

Graphical Logic \Leftrightarrow Logical Graphs

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Redundancy in Logic

Each of the following statements are inter-derivable:

$$(p_1 \wedge p_2) \Rightarrow q \tag{1}$$

$$(p_2 \wedge p_1) \Rightarrow q \tag{2}$$

$$p_2 \Rightarrow (p_1 \Rightarrow q) \tag{3}$$

$$p_1 \Rightarrow (p_2 \Rightarrow q) \tag{4}$$

- **hard to lookup/apply results**
(need to have *exact* formulation of result)
- **Symmetries in proof search**
- Rich/interesting concept of **modularity/composability?**

Multiplicative Intuitionistic Linear Logic (MILL)

Rules for 1: $\frac{\Gamma \vdash C}{\Gamma, 1 \vdash C} \quad \overline{\{\} \vdash 1}$

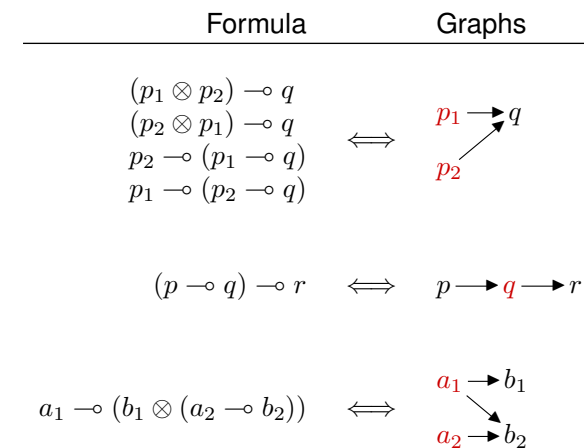
Rules for \otimes : $\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$

Rules for \multimap : $\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$

The cut rule: $\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$

We consider the \otimes/\multimap redundancy for MILL...

Logical Graphs - Informally



Logical Graphs - Formally

Labelling: A total surjective function $l_G : V_G \mapsto L_G$

Edge Relation: A binary relation, $E_G \subset (V_G \times V_G)$

$C(x)$ = conclusions of x and $A(x)$ = assumptions of x

- **Strictness:** the transitive closure of the relation forms a strict partial order.
- Well formed (**Stratified**):

$$A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y)$$

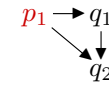
- Well formed (**Formula-Shaped**):

$$C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \vee C(y) \subset C(x))$$

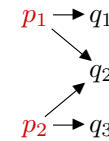
- Graphs are considered **upto vertex-isomorphism**.

Not Formula/Logical Graphs

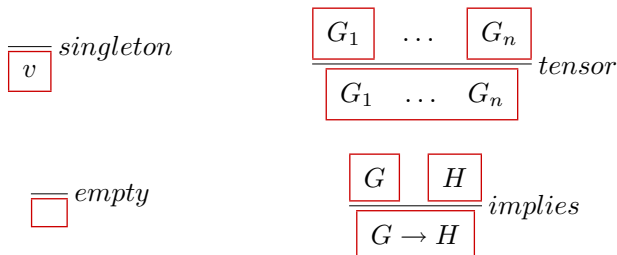
Stratified, $A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y)$, **not:**



Formula-shaped: $C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \vee C(y) \subset C(x))$, **not:**

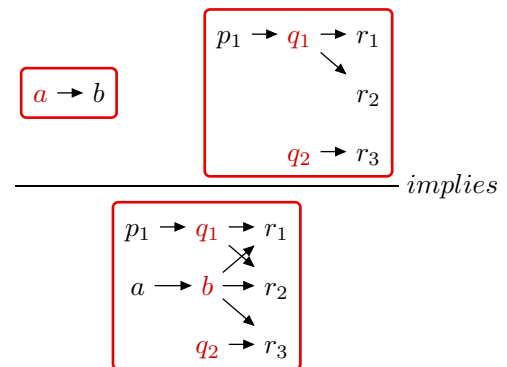


Well-Formed Logical Graphs, Graphically



- $G \rightarrow H$ = draw edges from every conclusion of G to every conclusion of H .
- Does not compare too badly to BNF: $F ::= v \mid F \otimes F \mid F \multimap F$

Example Construction



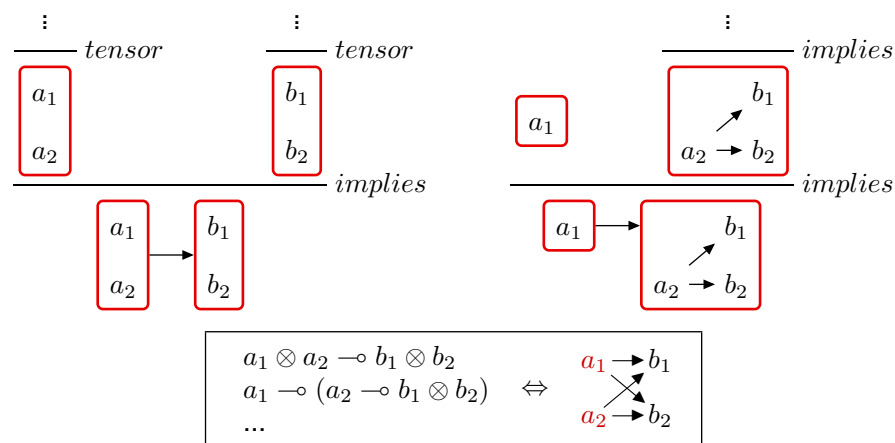
Graphs \Leftrightarrow Formulas

$$\begin{aligned} \llbracket v \rrbracket_f &= v \\ \llbracket G_1 \dots G_n \rrbracket_f &= \otimes_i (\llbracket G_i \rrbracket_f) \\ \llbracket G \rightarrow H \rrbracket_f &= \llbracket G \rrbracket_f \multimap \llbracket H \rrbracket_f \end{aligned}$$

$$\begin{aligned} \llbracket v \rrbracket_g &= v \\ \llbracket F_1 \otimes F_2 \rrbracket_g &= \llbracket F_1 \rrbracket_g \otimes \llbracket F_2 \rrbracket_g \\ \llbracket F_1 \multimap F_2 \rrbracket_g &= \llbracket F_1 \rrbracket_g \multimap \llbracket F_2 \rrbracket_g \end{aligned}$$

Lemma: graphs provide a normal-form; $\llbracket \llbracket G \rrbracket_f \rrbracket_g = G$;
Formulas are in bijective correspondence to graph-(de)compositions.

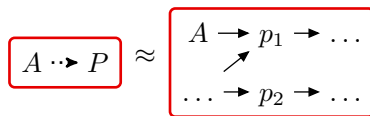
Graphs \Leftrightarrow Formulas (Examples)



More Graphical Notation

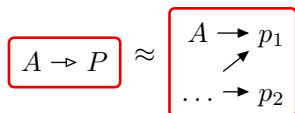
A is an *assumption subgraph*: $G = \boxed{A \multimap P}$

$$\Leftrightarrow (A \cap P = \emptyset) \wedge (A \subset G) \wedge (P \subset G) \wedge (\exists s \subset V_p. (A \rightarrow s) \cup P = G)$$

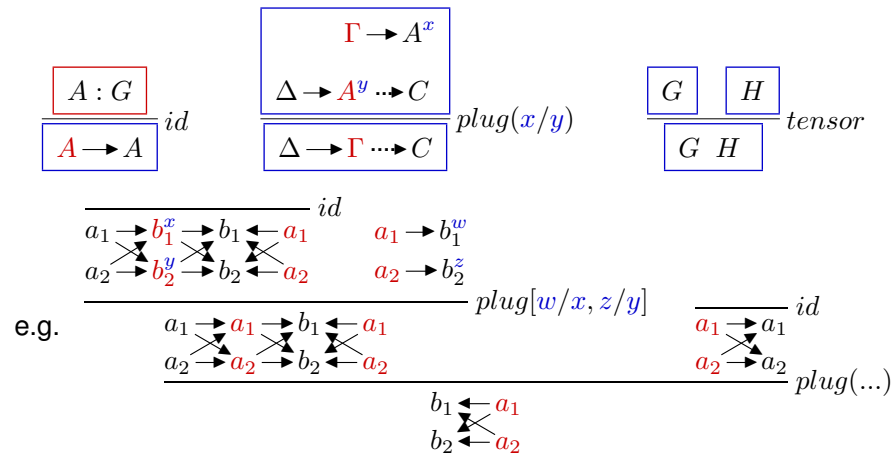


A is a *direct assumption subgraph*: $G = \boxed{A \rightarrow P}$

$$\Leftrightarrow (A \cap P = \emptyset) \wedge (A \subset G) \wedge (P \subset G) \wedge (\exists s \subset C(P). (A \rightarrow s) \cup P = G)$$

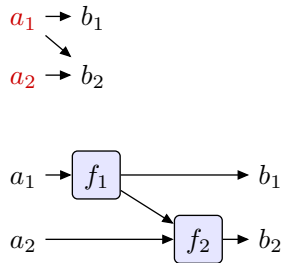


Graphical Proof



Proof Terms?

- Proof terms are symmetric monoidal categories



- plug = composition; id = id; tensor = tensor

Properties

Sound and complete: using basic schemes w.r.t. MILL

Deduction Theorem: $\frac{A}{B} \Leftrightarrow \overline{A \rightarrow B}$

Symmetric monoidal category have equivalences by graph-isomorphisms.

Higher-order graphs: $\frac{\forall X}{_ \rightarrow _} = \boxed{X \rightarrow X}$

Summary

- **Logical graphs** absorbing \otimes / \multimap symmetry; \approx MILL, allows deep inference.
- **Related work:** Games and Proof-nets for ILL (polarity): Lamarche'08; Categorical DP for MILL: Soloviev'02; but nothing *higher-order*
- ... **lots lots more to do...** schemes/definitions with higher-order graphs:

