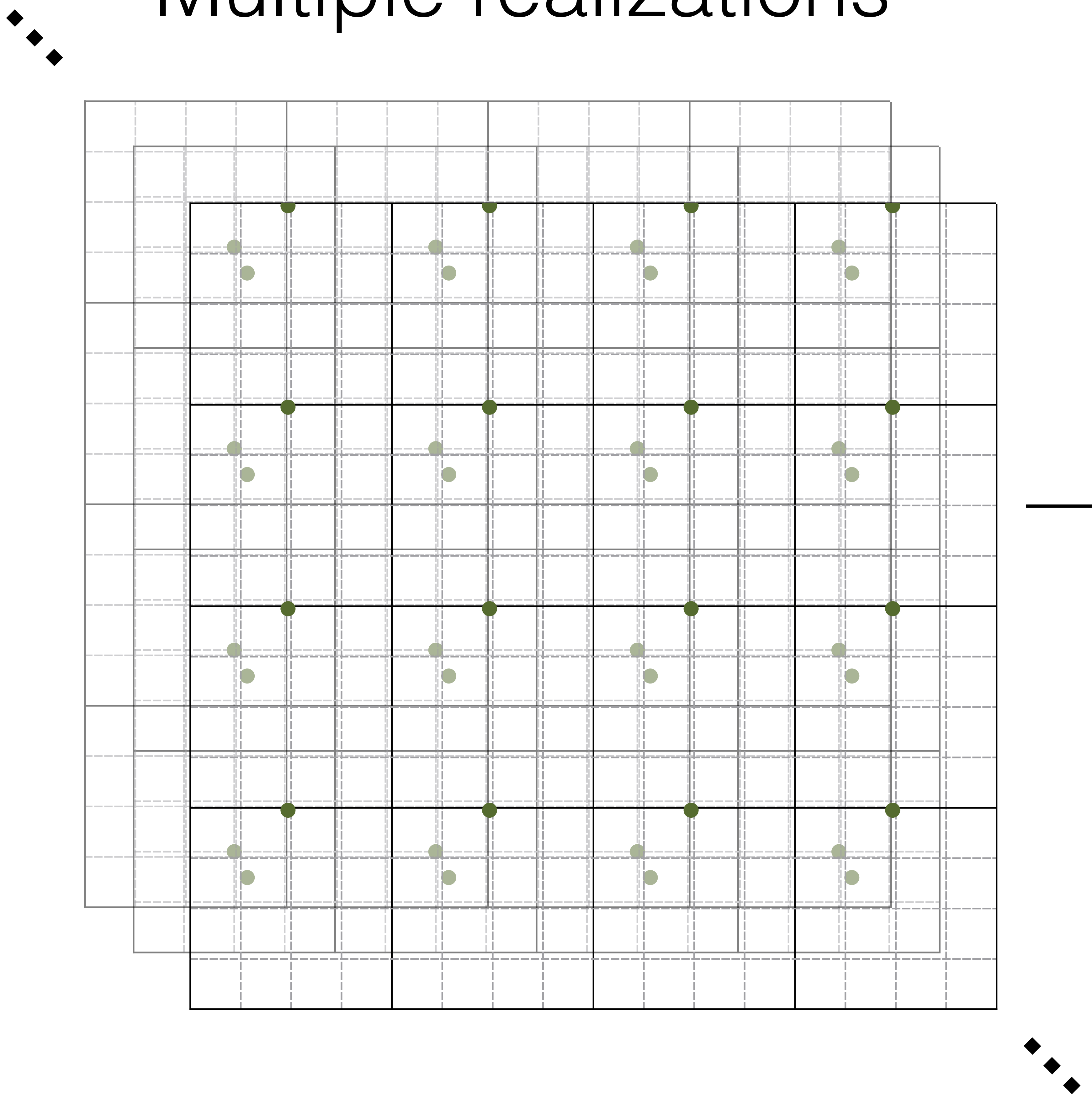

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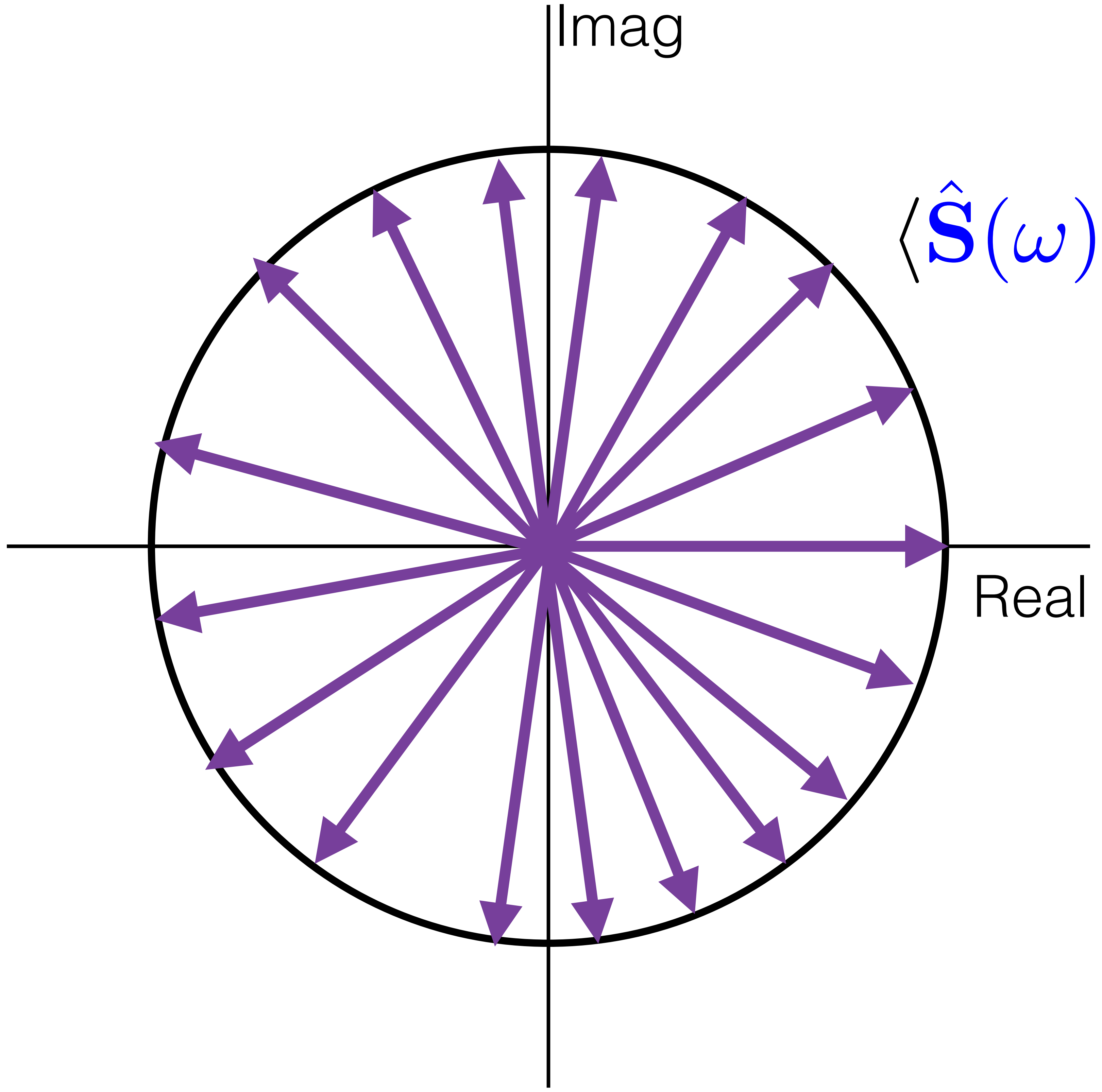

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \;\; \forall \omega \neq 0$$

$$\mathrm{Error} = \mathrm{Bias}^2 + \mathrm{Variance}$$

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- Homogenization allows representation of error only in terms of variance

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- Homogenization allows representation of error only in terms of variance

- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

Variance in the Fourier domain

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Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

where,

$$P_f(\omega) = |\hat{f}^*(\omega)|^2 \quad \text{Power Spectrum}$$

Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

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Subr and Kautz [2013]

Variance in the Fourier domain

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega}$$

Subr and Kautz [2013]

This is a general form, both for homogenised as well as non-homogenised sampling patterns

Variance in the Fourier domain

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Fredo Durand [2011]

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Pilleboue et al. [2015]

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Variance using Homogenized Samples

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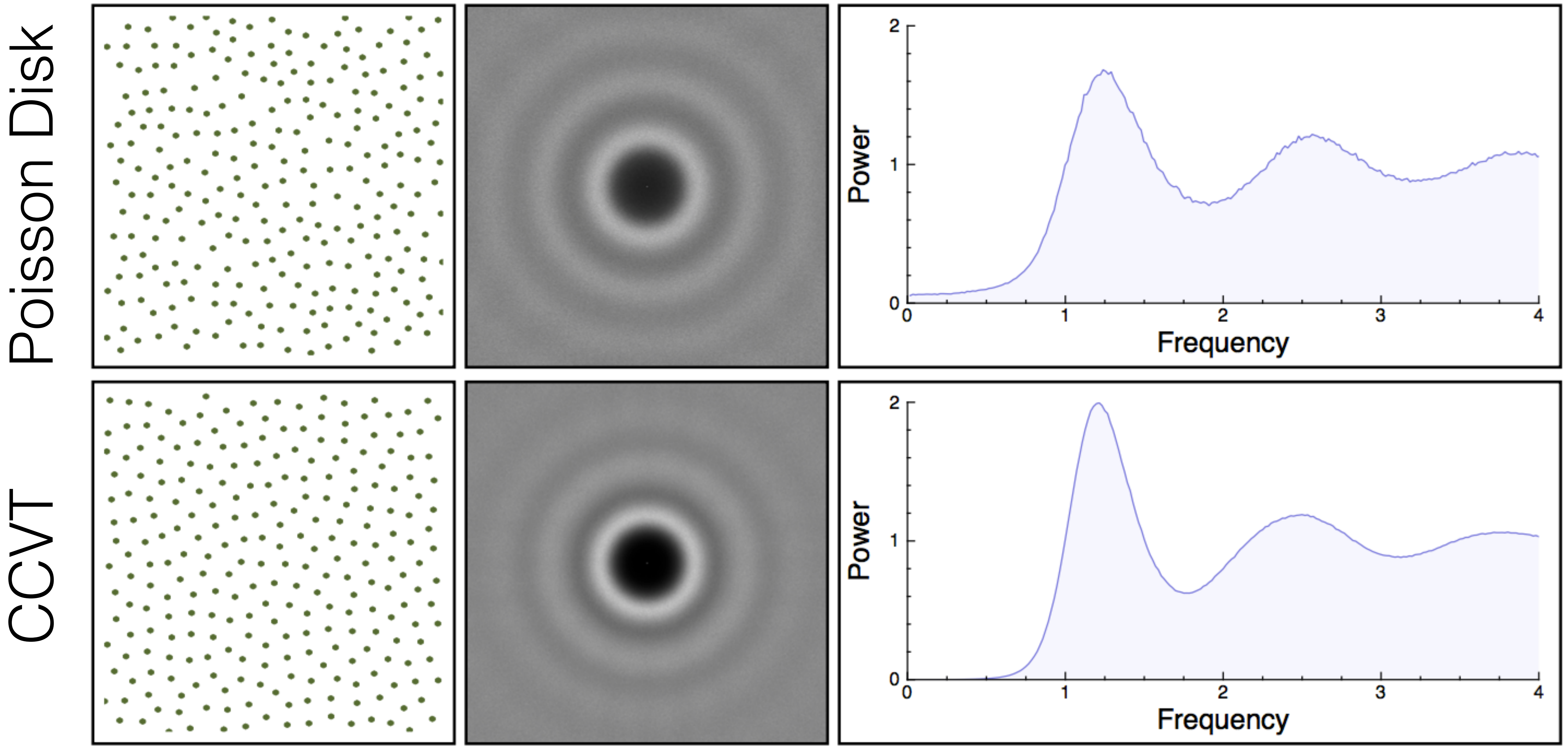
Variance in terms of n-dimensional Power Spectra

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \langle P_S(\omega) \rangle \, d\omega$$



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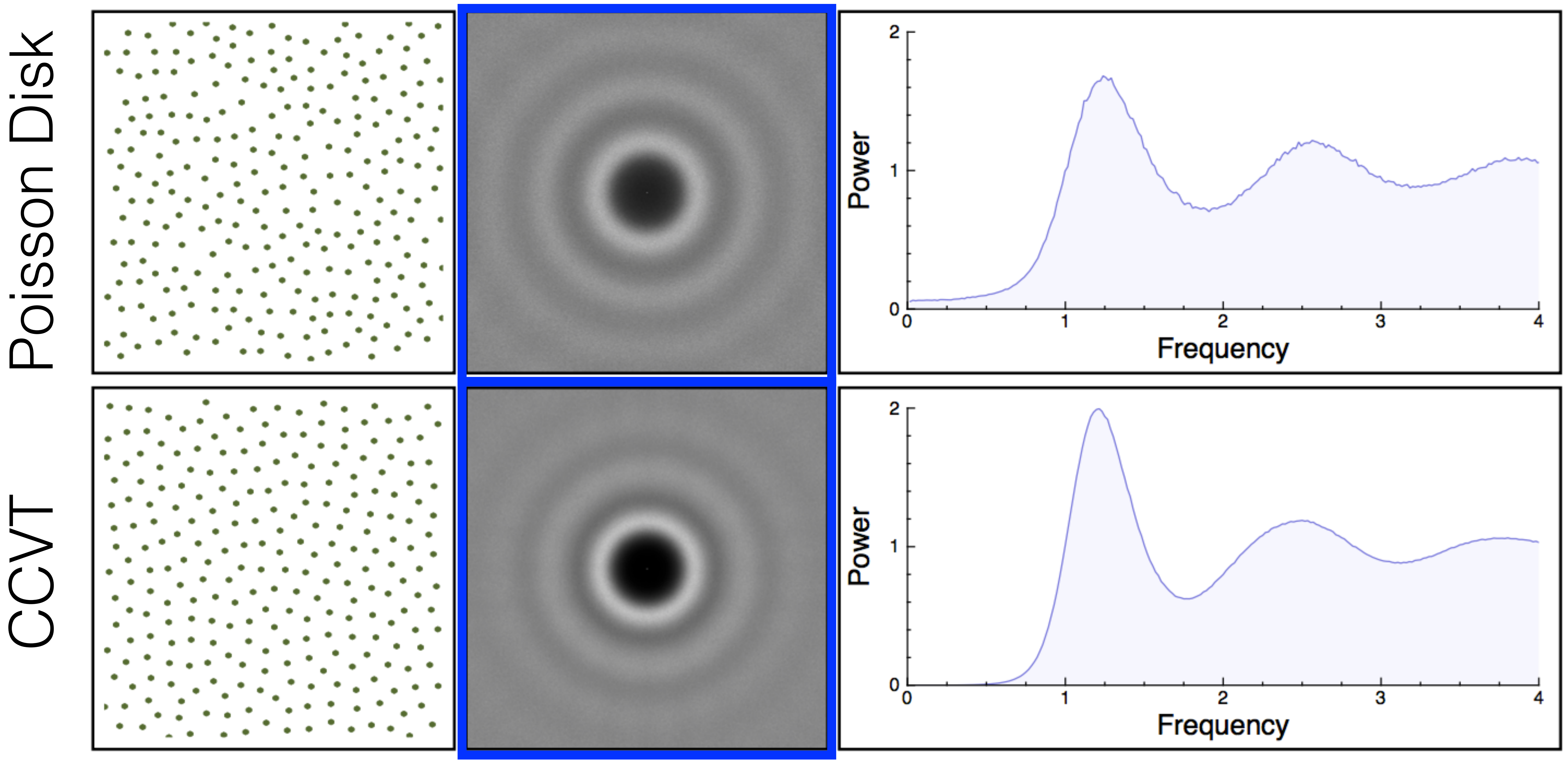

Variance in the Polar Coordinates

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$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho \mathbf{n})\, \langle P_{\mathbf{S}}(\rho \mathbf{n}) \rangle\, d\mathbf{n}\, d\rho$$

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Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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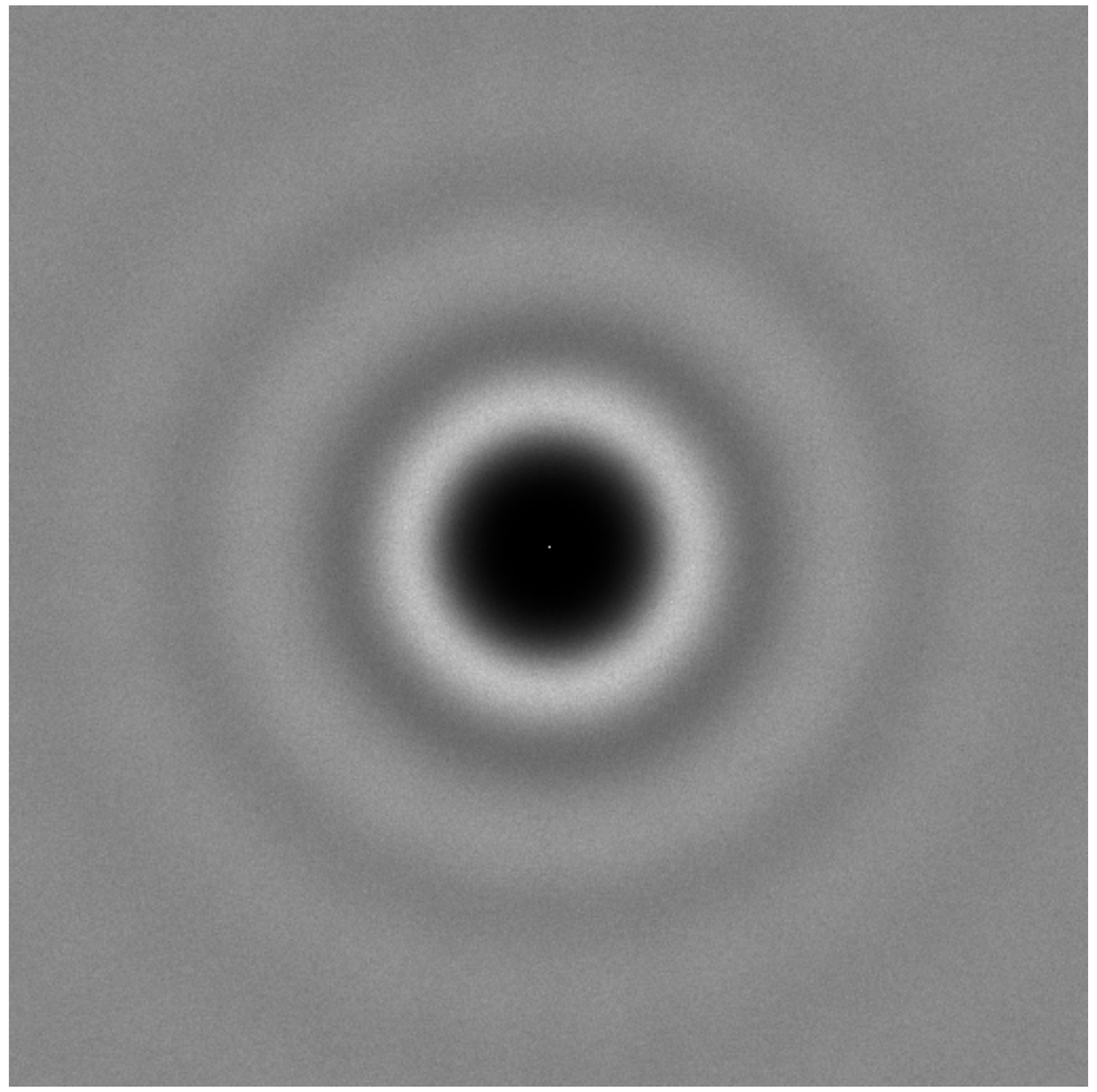

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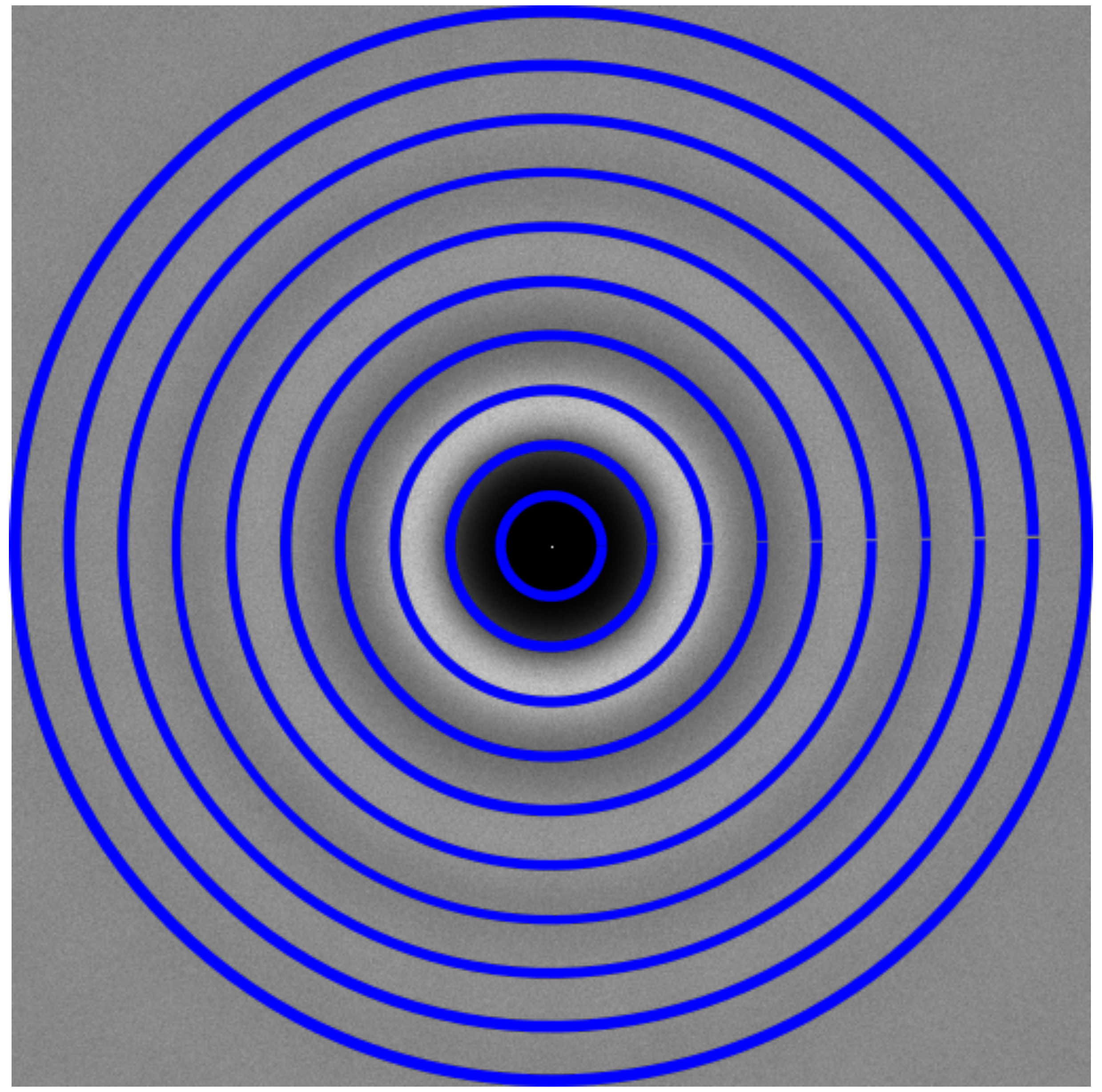

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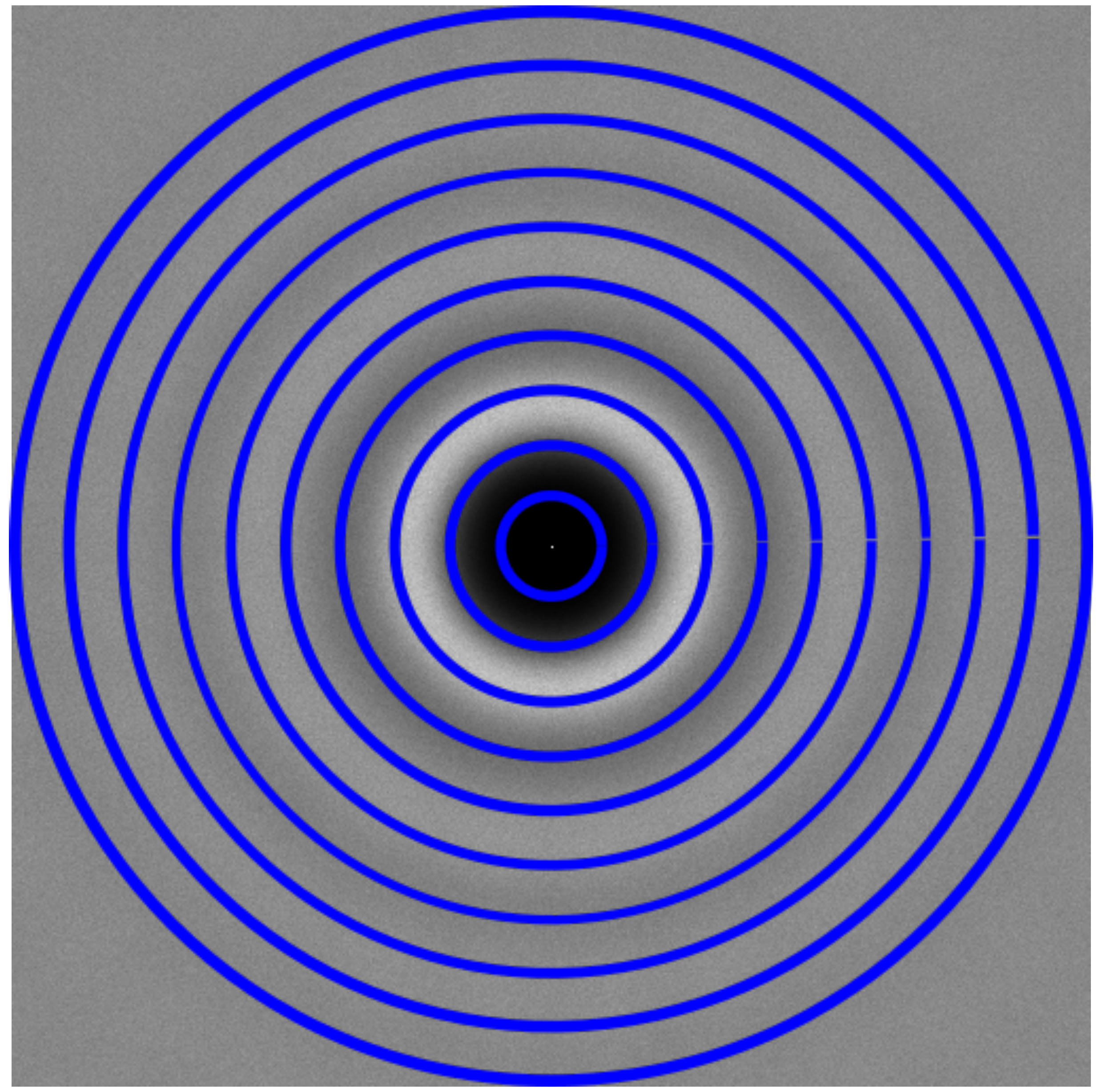

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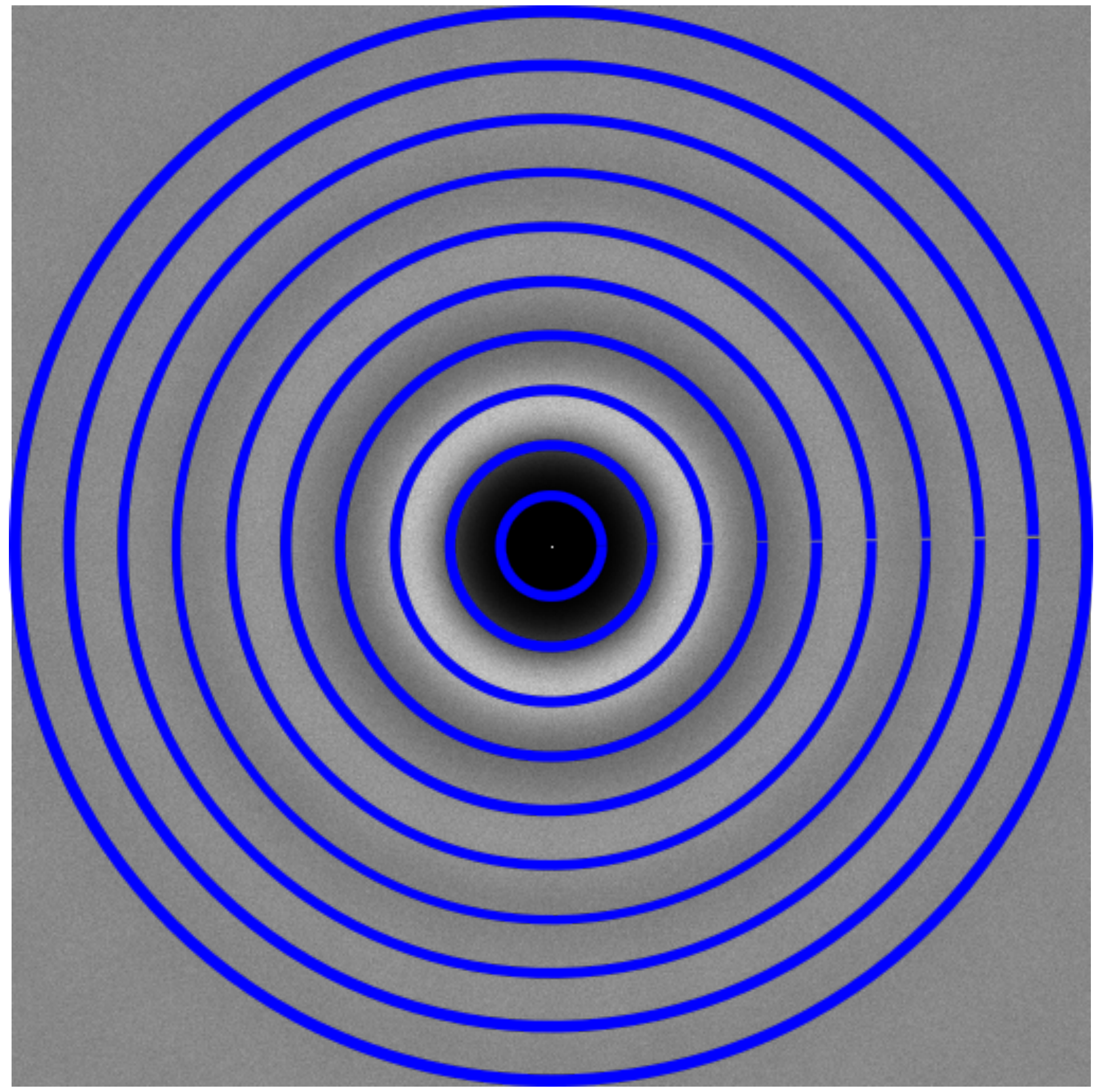
For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$



Variance for Isotropic Power Spectra

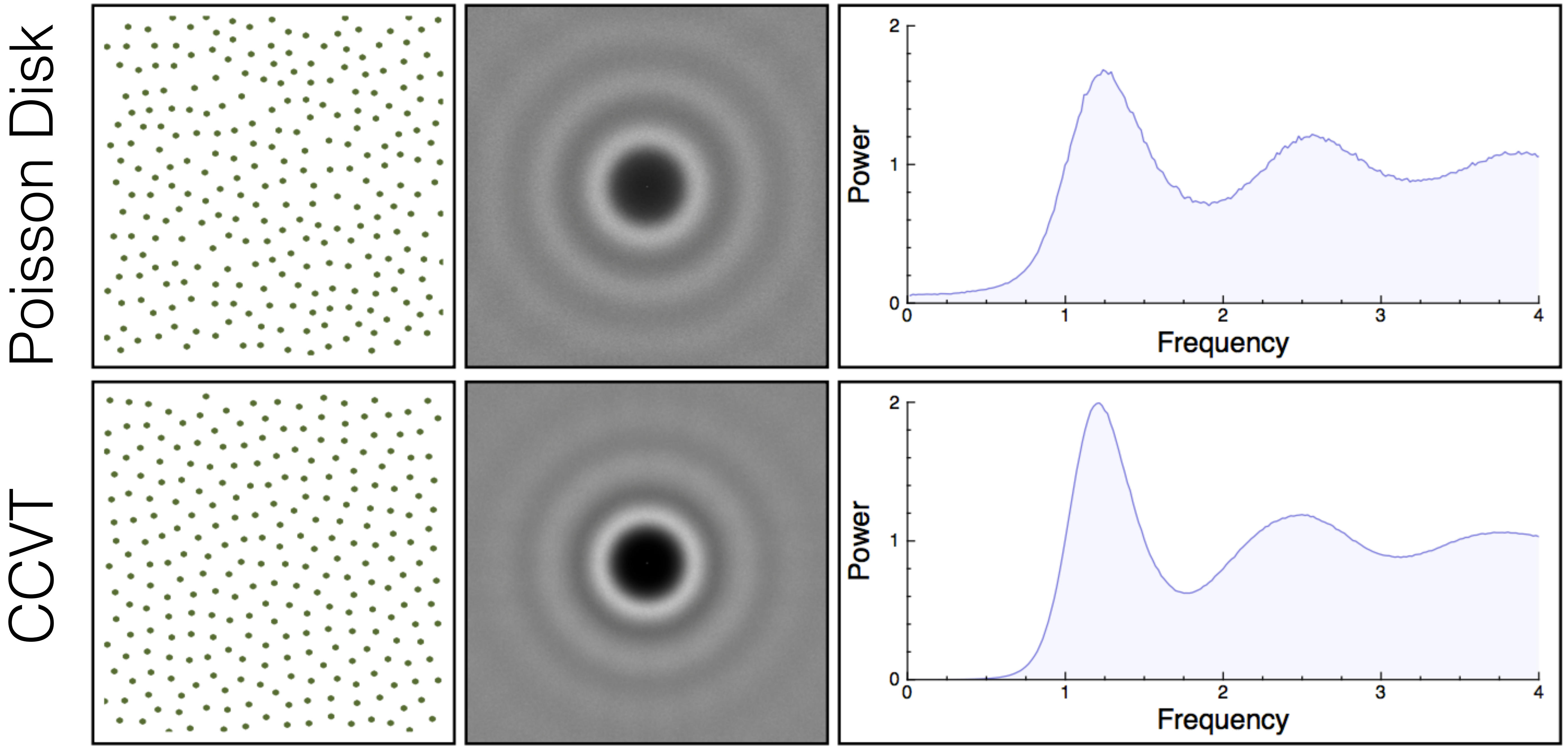
$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \left\langle P_{\mathbf{S}}(\rho\mathbf{n}) \right\rangle d\mathbf{n}\, d\rho$$

For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$

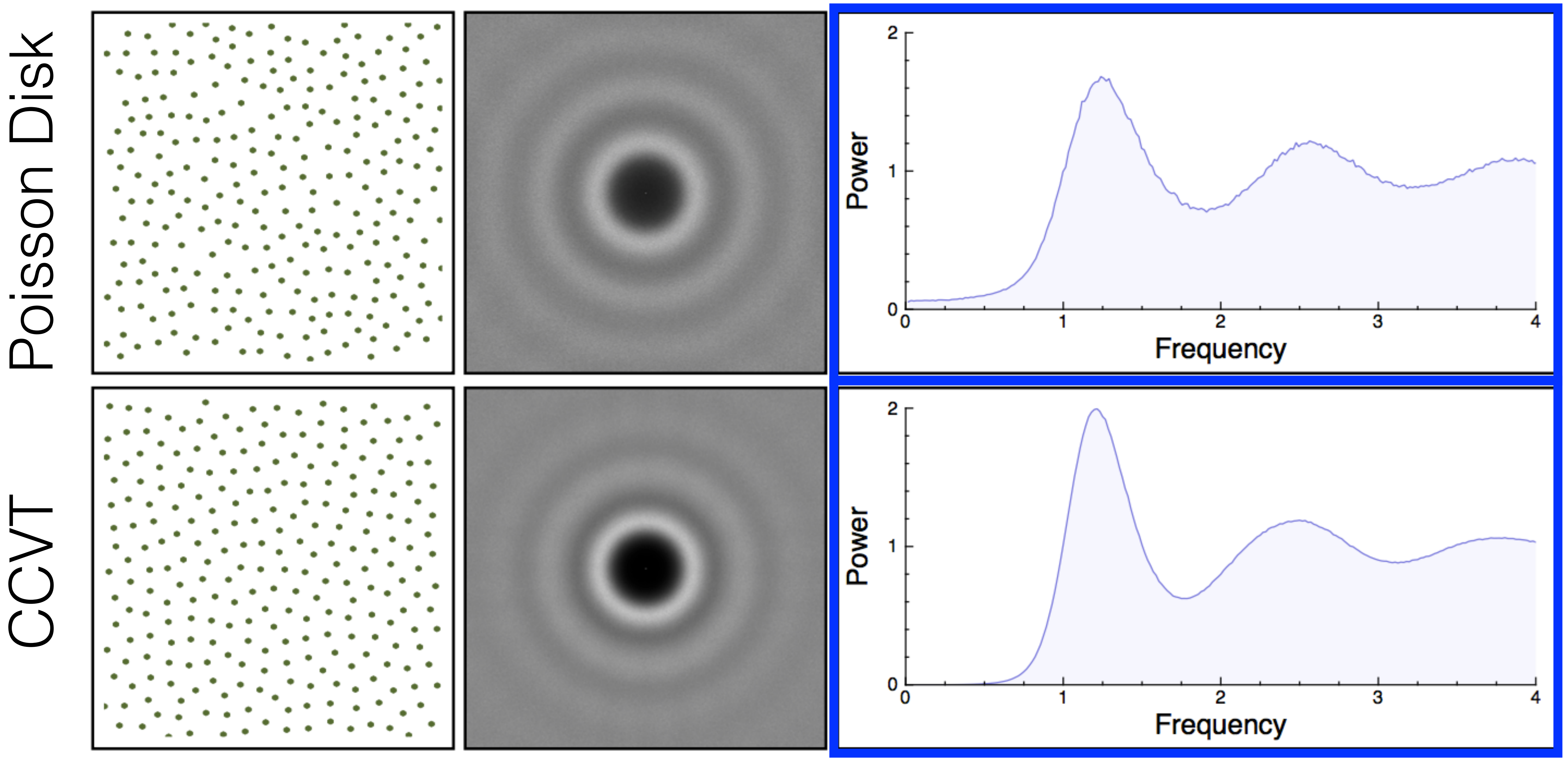
Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{s}}(\rho) \rangle \, d\rho$$



Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$

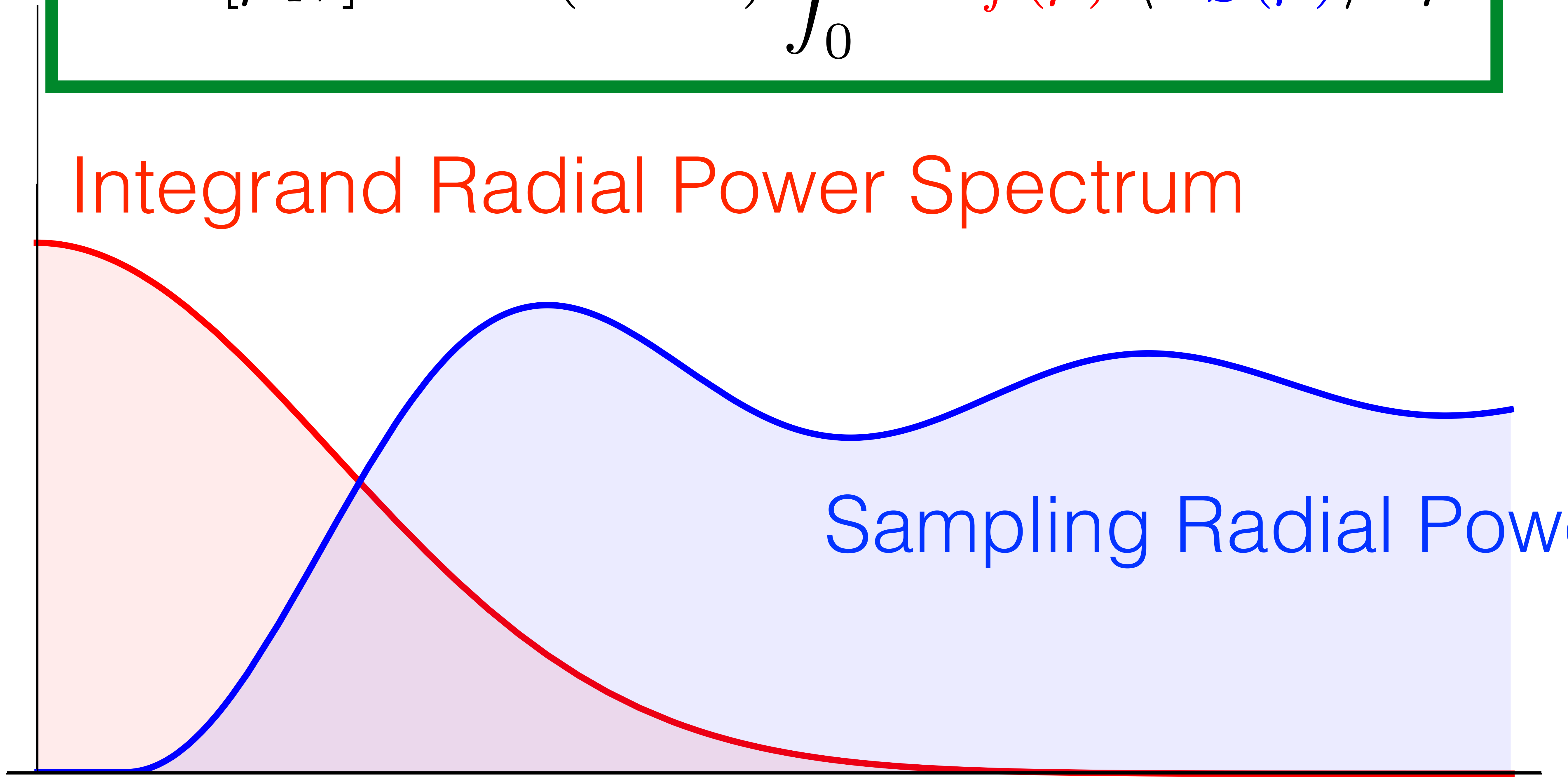


Variance: Integral over Product of Power Spectra

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$

Variance: Integral over Product of Power Spectra

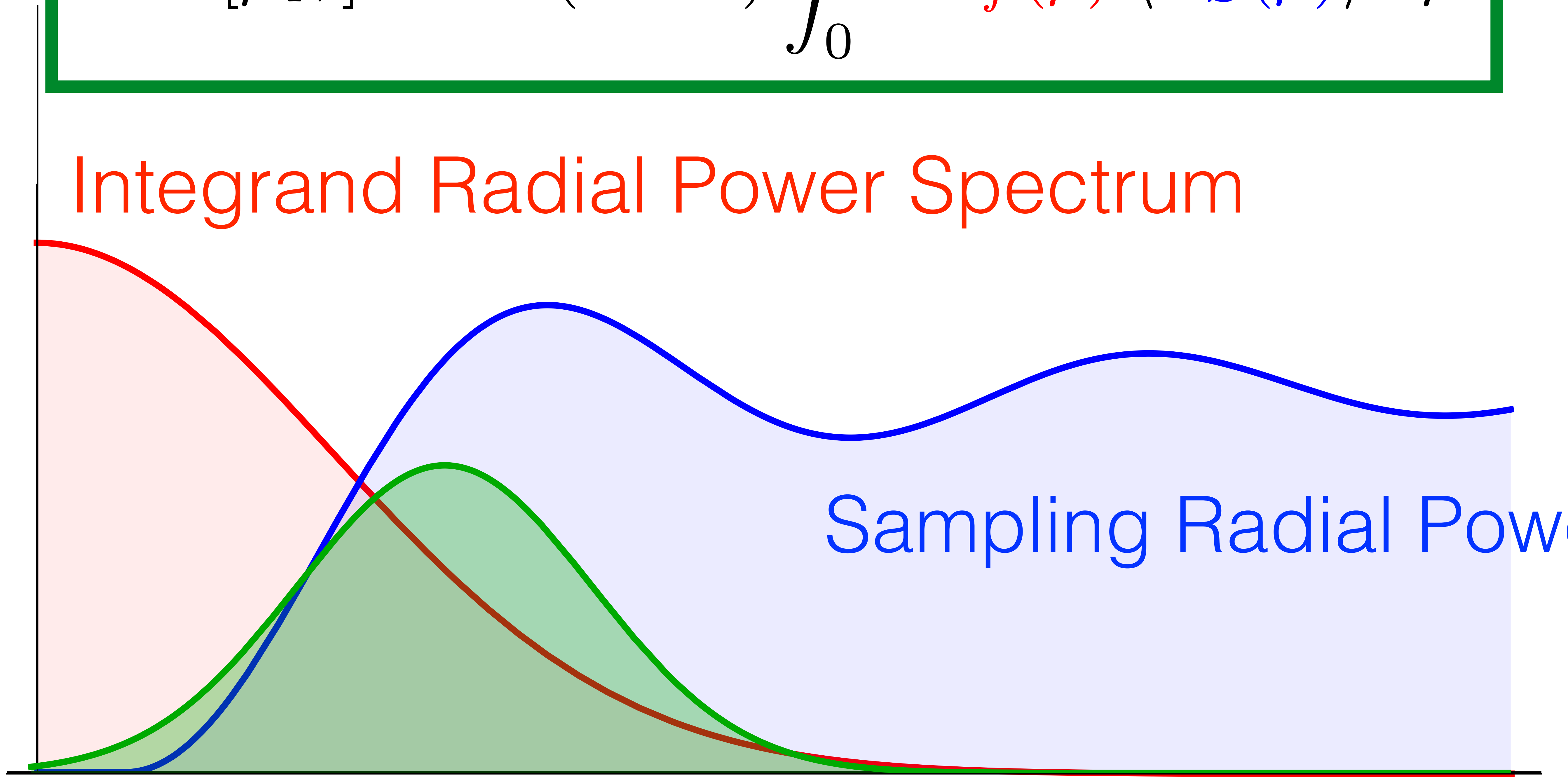
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$



For given number of Samples

Variance: Integral over Product of Power Spectra

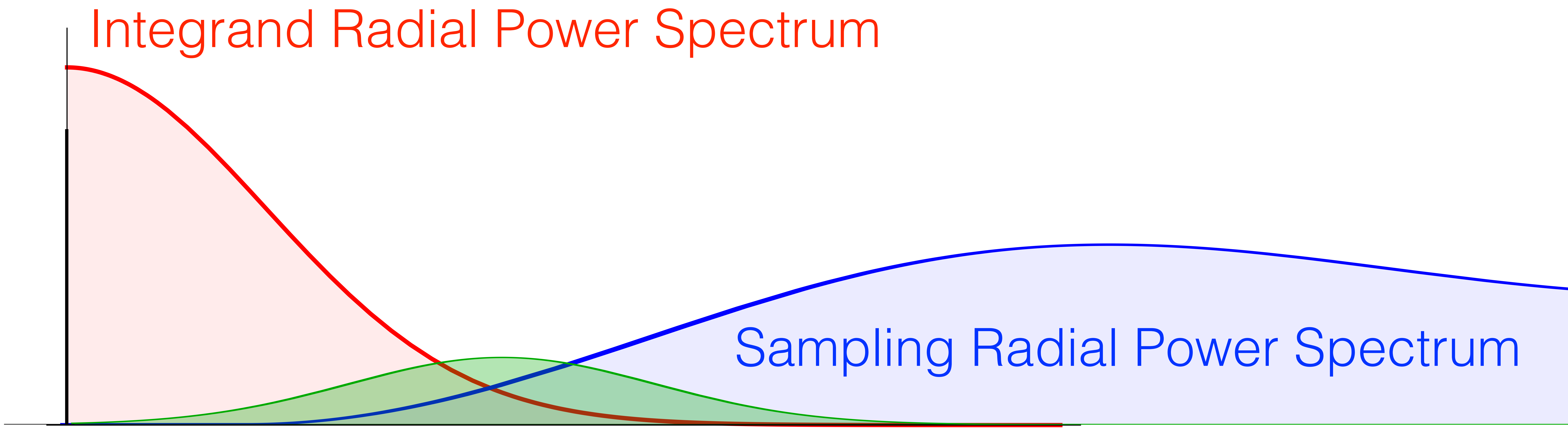
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho)\, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho}$$



For given number of Samples

Variance: Integral over Product of Power Spectra

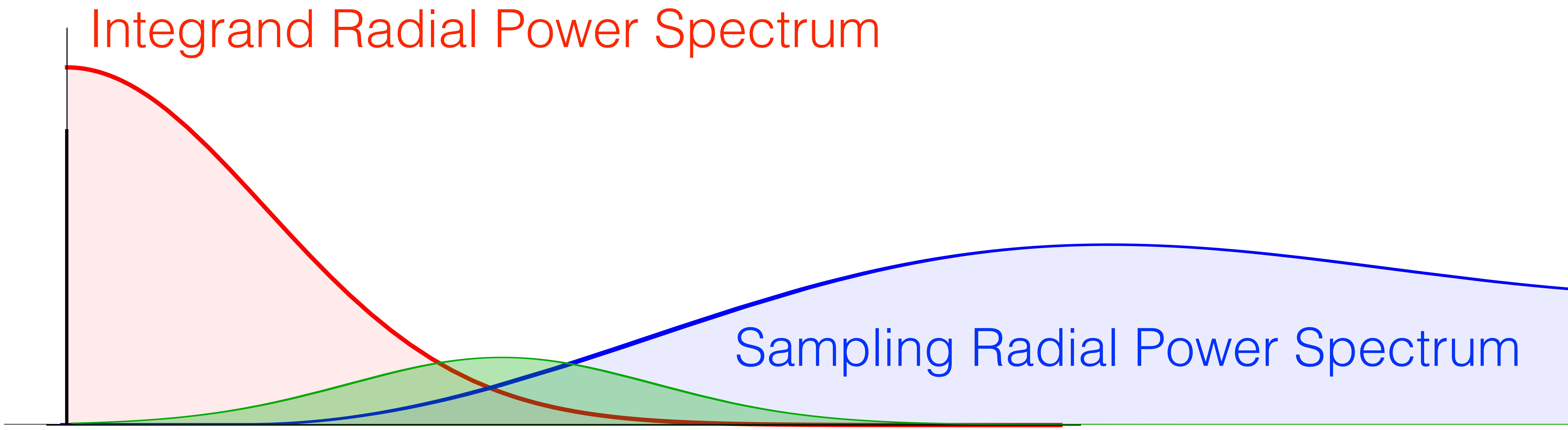
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{s}}(\rho) \right\rangle d\rho}$$



For given number of Samples

Variance: Integral over Product of Power Spectra

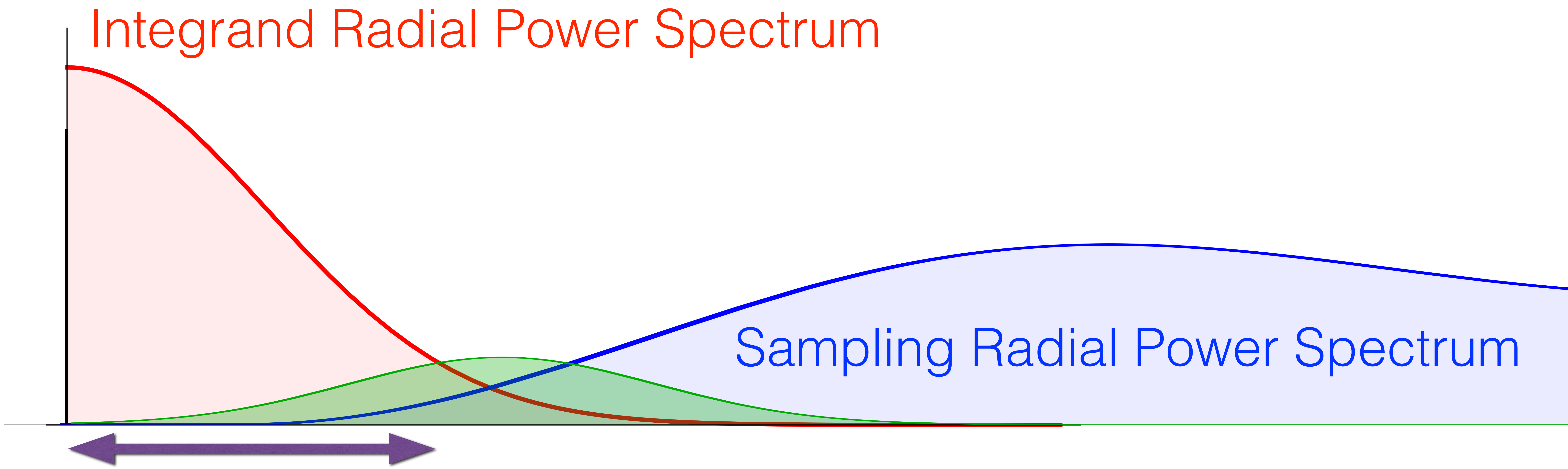
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho)\,\langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho}$$



For given number of Samples

Variance: Integral over Product of Power Spectra

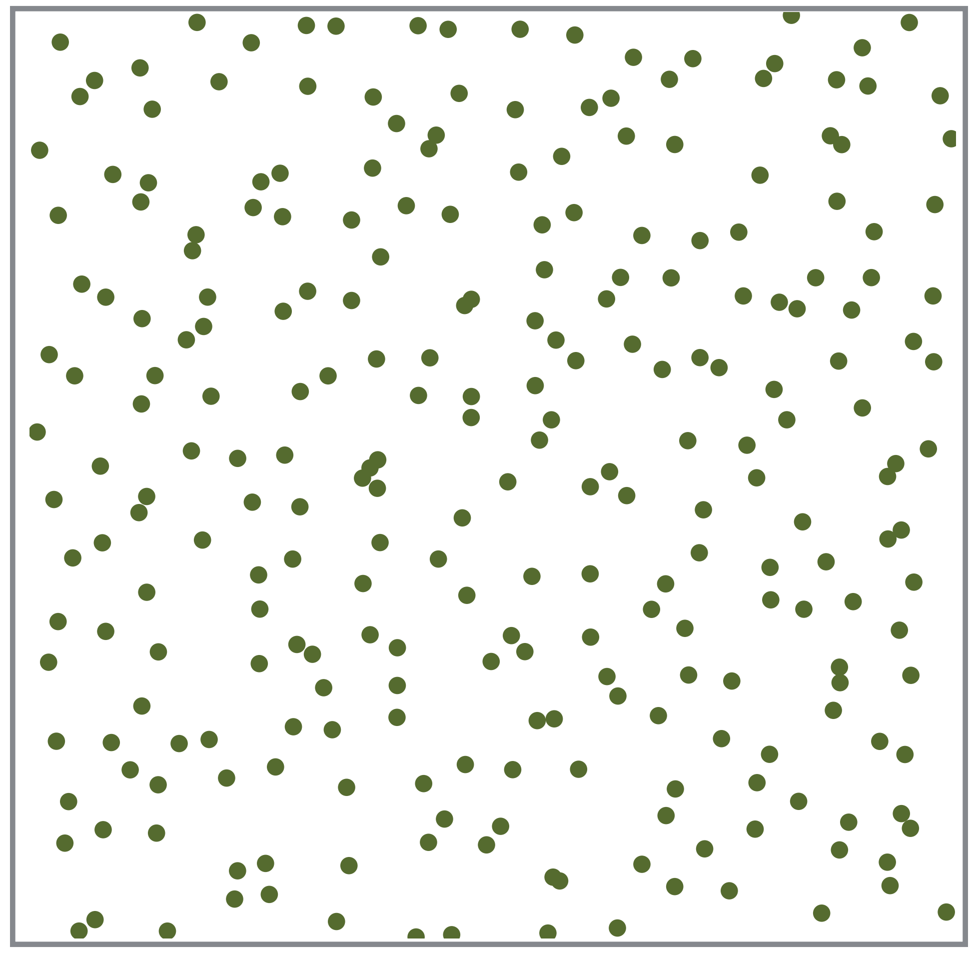
$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho}$$

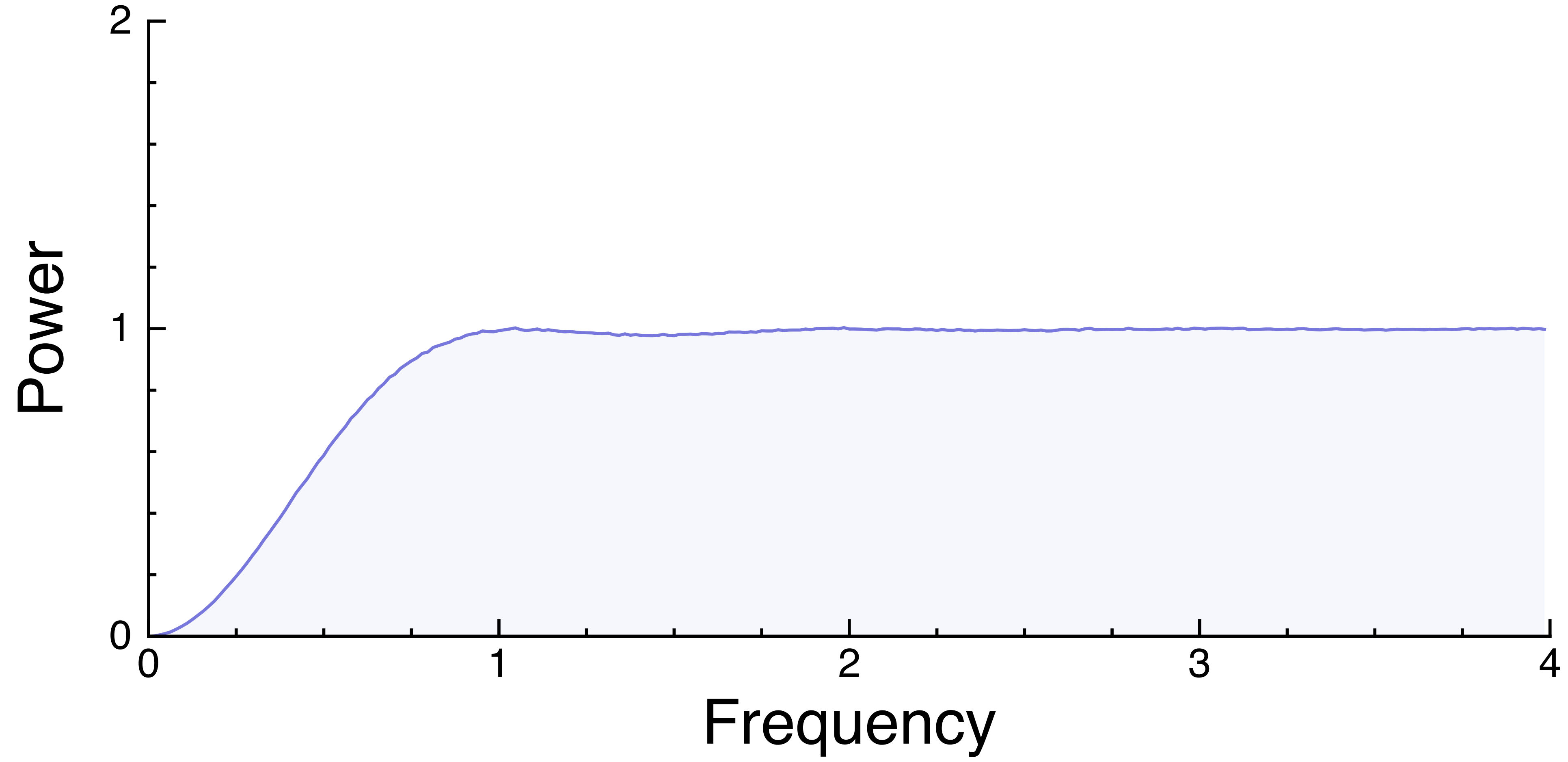


For given number of Samples

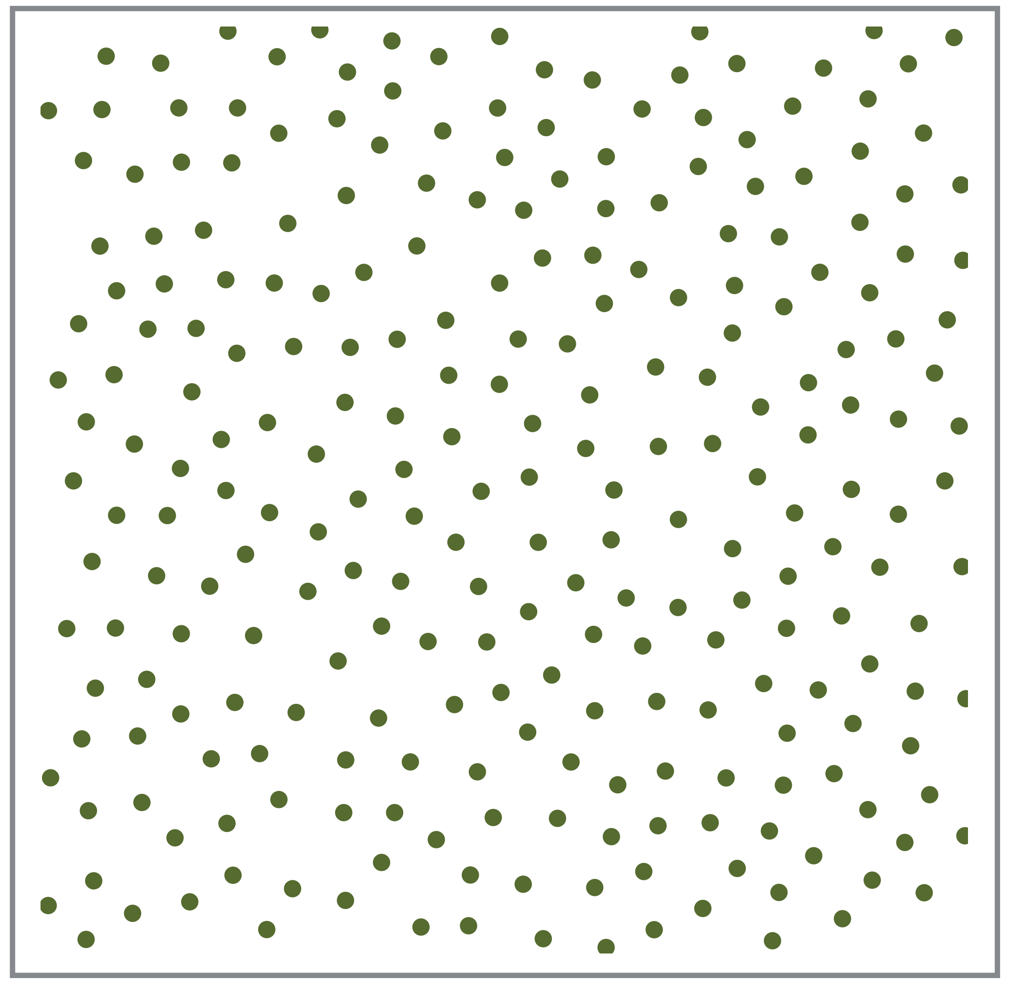
Spatial Distribution vs Radial Mean Power Spectra

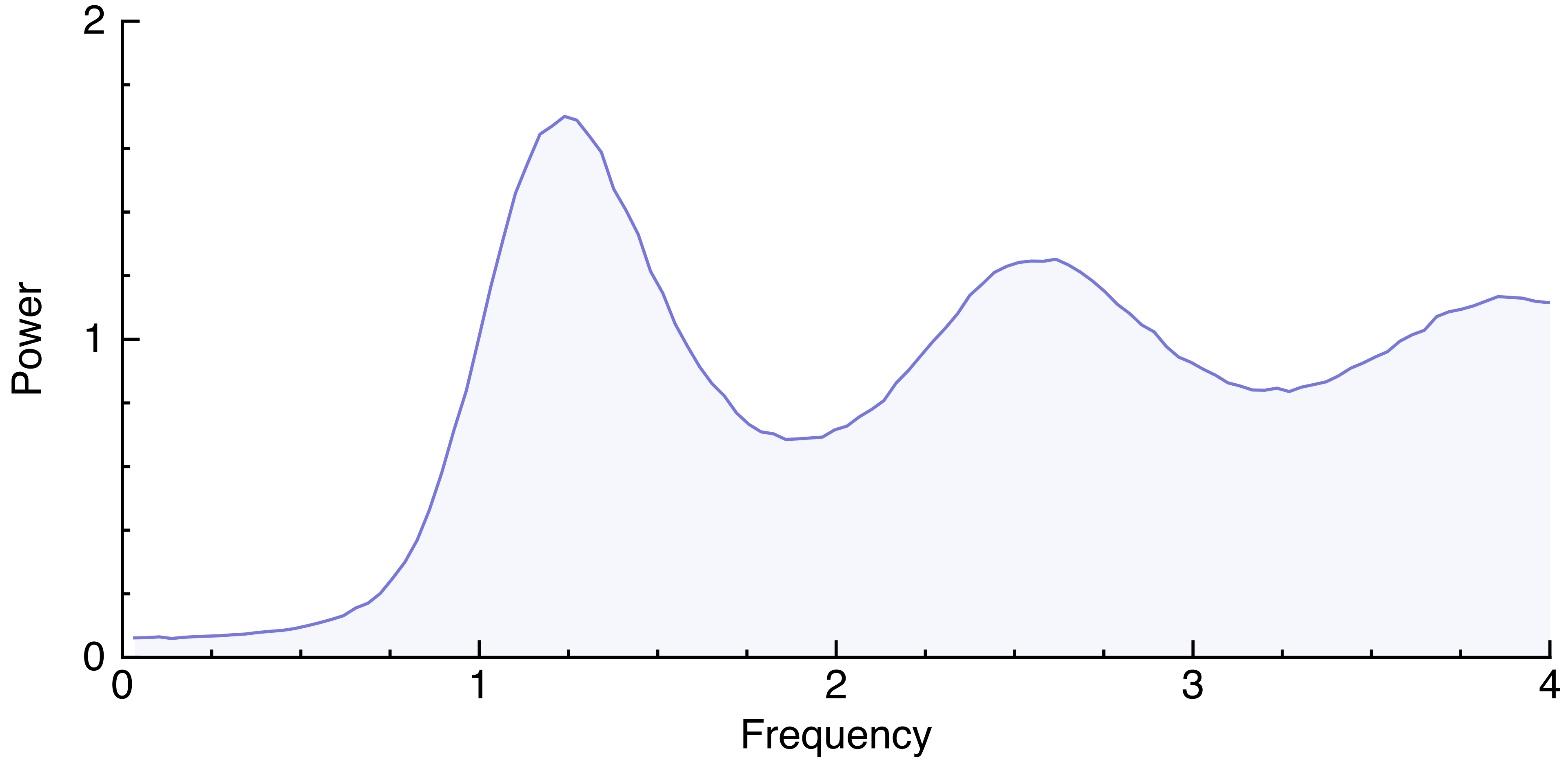
Jitter





Poisson Disk





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For 2-dimensions

Samplers	Worst Case	Best Case
Random		
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	

Pilleboue et al. [2015]

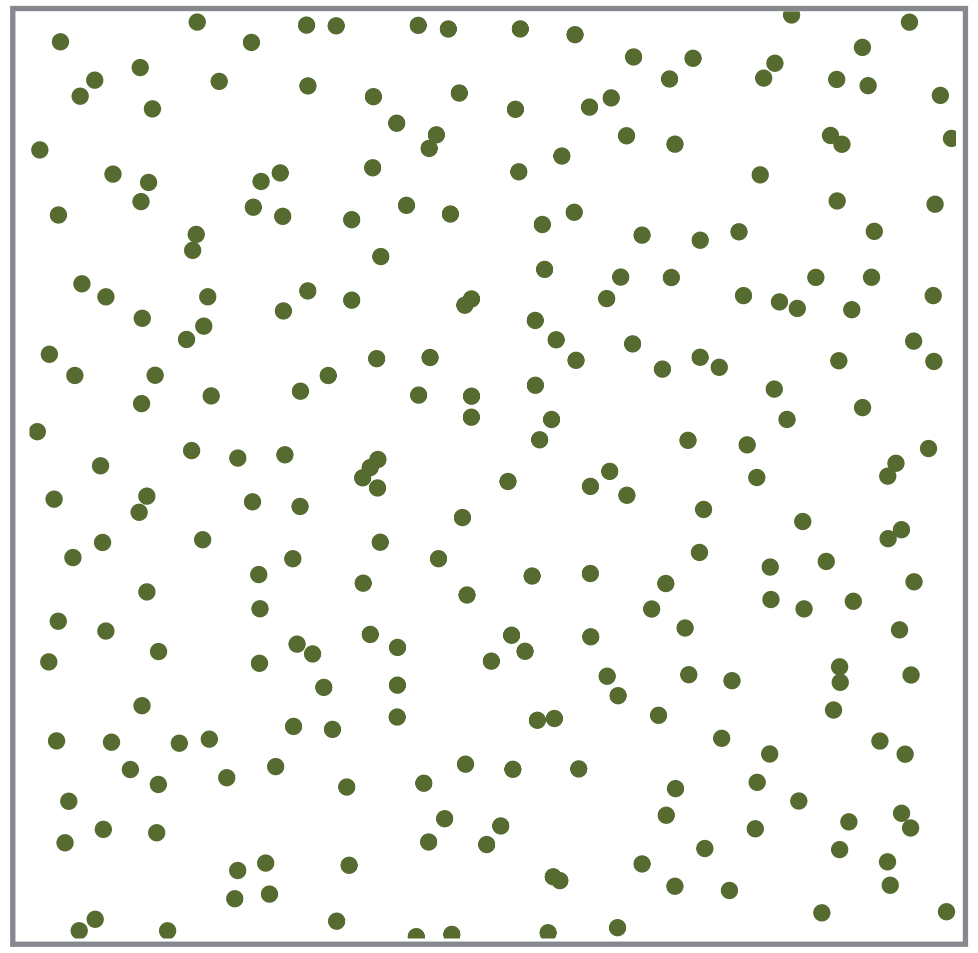
For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

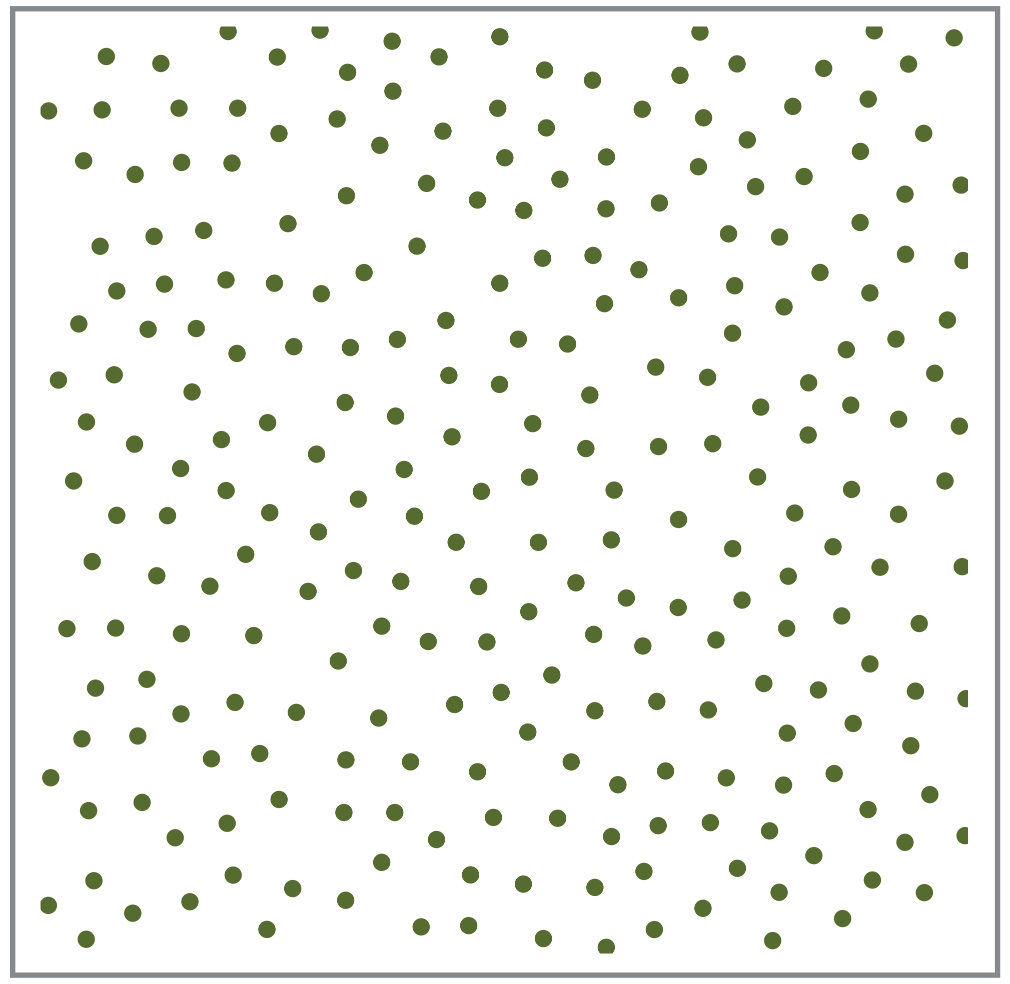
Pilleboue et al. [2015]

For 2-dimensions

Jitter



Poisson Disk

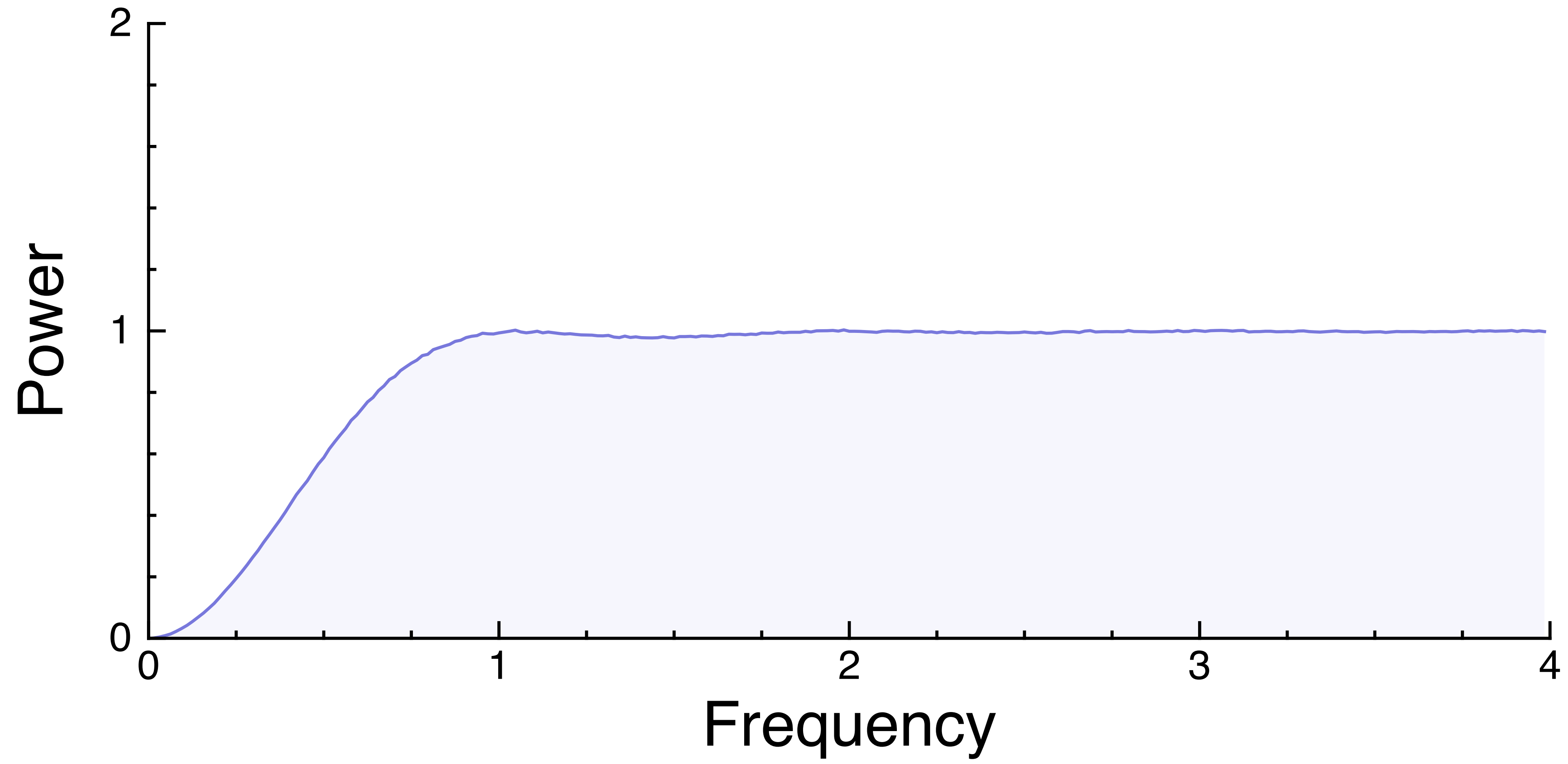


Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

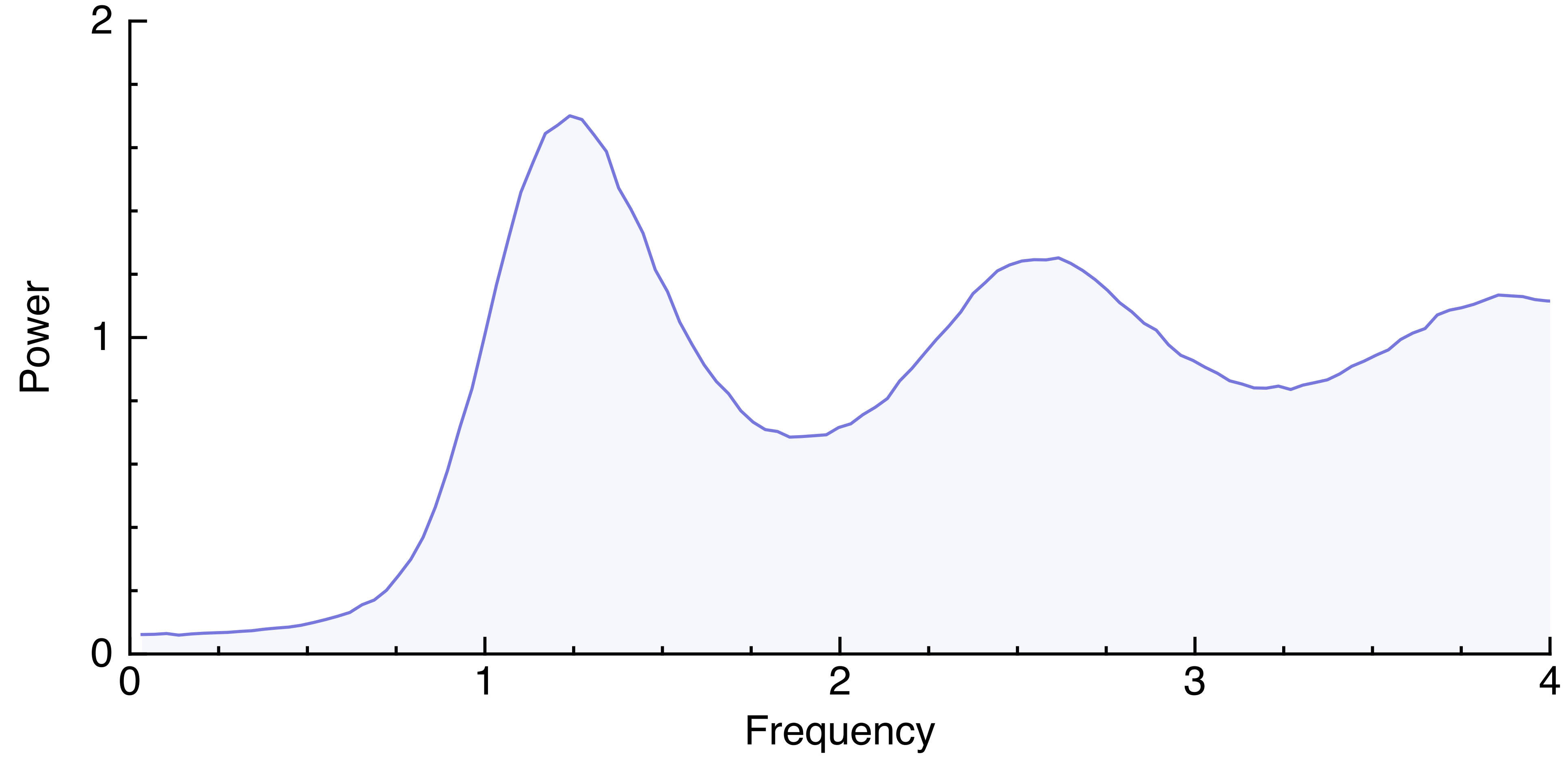
Pilleboue et al. [2015]

Low Frequency Region

Jitter

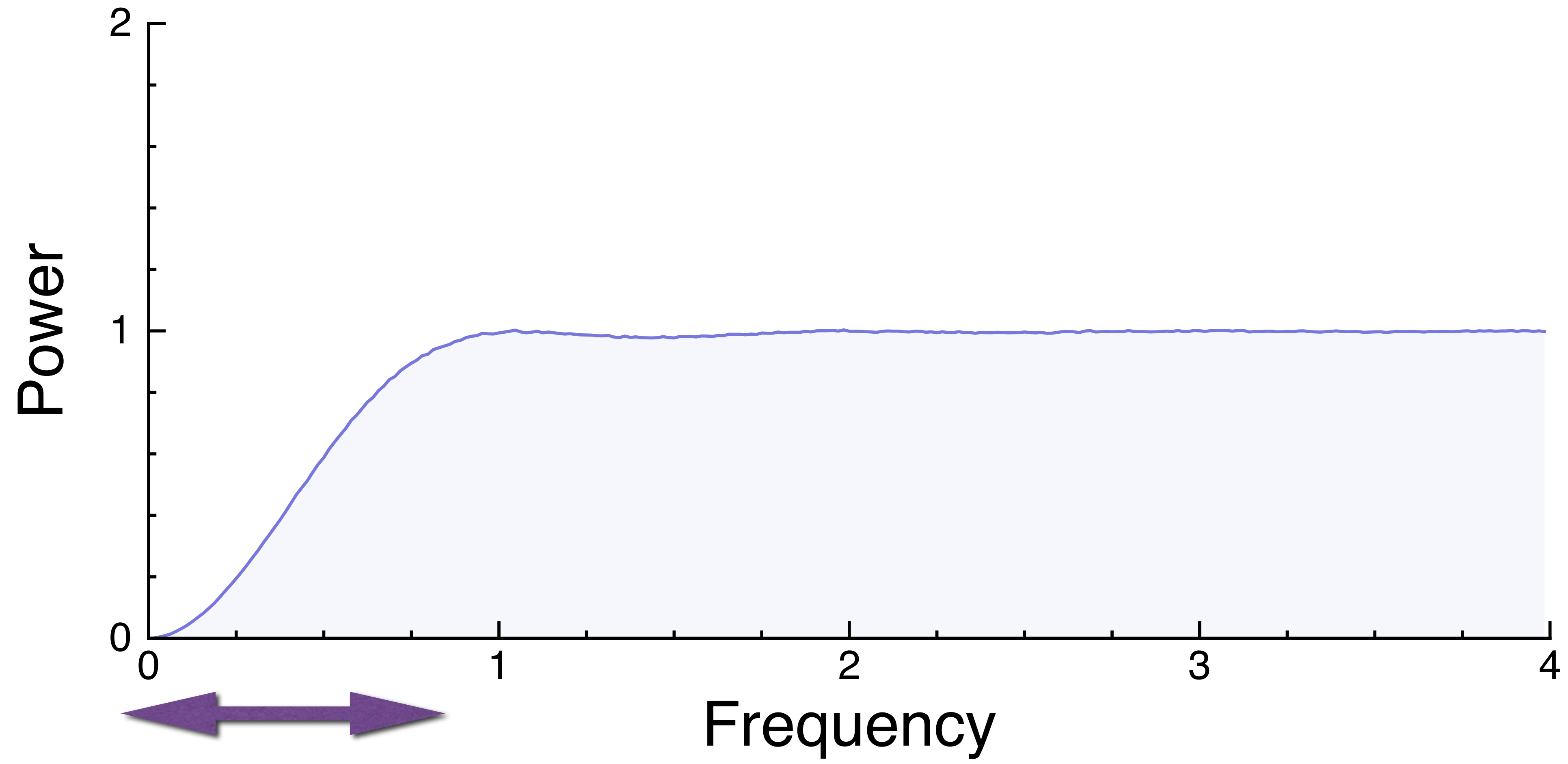


Poisson Disk

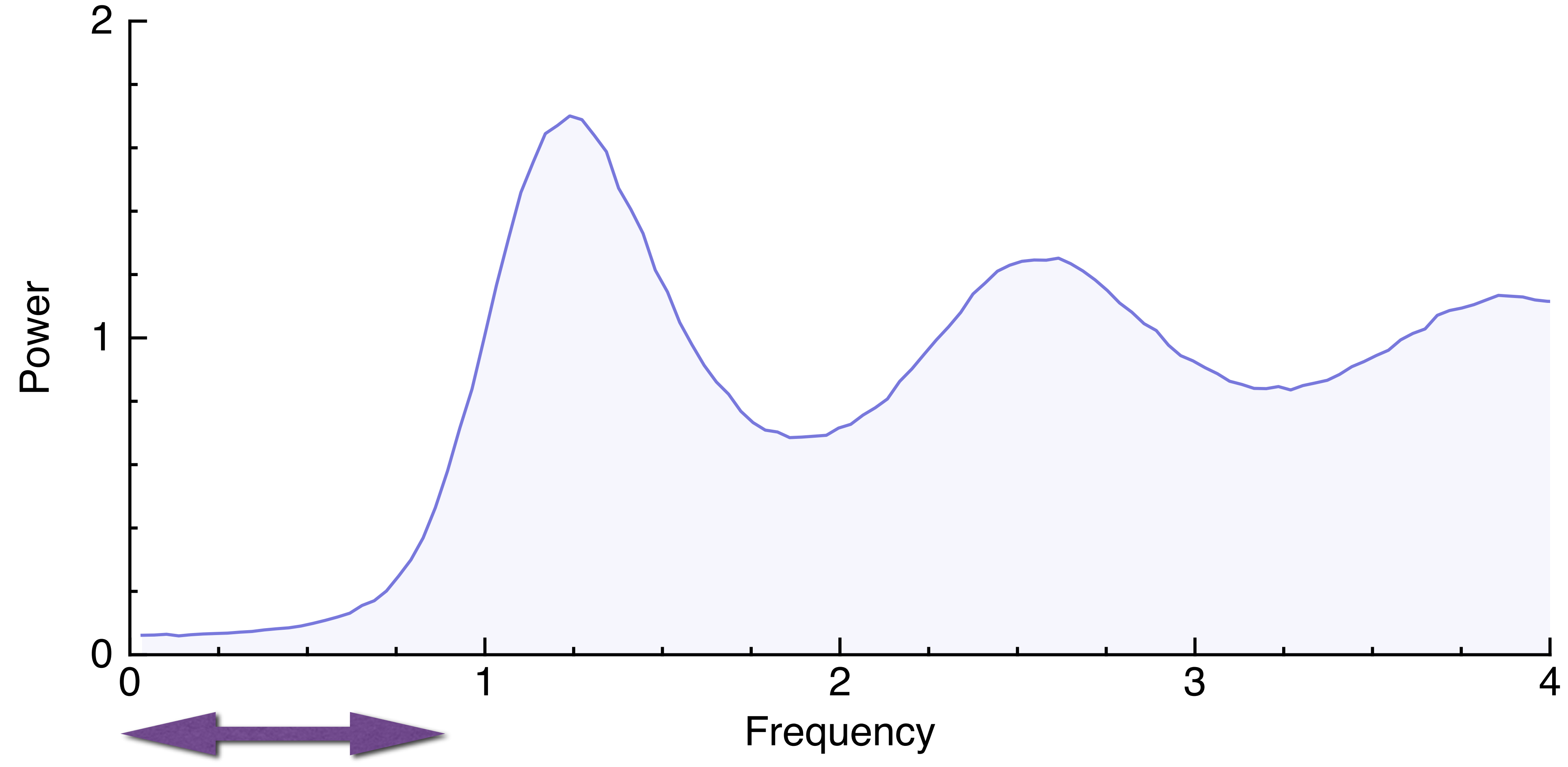


Low Frequency Region

Jitter



Poisson Disk



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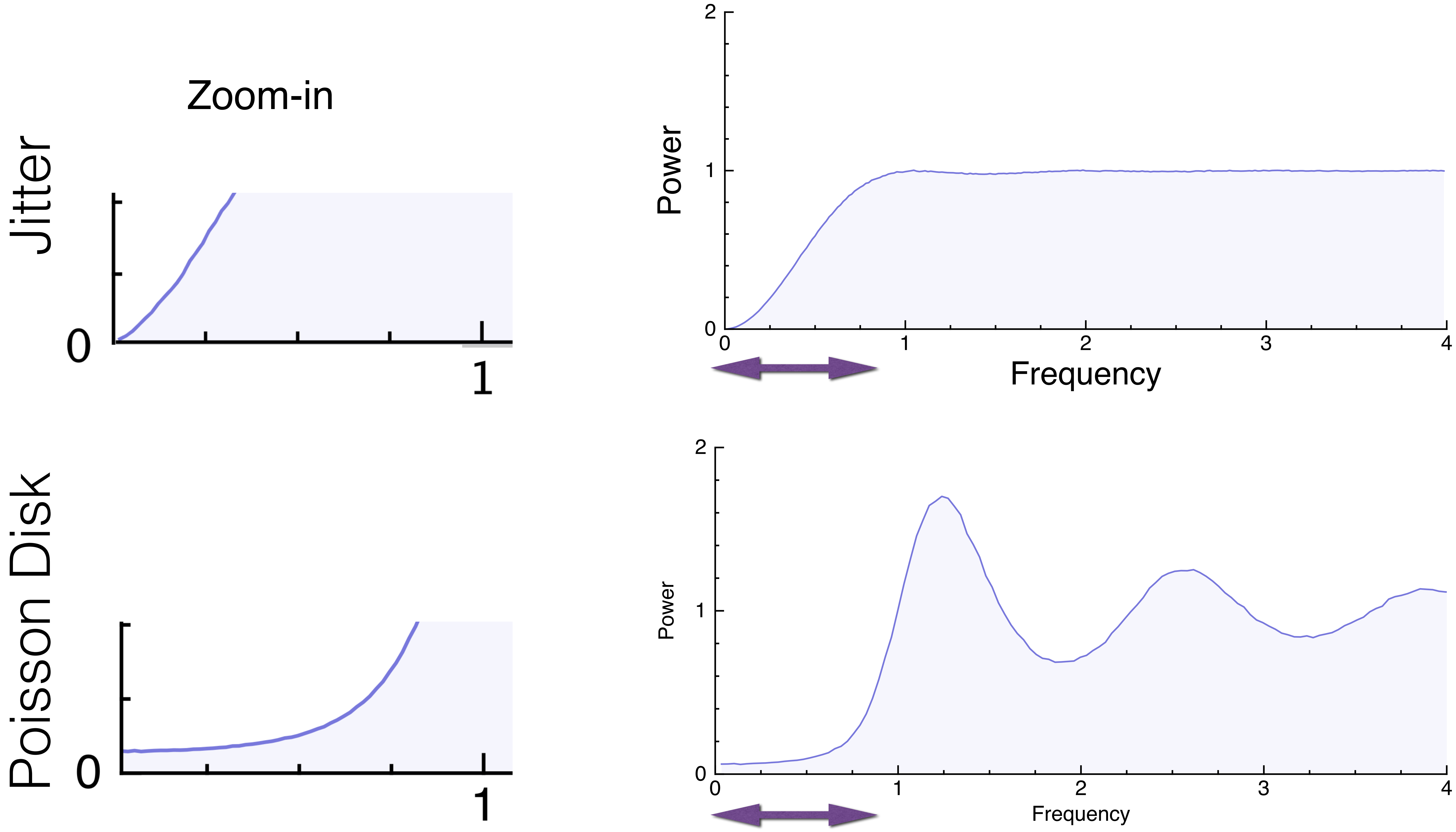
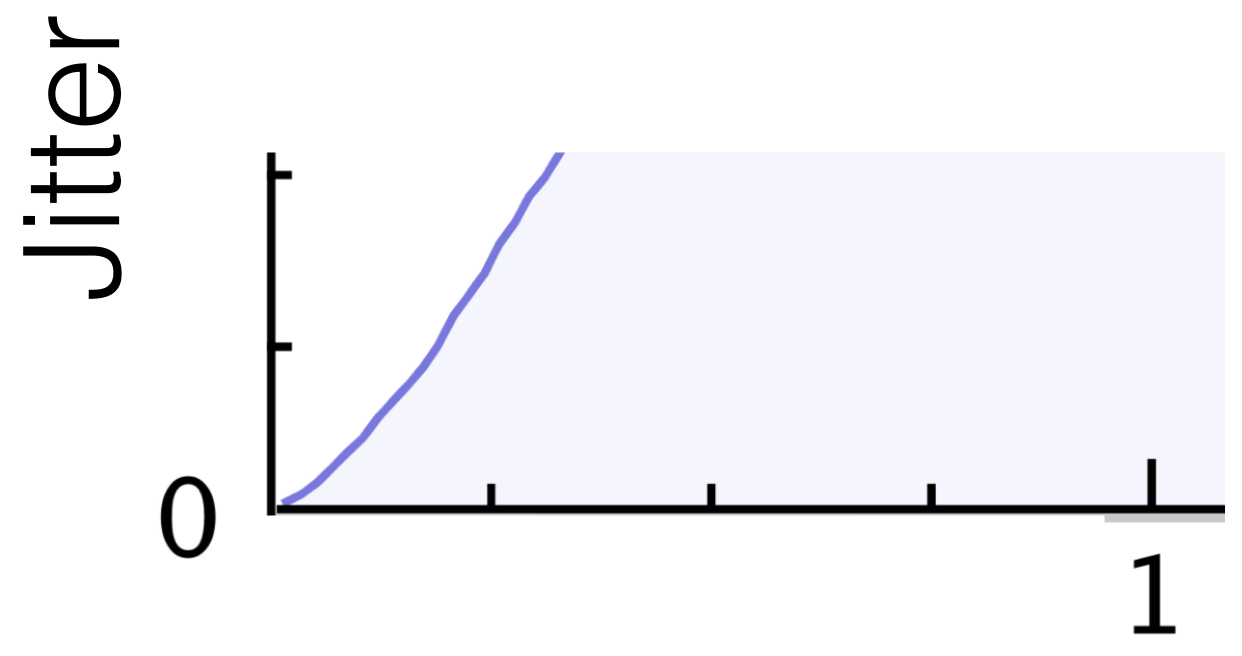
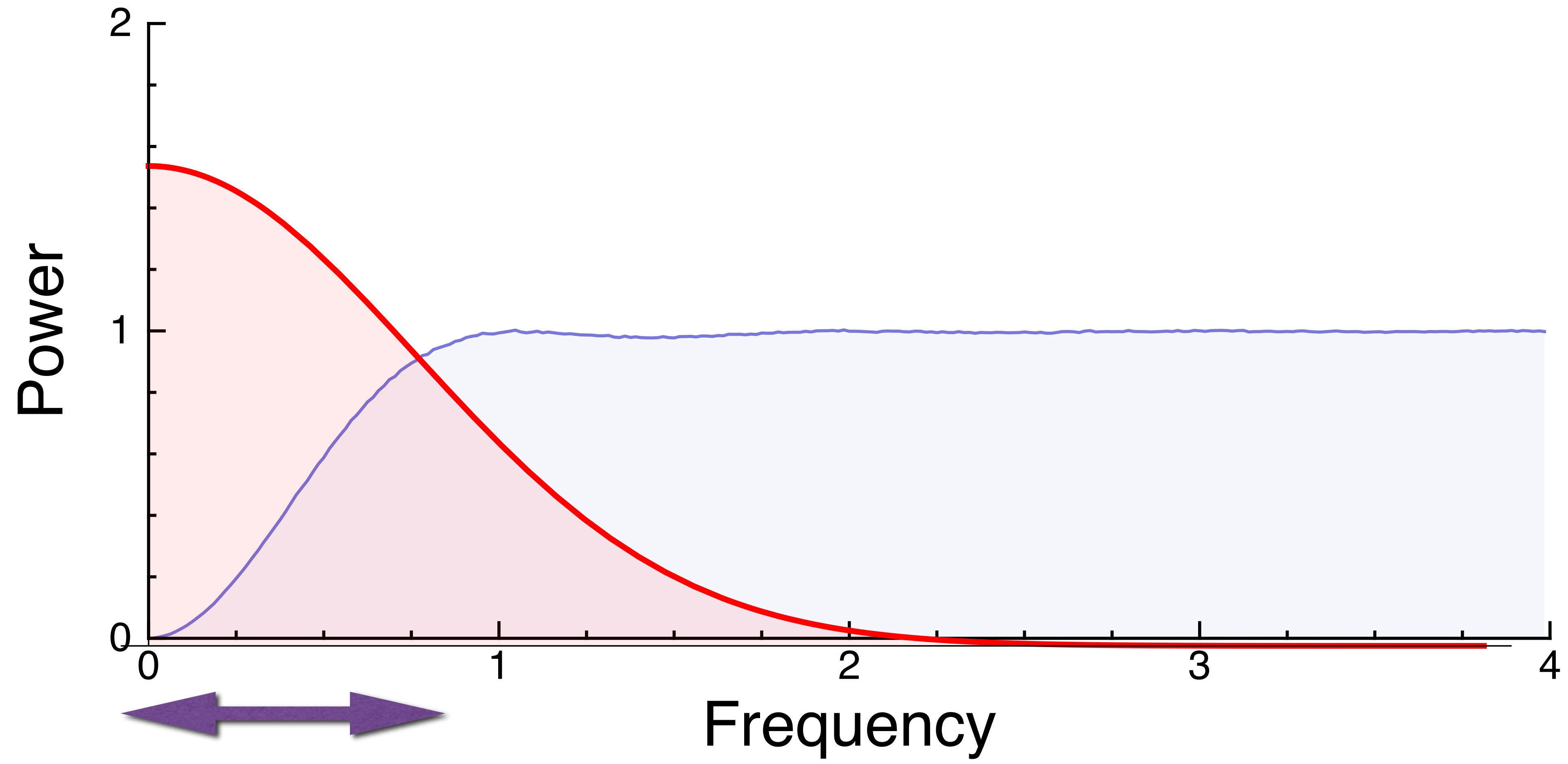
Low Frequency Region

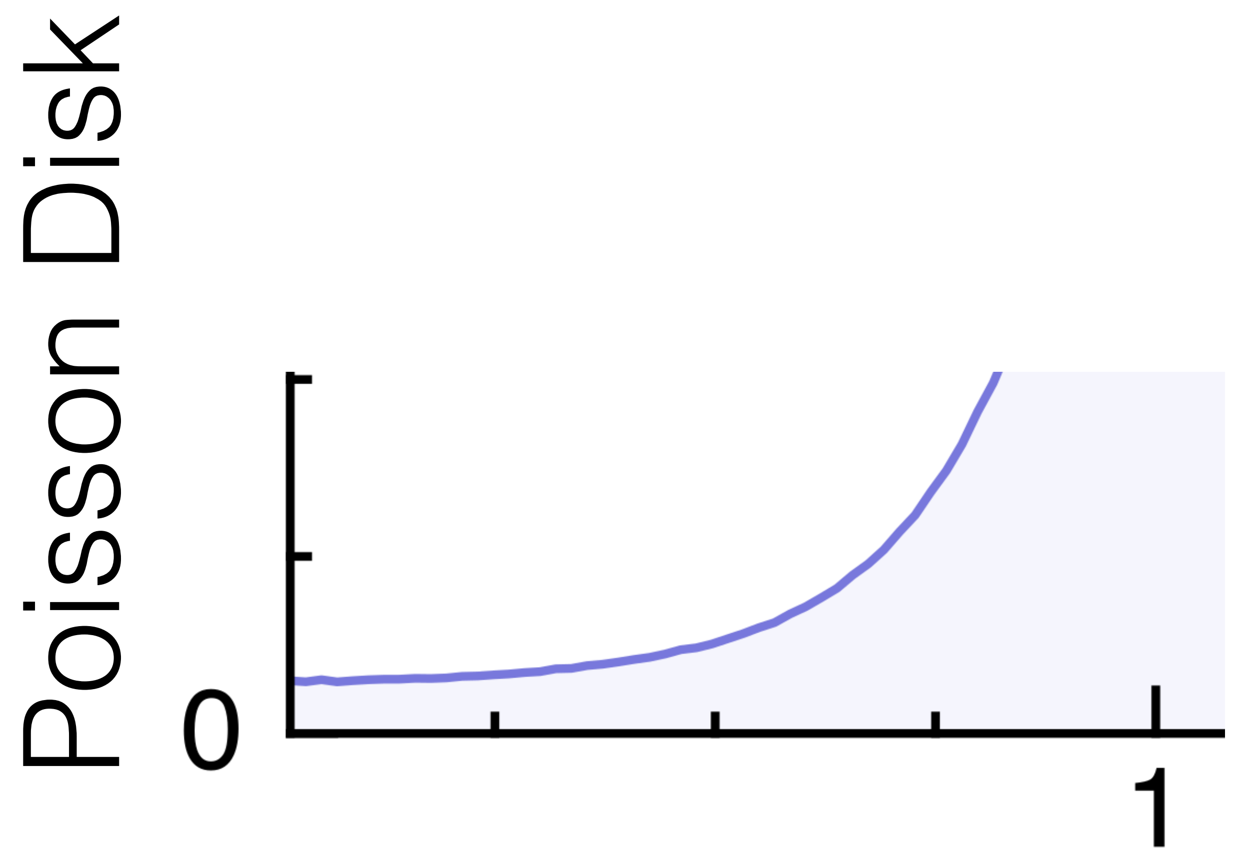


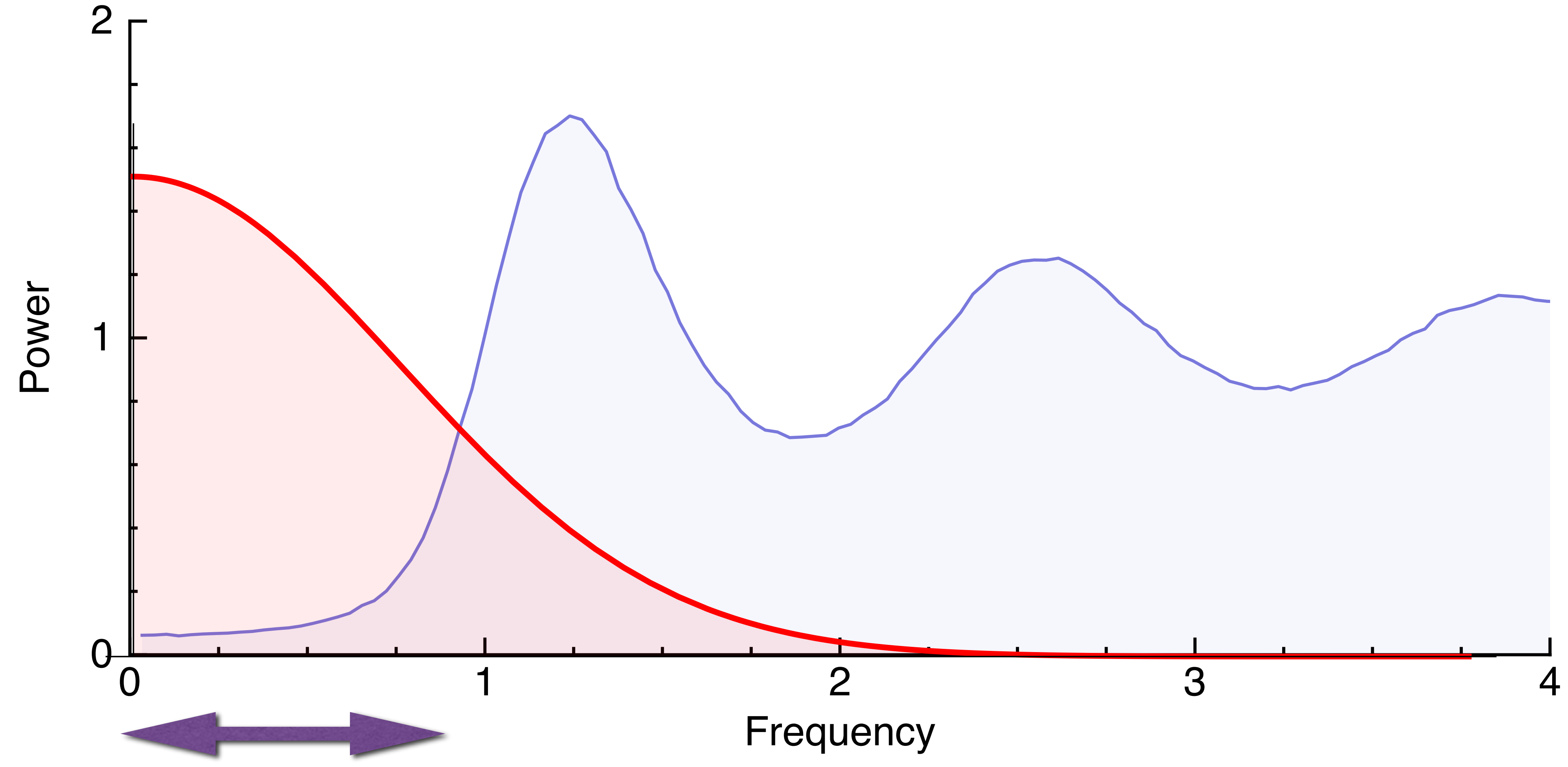
Variance for Low Sample Count

Zoom-in

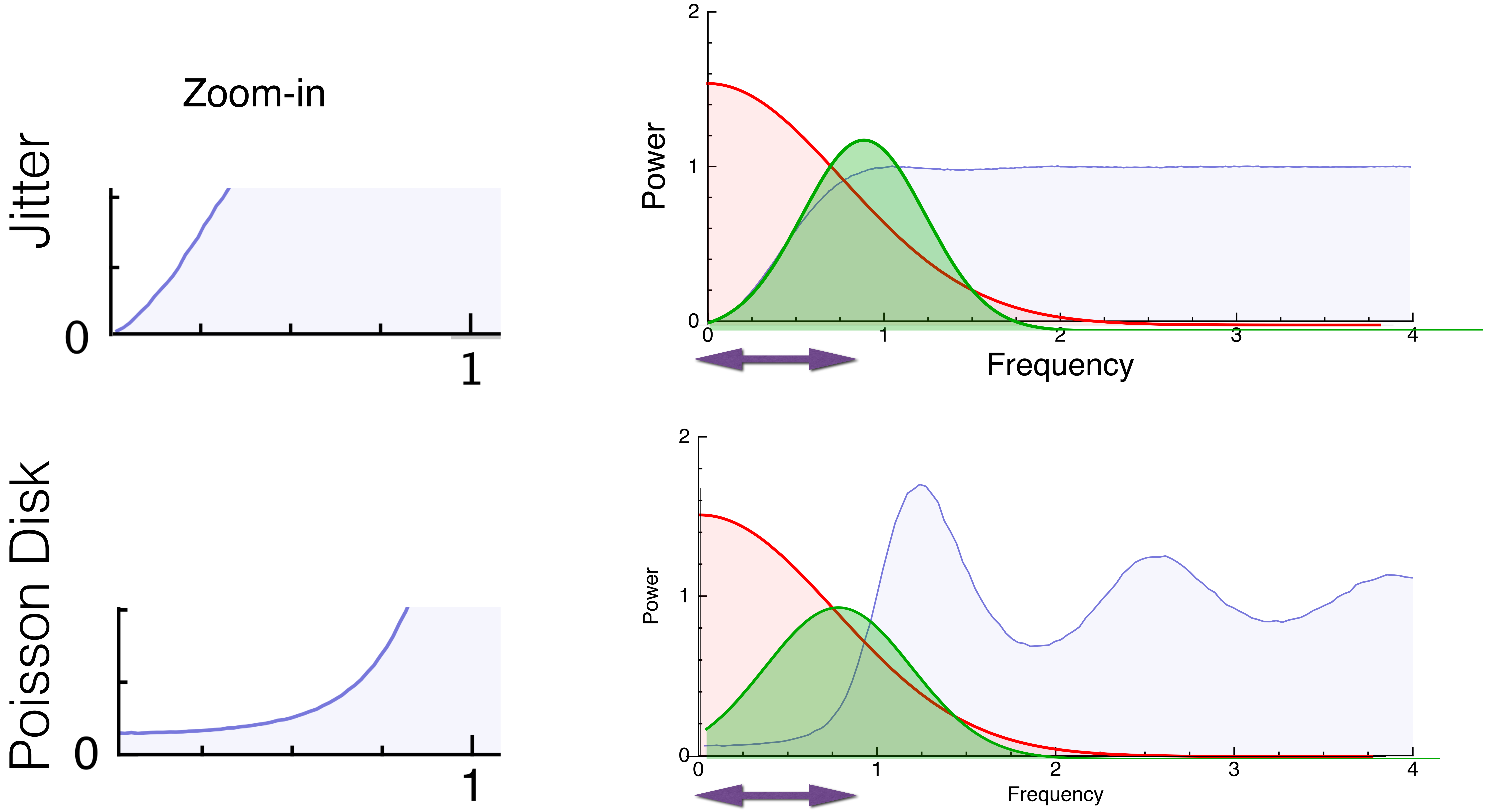




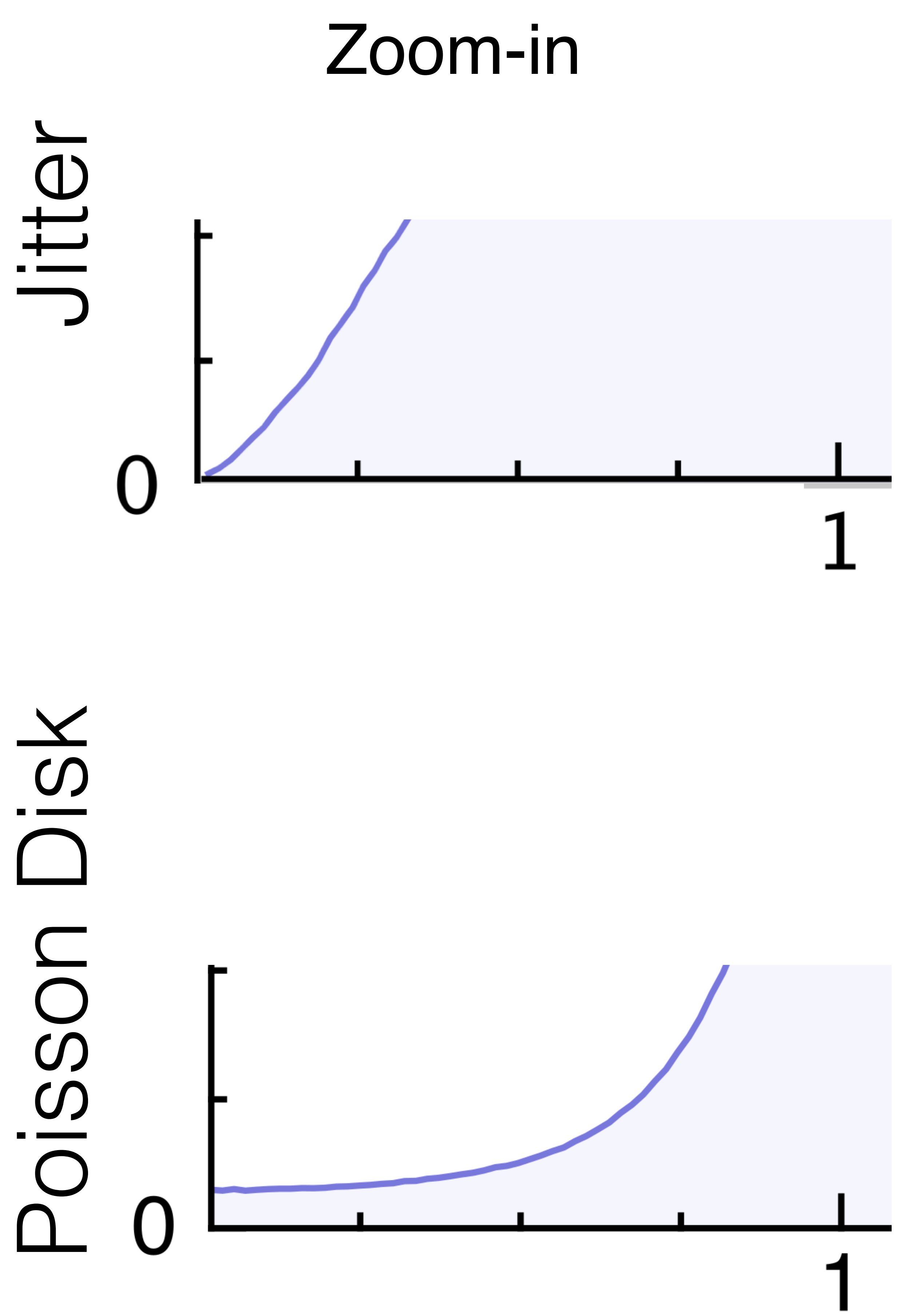


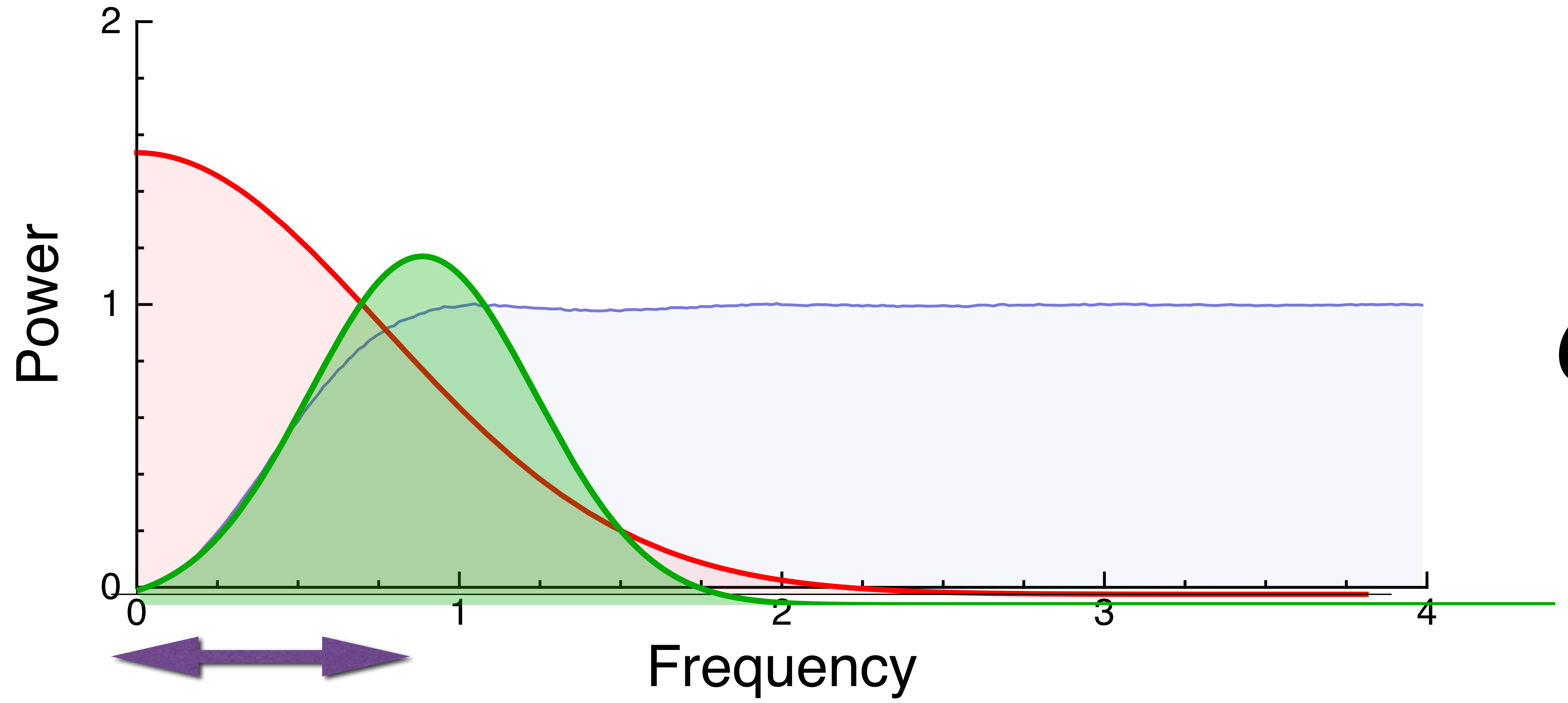


Variance for Low Sample Count

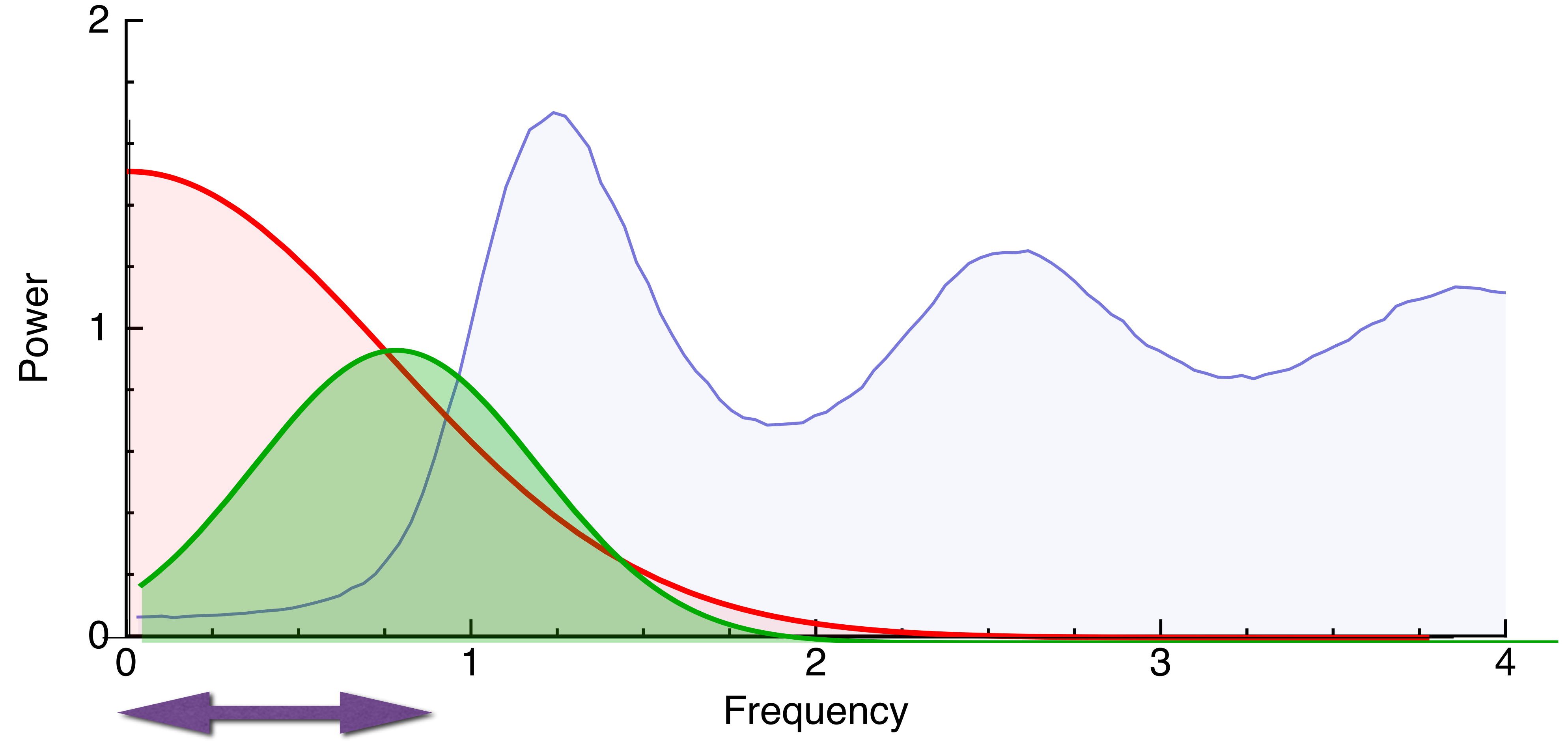


Variance for Increasing Sample Count





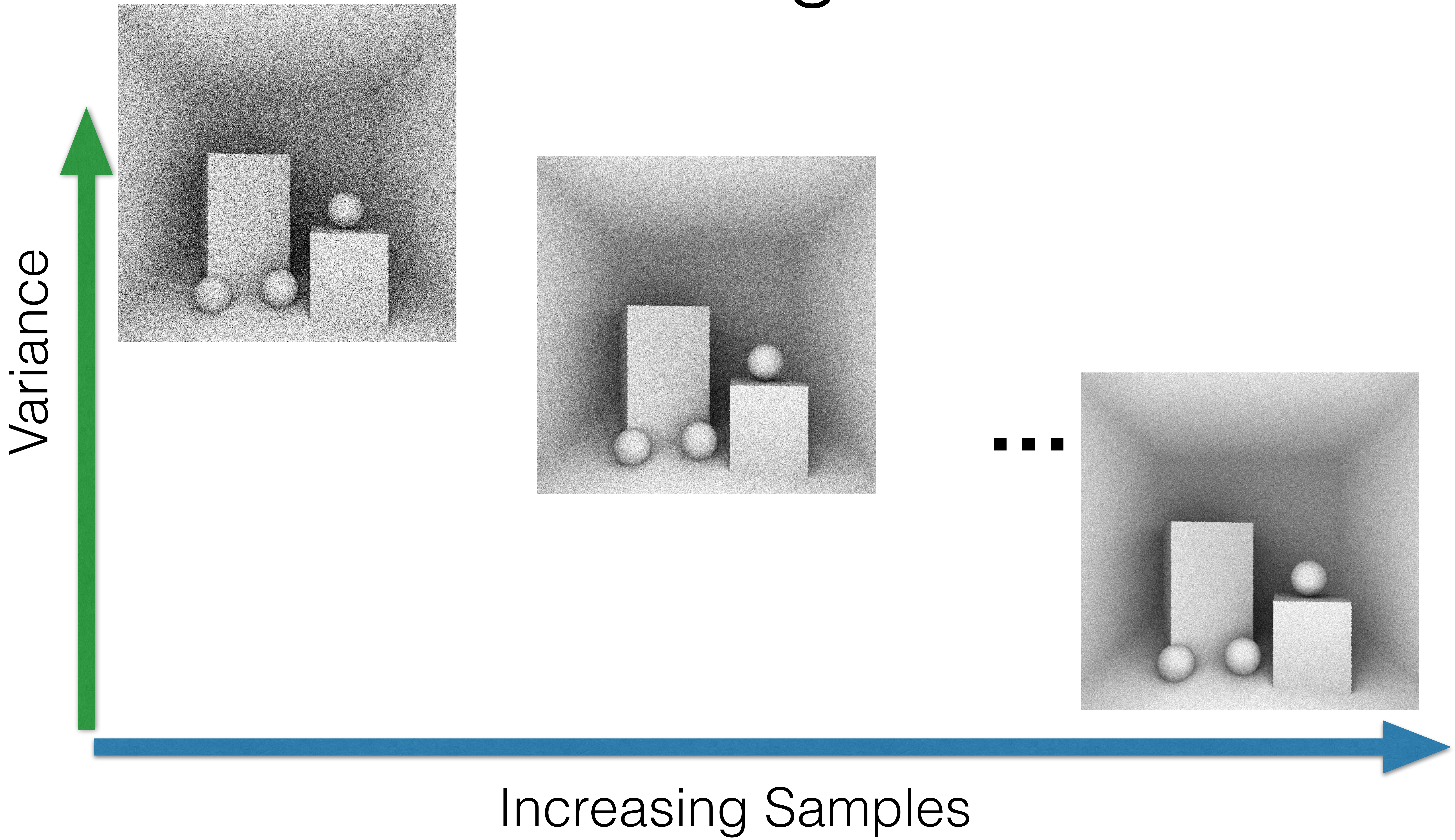
$\mathcal{O}(N^{-2})$



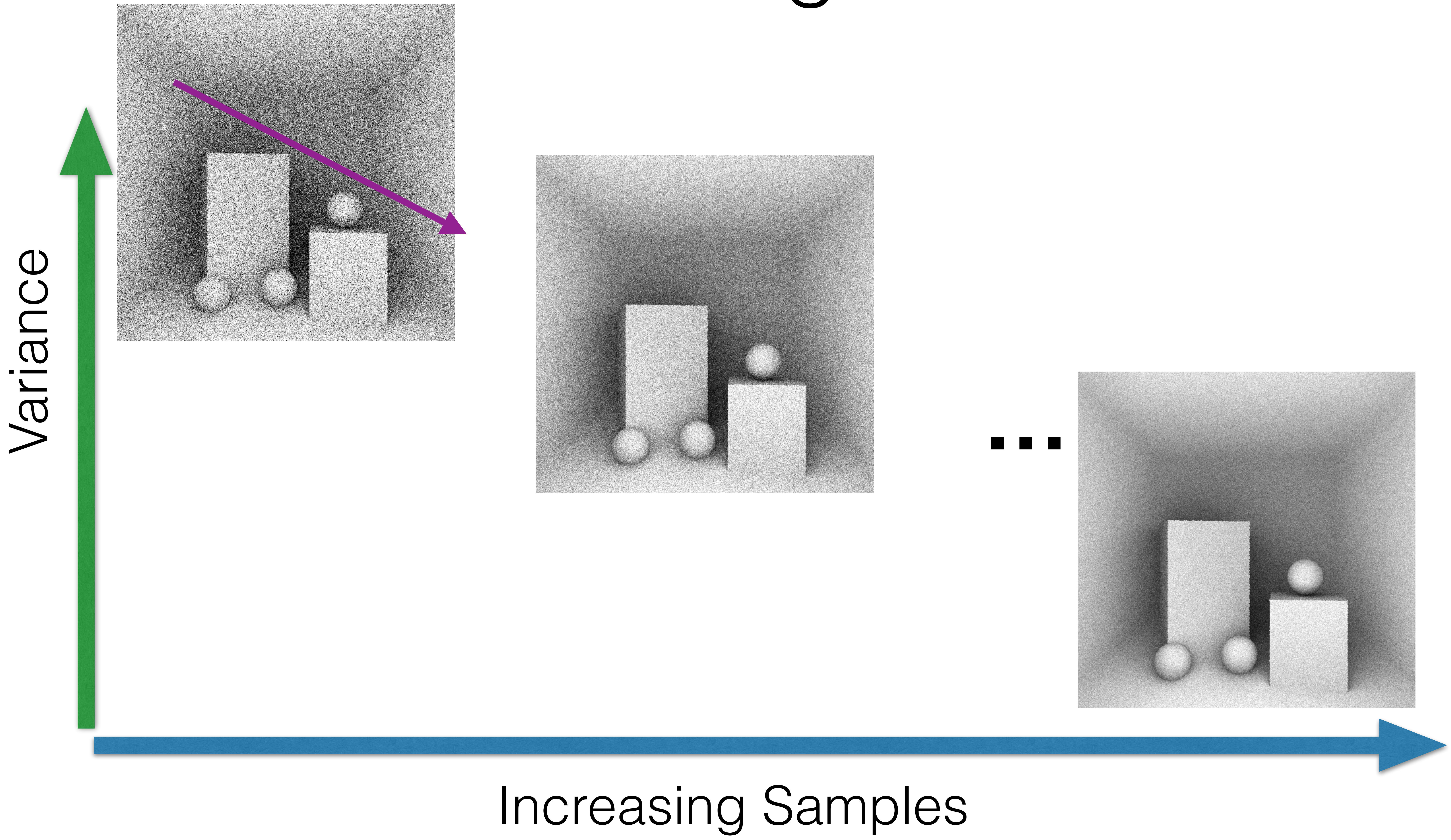
$\mathcal{O}(N^{-1})$

Experimental Verification

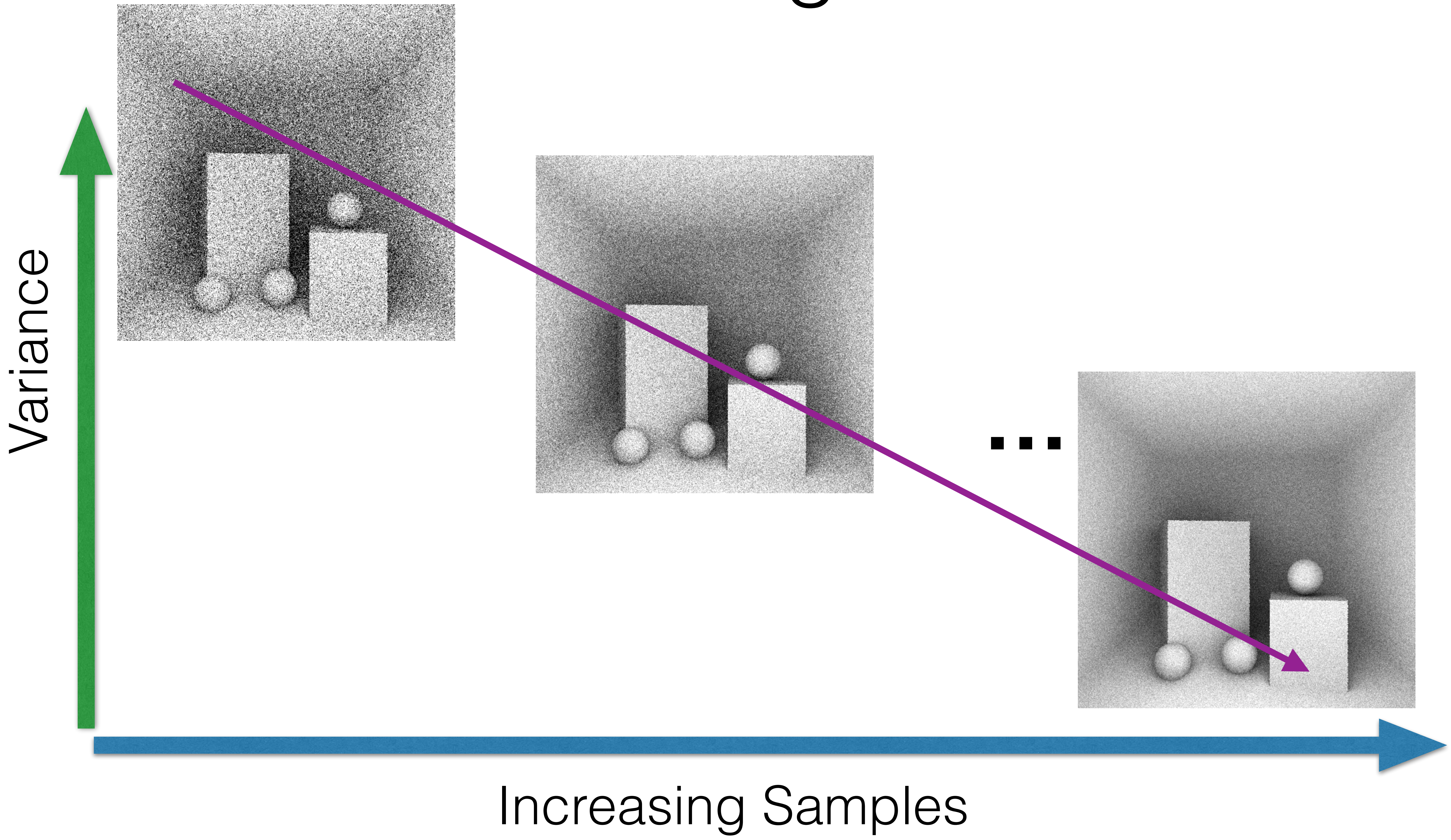
Convergence rate



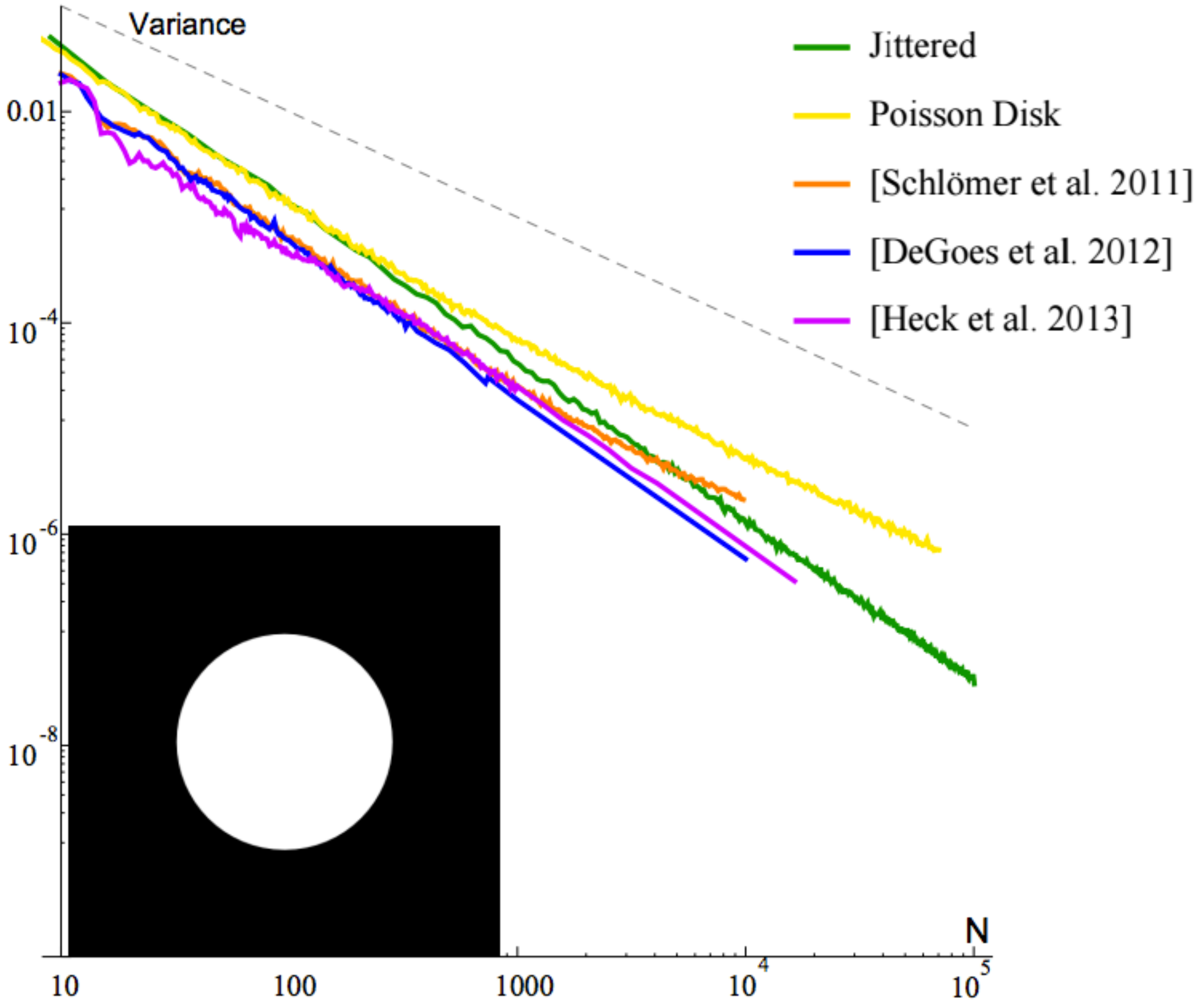
Convergence rate



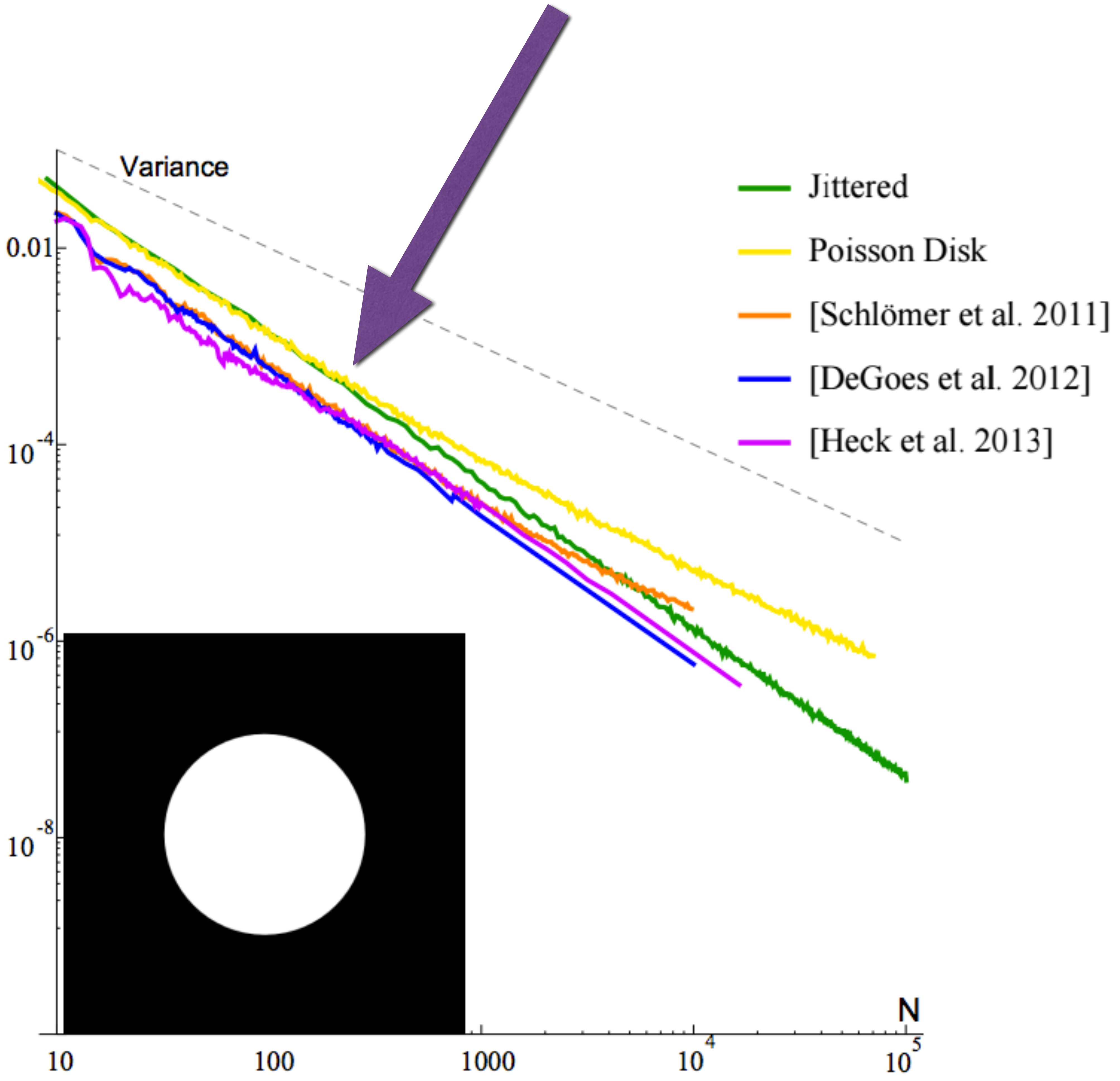
Convergence rate



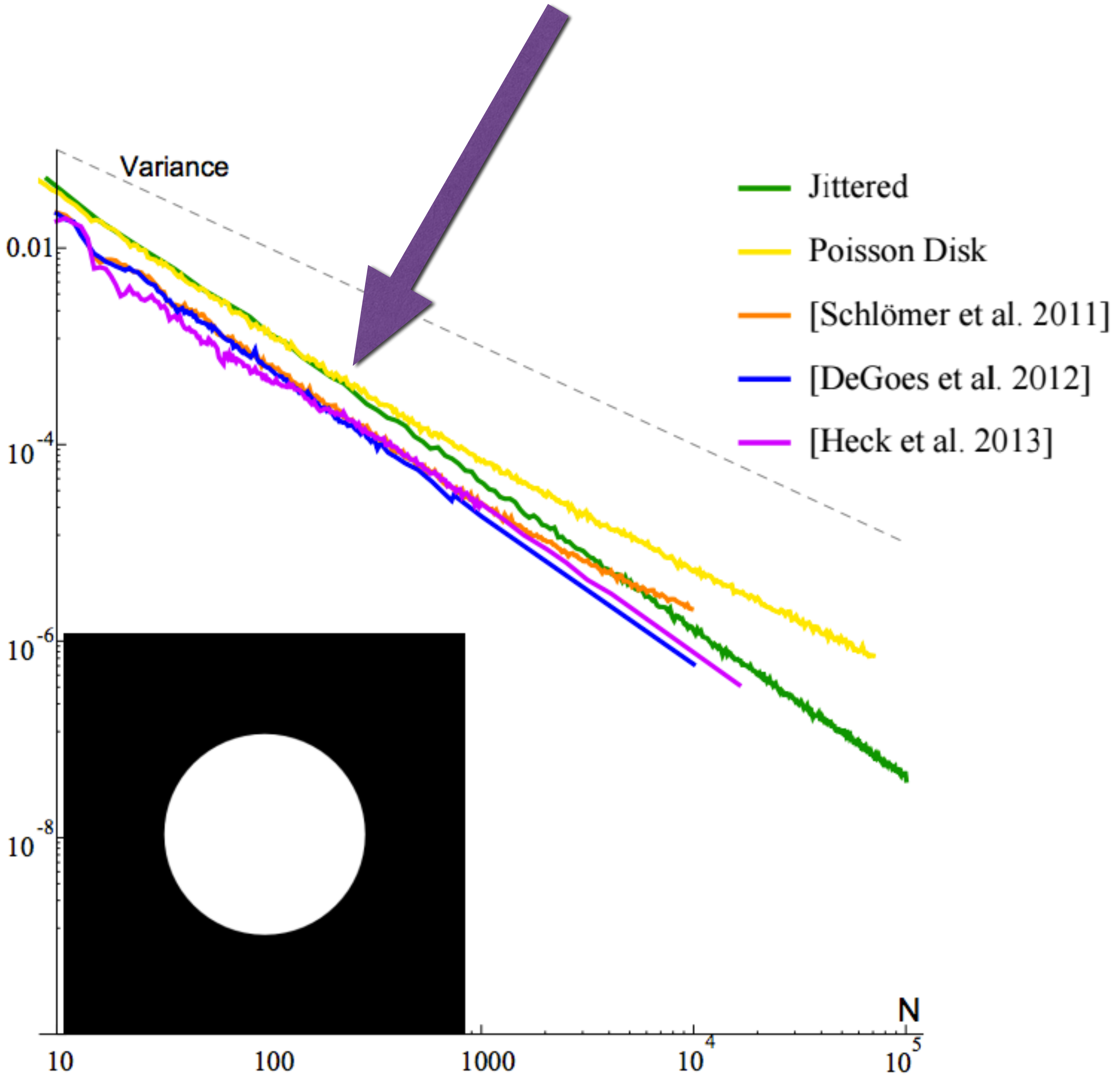
Disk Function as Worst Case



Disk Function as Worst Case



Disk Function as Worst Case



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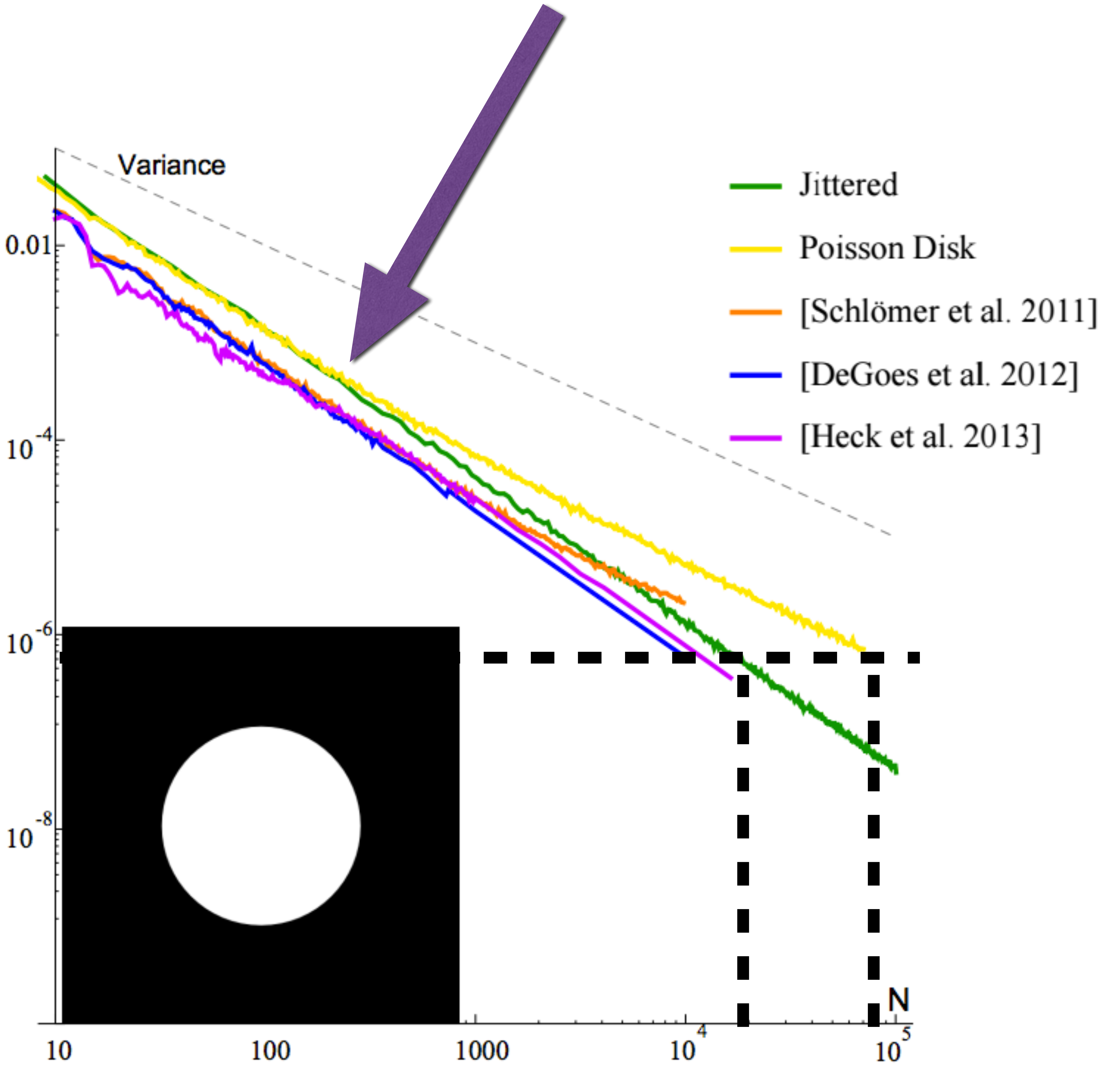
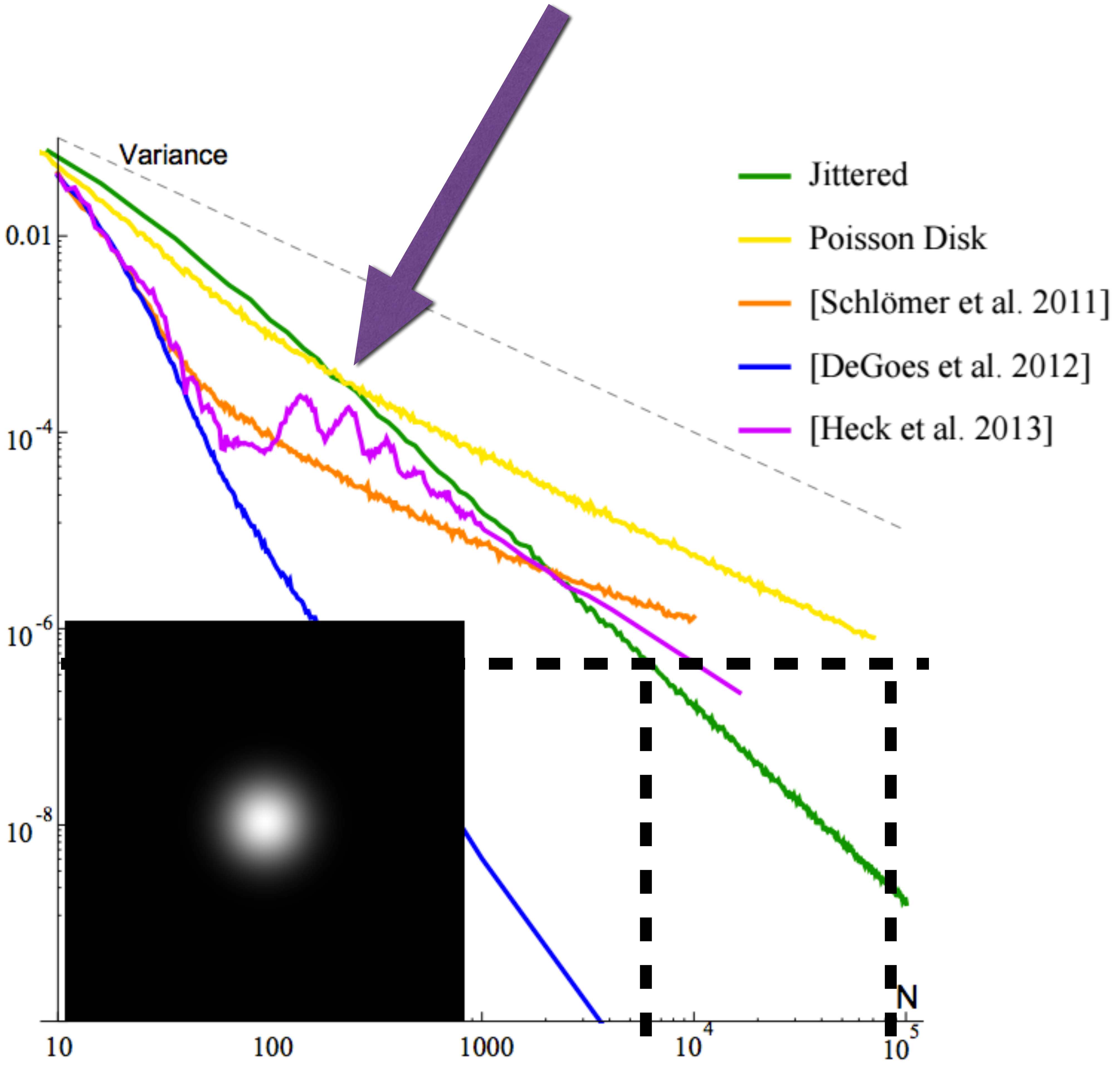
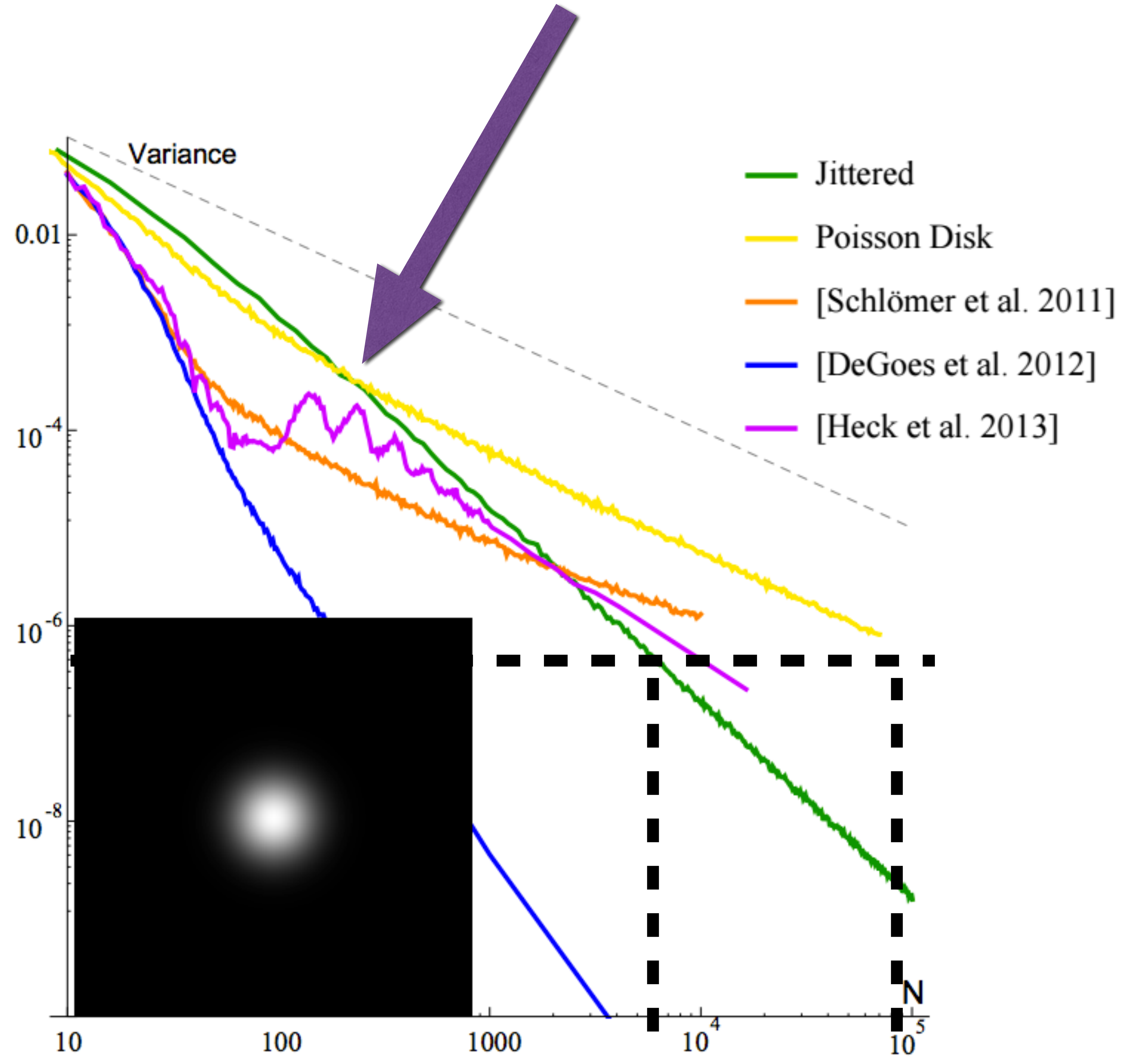
Disk Function as Worst Case



Gaussian as Best Case



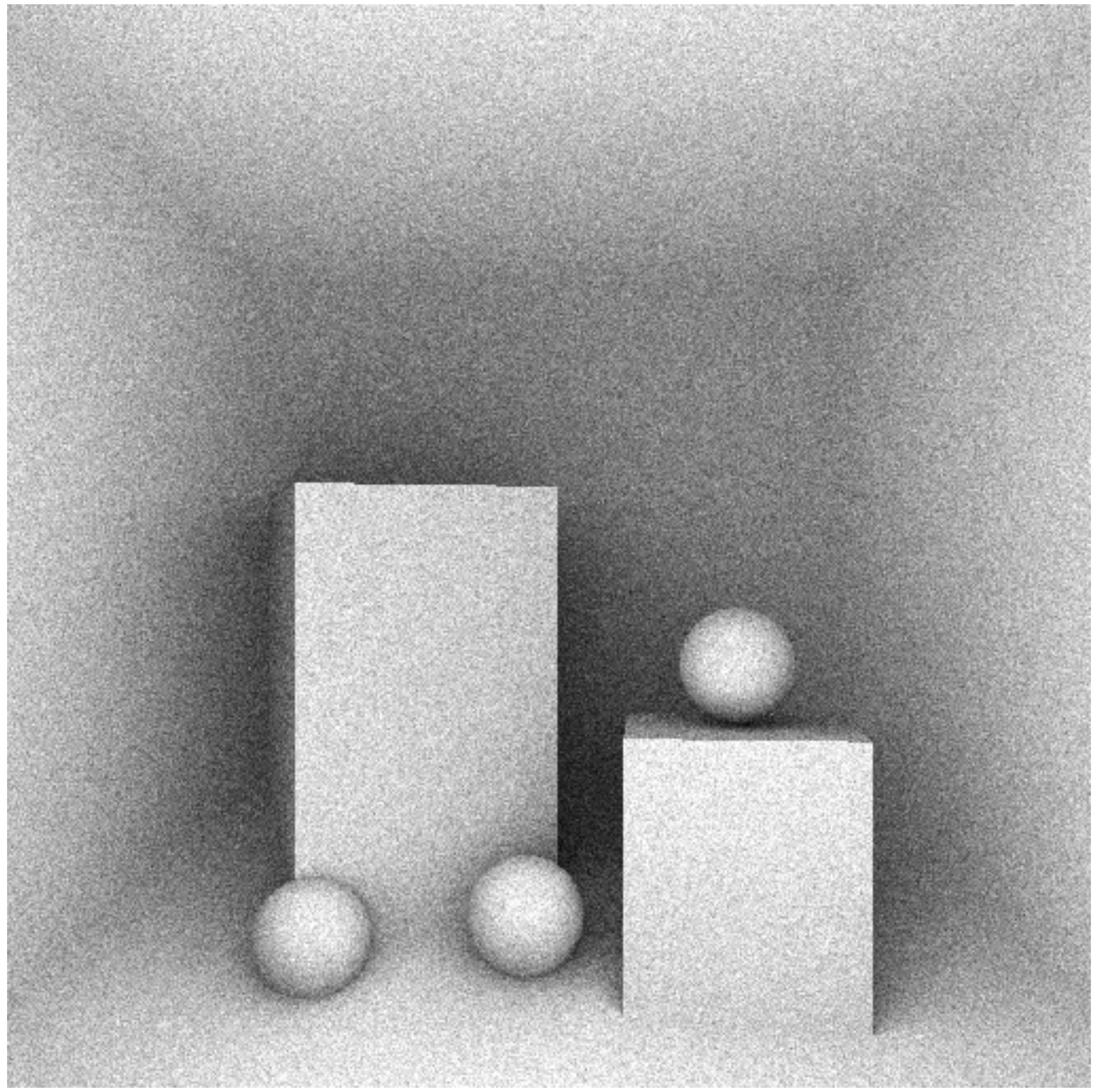
Gaussian as Best Case



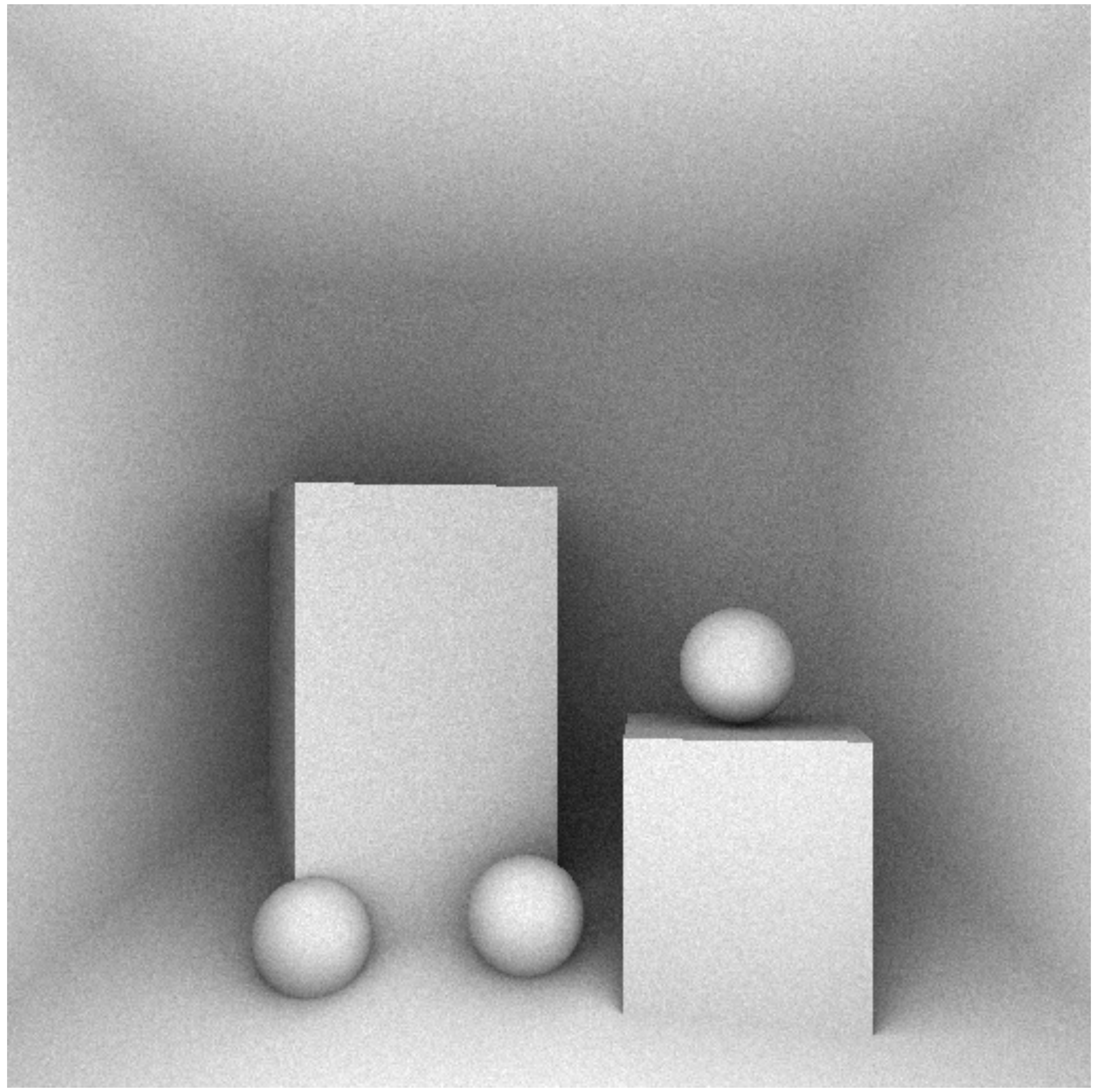
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



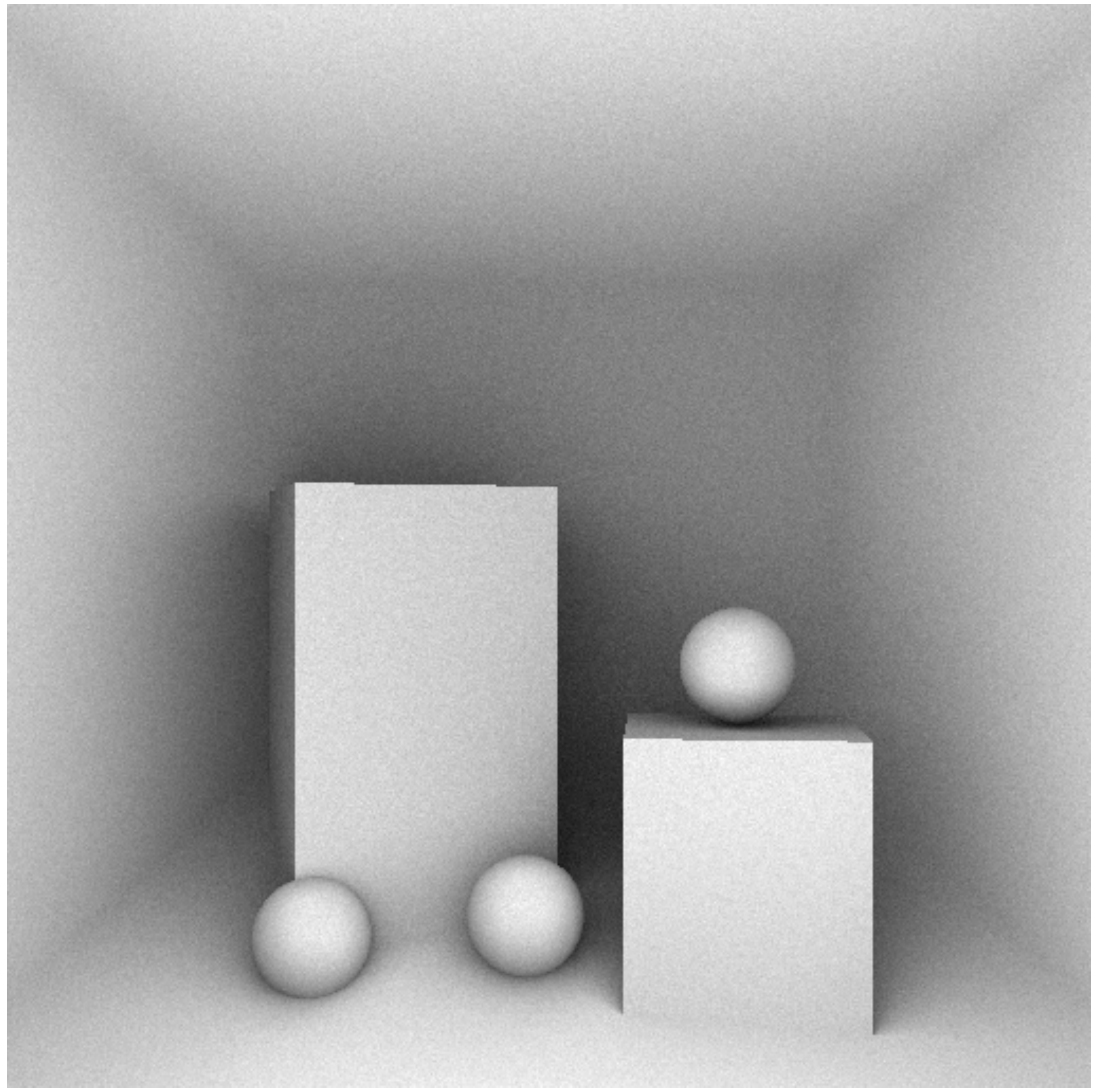
MSE: 4.74 x 10e-3



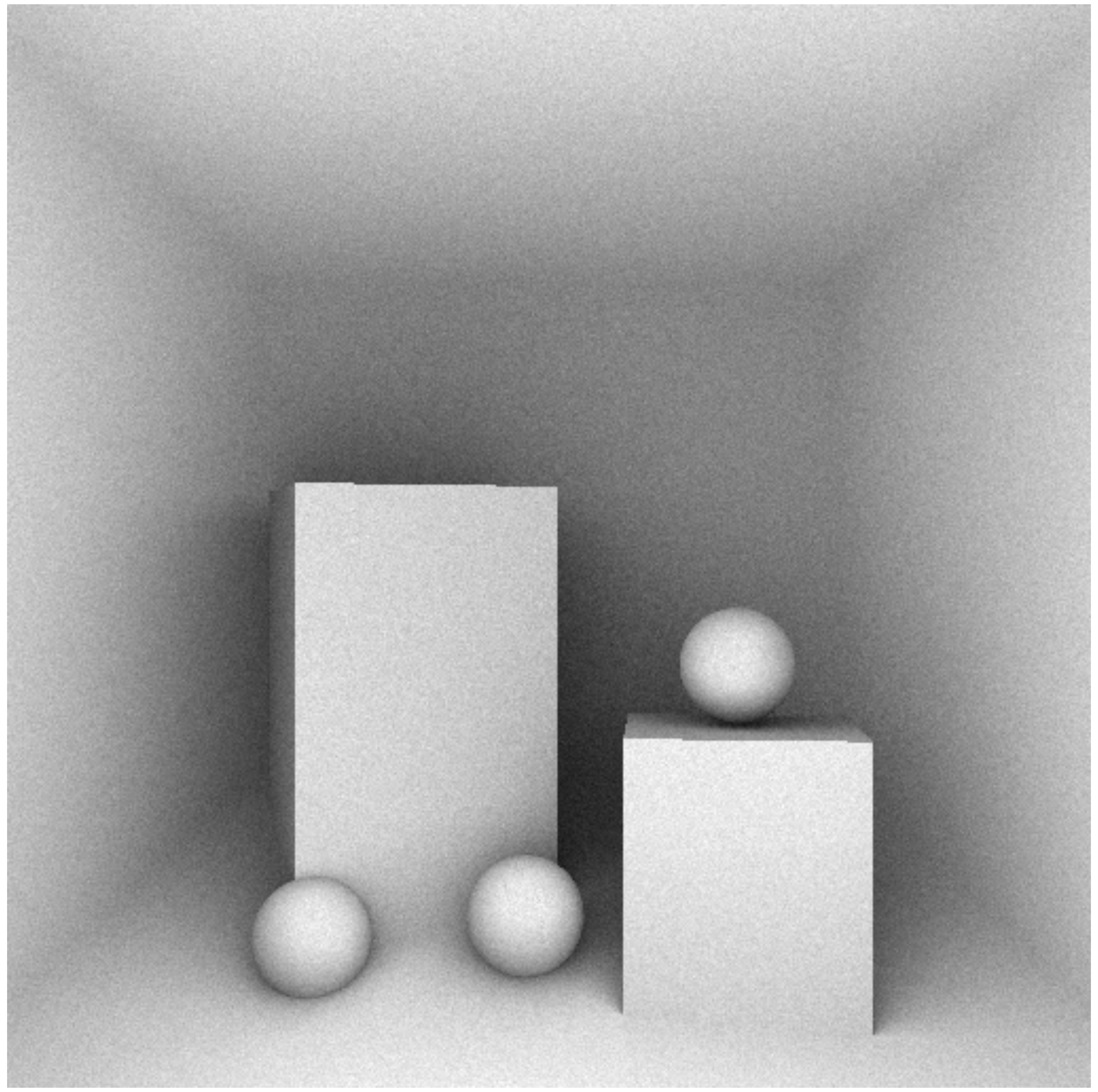
MSE: 8.56 x 10e-4

CCVT vs. Poisson Disk

96 Secondary Rays

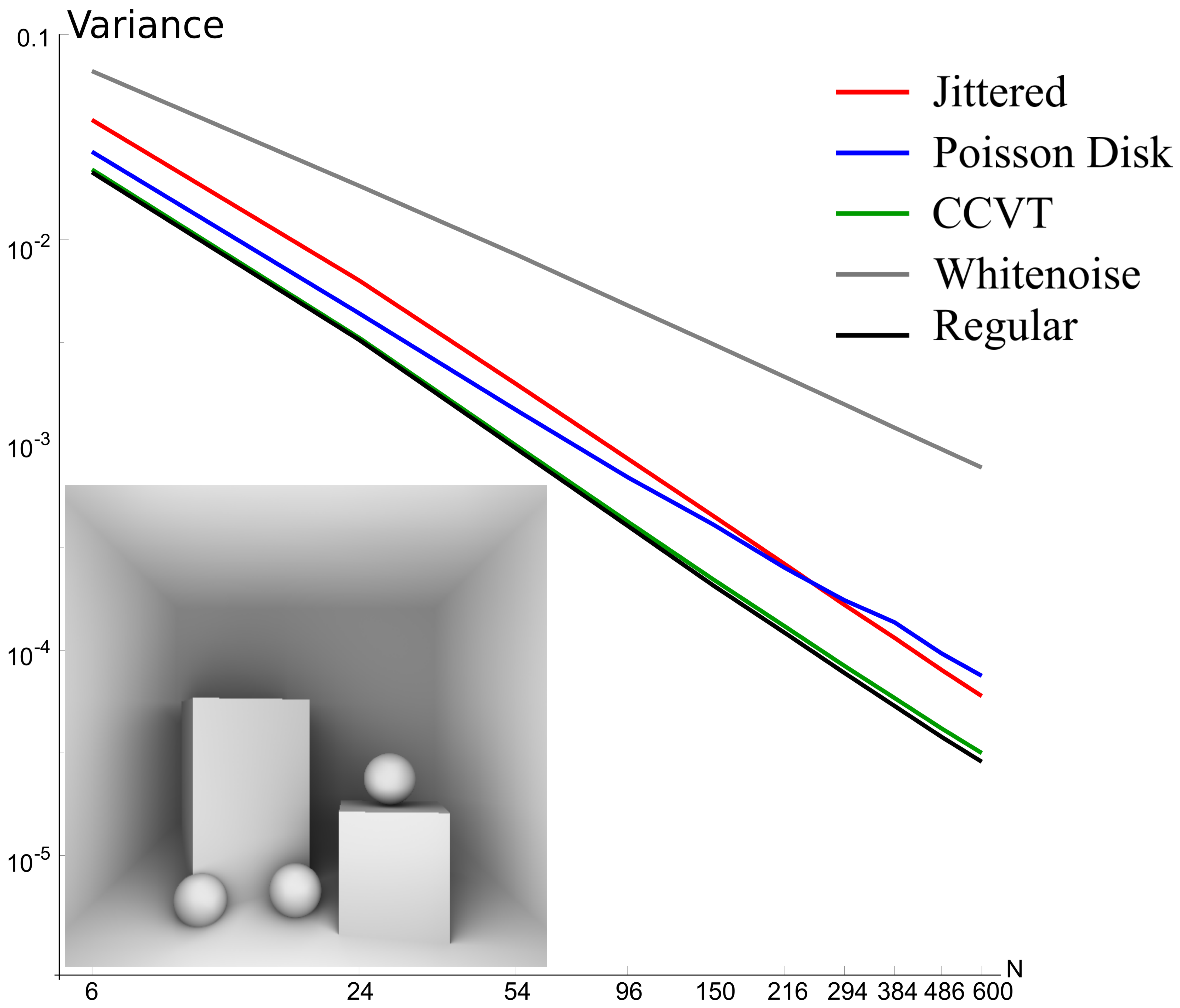


MSE: 4.24 x 10e-4

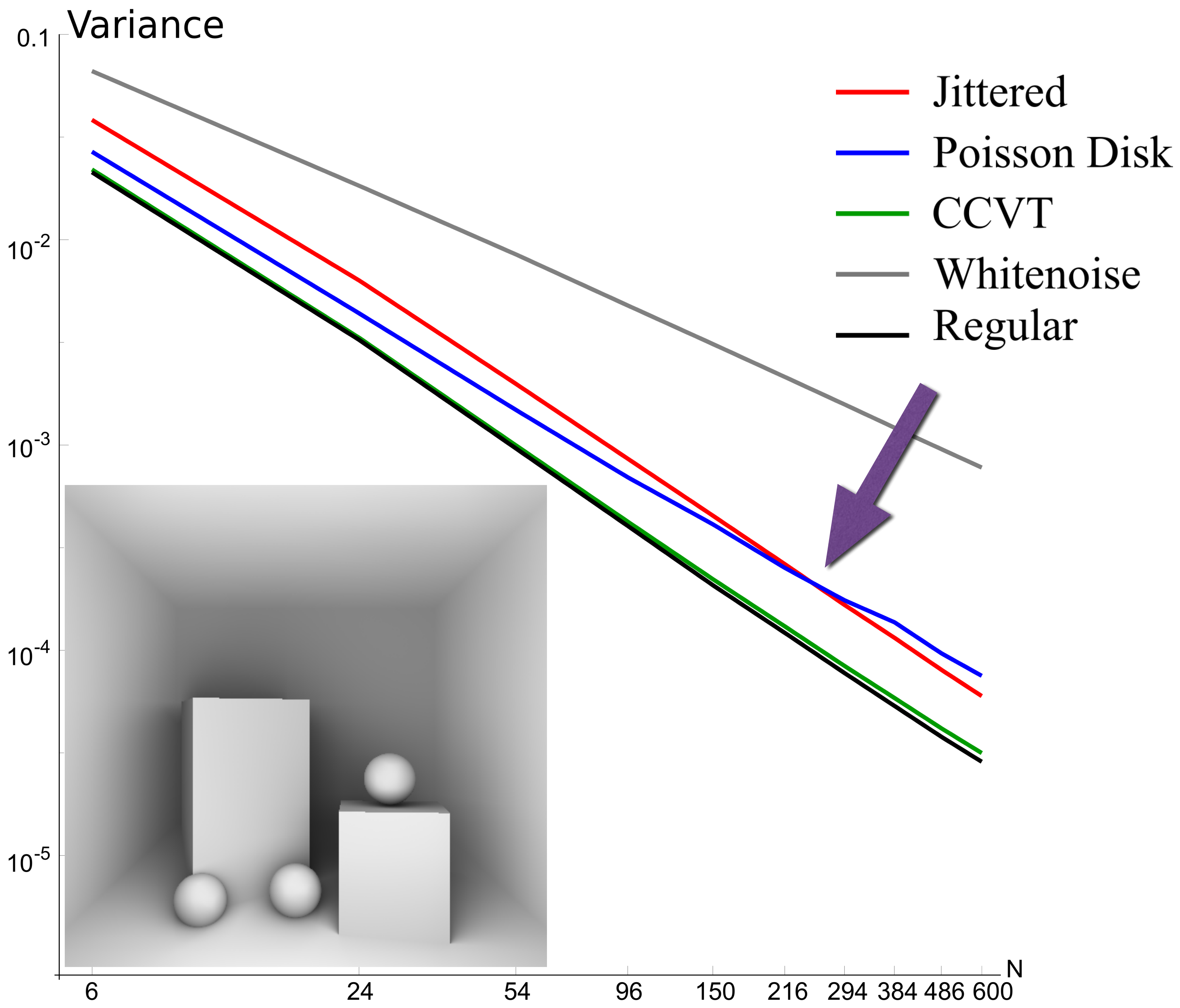


MSE: 6.95 x 10e-4

Convergence rates



Convergence rates



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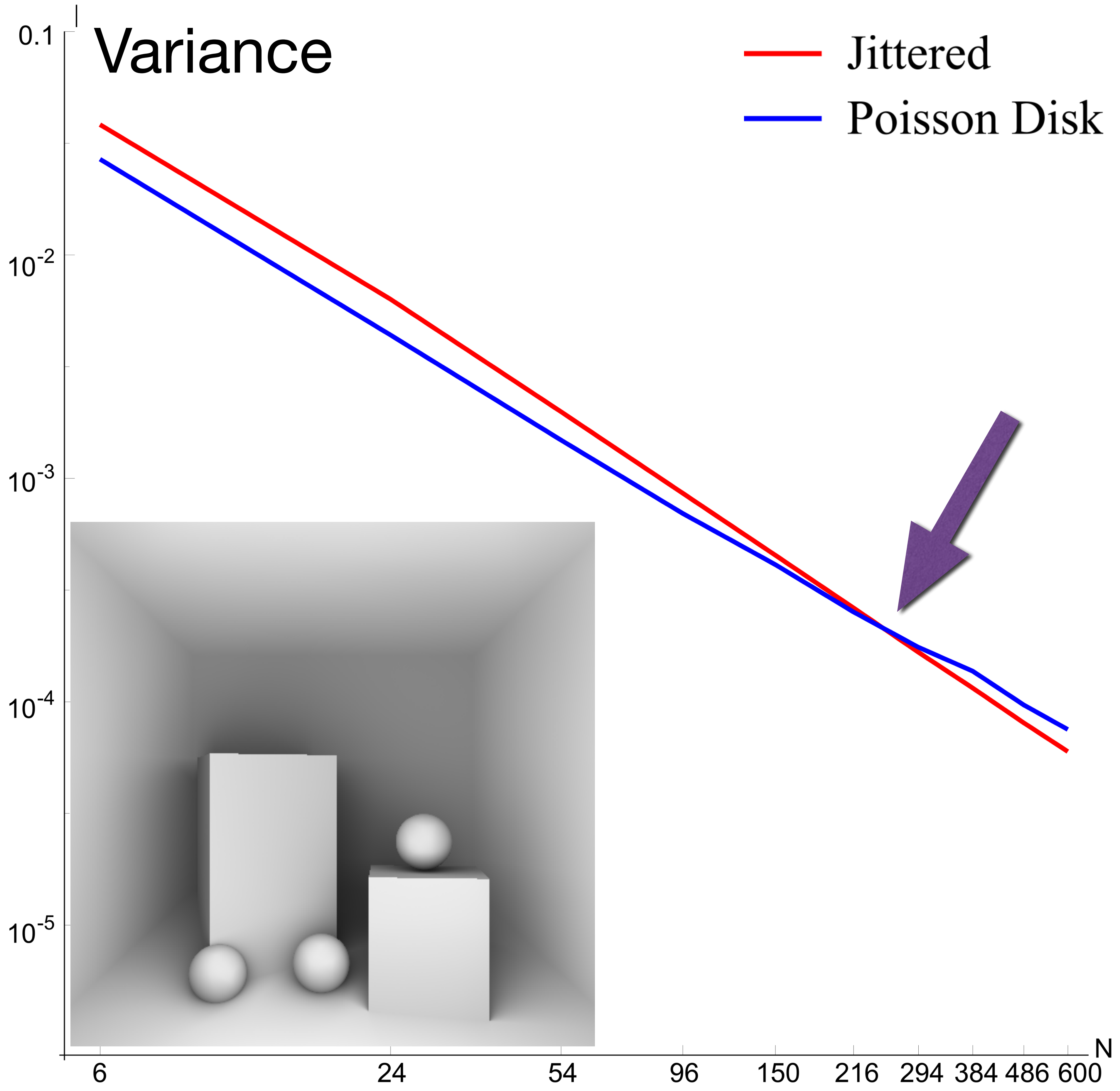
Jittered vs Poisson Disk



Render the Possibilities
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What are the benefits of this analysis ?

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- For offline rendering, analysis tells which samplers would converge faster.

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- For offline rendering, analysis tells which samplers would converge faster.
- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples