Graphical Logic ⇔ Logical Graphs

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Redundancy in Logic

Each of the following statements are inter-derivable:

$$(p_1 \wedge p_2) \Rightarrow q \quad (1)$$
$$(p_2 \wedge p_1) \Rightarrow q \quad (2)$$
$$p_2 \Rightarrow (p_1 \Rightarrow q) \quad (3)$$
$$p_1 \Rightarrow (p_2 \Rightarrow q) \quad (4)$$

- **hard to lookup/apply results**
  (need to have *exact* formulation of result)
- **Symmetries in proof search**
- Rich/interesting concept of **modularity/composability**?

Multiplicative Intuitionistic Linear Logic (MILL)

Rules for 1: $\dfrac{\Gamma \vdash C}{\Gamma,\ 1 \vdash C}$ $\qquad \dfrac{}{\{\} \vdash 1}$

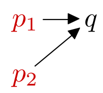
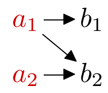
Rules for $\otimes$: $\dfrac{\Gamma,\ A,\ B \vdash C}{\Gamma,\ A \otimes B \vdash C}$ $\qquad \dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma,\ \Delta \vdash A \otimes B}$

Rules for $\multimap$: $\dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma,\ \Delta,\ A \multimap B \vdash C}$ $\qquad \dfrac{\Gamma,\ A \vdash B}{\Gamma \vdash A \multimap B}$

The cut rule: $\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma,\ \Delta \vdash B}$

We consider the $\otimes/\multimap$ redundancy for MILL...

Logical Graphs - Informally

Formula		Graphs
$(p_1 \otimes p_2) \multimap q$ $(p_2 \otimes p_1) \multimap q$ $p_2 \multimap (p_1 \multimap q)$ $p_1 \multimap (p_2 \multimap q)$	$\Longleftrightarrow$	
$(p \multimap q) \multimap r$	$\Longleftrightarrow$	
$a_1 \multimap (b_1 \otimes (a_2 \multimap b_2))$	$\Longleftrightarrow$	

Logical Graphs - Formally

Labelling: A total surjective function $l_G : V_G \mapsto L_G$

Edge Relation: A binary relation, $E_G \subset (V_G \times V_G)$
$C(x)$ = conclusions of x and $A(x)$ = assumptions of x

- **Strictness:** the transitive closure of the relation forms a strict partial order.
- Well formed (**Stratified**):

$$A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y)$$

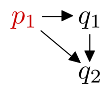
- Well formed (**Formula-Shaped**):

$$C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \vee C(y) \subset C(x))$$

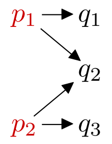
- Graphs are considered **upto vertex-isomorphism**.

Not Formula/Logical Graphs

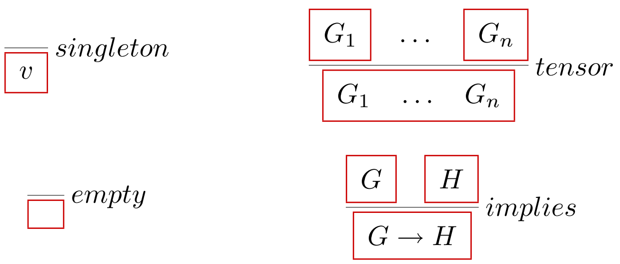
Stratified, $A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y)$, **not:**



Formula-shaped: $C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \vee C(y) \subset C(x))$, **not:**

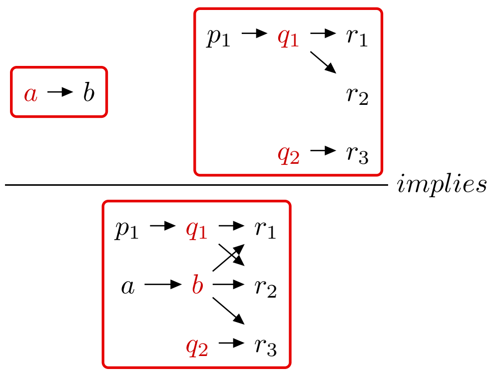


Well-Formed Logical Graphs, Graphically



- $G \rightarrow H$ = draw edges from every conclusion of G to every conclusion of H.
- Does not compare too badly to BNF: `F ::= v | F` $\otimes$ `F | F` $\multimap$ `F`

Example Construction



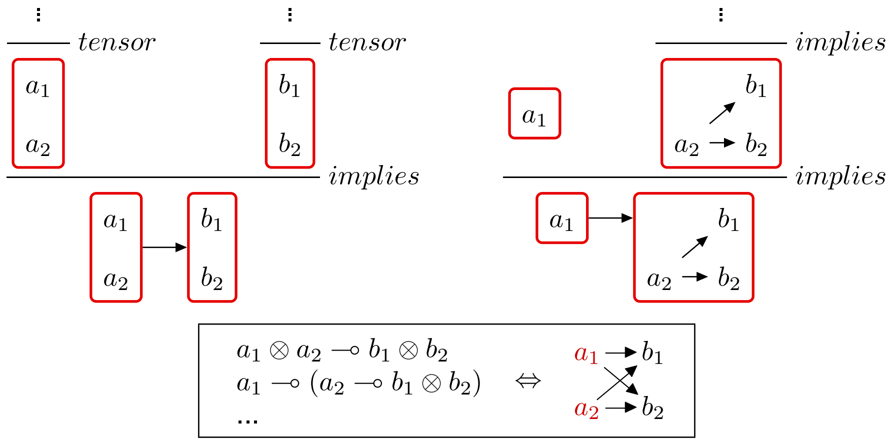
Graphs ⇔ Formulas

$$[\![v]\!]_f = v$$
$$[\![G_1 \dots G_n]\!]_f = \otimes_i([\![G_i]\!]_f)$$
$$[\![G \rightarrow H]\!]_f = [\![G]\!]_f \multimap [\![H]\!]_f$$

$$[\![v]\!]_g = v$$
$$[\![F_1 \otimes F_2]\!]_g = [\![F_1]\!]_g \quad [\![F_2]\!]_g$$
$$[\![F_1 \multimap F_2]\!]_g = [\![F_1]\!]_g \rightarrow [\![F_2]\!]_g$$

Lemma: graphs provide a normal-form; $[\![[\![G]\!]_f]\!]_g = G$;
Formulas are in bijective correspondence to graph-(de)compositions.

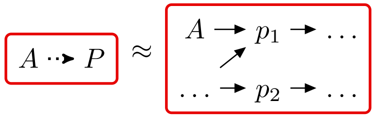
Graphs ⇔ Formulas (Examples)



More Graphical Notation

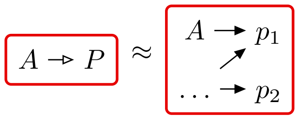
A is an *assumption subgraph*: $G = A \dashrightarrow P$

$\Longleftrightarrow \ (A \cap P = \emptyset) \wedge (A \subset G) \wedge (P \subset G) \wedge (\exists s \subset V_p.\ (A \rightarrow s) \cup P = G)$

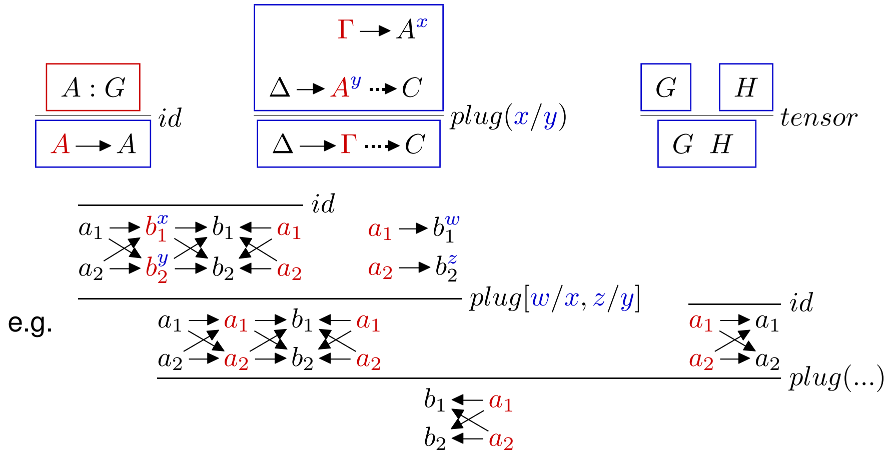


A is a *direct assumption subgraph*: $G = A \rightarrow P$

$\Longleftrightarrow \ (A \cap P = \emptyset) \wedge (A \subset G) \wedge (P \subset G) \wedge (\exists s \subset C(P).\ (A \rightarrow s) \cup P = G)$

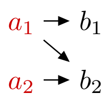


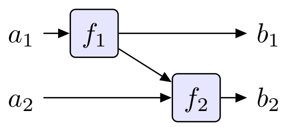
Graphical Proof



Proof Terms?

- Proof terms are symmetric monoidal categories

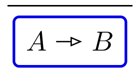




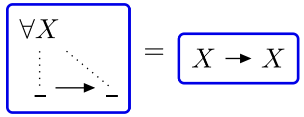
- plug = composition; id = id; tensor = tensor

Properties

Sound and complete: using basic schemes w.r.t. MILL

Deduction Theorem: $\frac{A}{B} \iff$ 

Symmetric monoidal category have equivalences by graph-isomorphisms.

Higher-order graphs: 

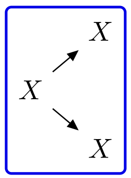
Summary

- **Logical graphs** absorbing $\otimes / \multimap$ symmetry; $\approx$ MILL, allows deep inference.

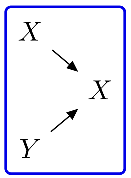
- **Related work**: Games and Proof-nets for ILL (polarity): Lamarche'08; Categocial DP for MILL: Soloviev'02; but nothing *higher-order*

- ... **lots lots more to do**... schemes/definitions with higher-order graphs:

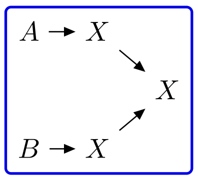
Contraction



Weakening



$A \oplus B$



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