Graphical Logic ⇔ Logical Graphs

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Redundancy in Logic

Each of the following statements are inter-derivable:

1. \((p_1 \land p_2) \Rightarrow q\)
2. \((p_2 \land p_1) \Rightarrow q\)
3. \(p_2 \Rightarrow (p_1 \Rightarrow q)\)
4. \(p_1 \Rightarrow (p_2 \Rightarrow q)\)

- hard to lookup/apply results
  (need to have exact formulation of result)
- Symmetries in proof search
- Rich/interesting concept of modularity/composability?

Multiplicative Intuitionistic Linear Logic (MILL)

Rules for 1:

\[
\Gamma, 1 \vdash C \\
\{ \} \vdash 1
\]

Rules for \(\otimes\):

\[
\begin{align*}
\Gamma, A, B & \vdash C \\
\Gamma, A \otimes B & \vdash C \\
\Gamma, A \Delta & \vdash B
\end{align*}
\]

Rules for \(\multimap\):

\[
\begin{align*}
\Gamma & \vdash A \\
\Delta, A \multimap B & \vdash C \\
\Gamma & \vdash A \Delta \multimap B
\end{align*}
\]

The cut rule:

\[
\begin{align*}
\Gamma & \vdash A \\
\Delta, A \multimap B & \vdash C \\
\Gamma, \Delta & \vdash B
\end{align*}
\]

We consider the \(\otimes/\multimap\) redundancy for MILL...

Logical Graphs - Informally

<table>
<thead>
<tr>
<th>Formula</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p_1 \otimes p_2) \multimap q)</td>
<td>((p_1 \otimes q) \multimap q) (p_1 \rightarrow q) and (p_2 \rightarrow q)</td>
</tr>
<tr>
<td>(p_1 \multimap (p_1 \Rightarrow q))</td>
<td>(p_2 \multimap (p_2 \Rightarrow q))</td>
</tr>
<tr>
<td>((p \Rightarrow q) \Rightarrow r)</td>
<td>(p \Rightarrow q \Rightarrow r)</td>
</tr>
<tr>
<td>(a_1 \Rightarrow (b_1 \otimes (a_2 \Rightarrow b_2)))</td>
<td>(a_1 \rightarrow b_1) and (a_2 \rightarrow b_2)</td>
</tr>
</tbody>
</table>
Logical Graphs - Formally

Labelling: A total surjective function $l_G : V_G \mapsto L_G$

Edge Relation: A binary relation, $E_G \subset (V_G \times V_G)$

- $C(x) = \text{conclusions of } x$ and $A(x) = \text{assumptions of } x$
- Strictness: the transitive closure of the relation forms a strict partial order.
- Well formed (Stratified):
  \[ A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y) \]
- Well formed (Formula-Shaped):
  \[ C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \lor C(y) \subset C(x)) \]
- Graphs are considered up to vertex-isomorphism.

Not Formula/Logical Graphs

Stratified, $A(x) \cap A(y) \neq \emptyset \Rightarrow C(x) = C(y)$, not:

Formula-shaped: $C(x) \cap C(y) \neq \emptyset \Rightarrow (C(x) \subset C(y) \lor C(y) \subset C(x))$, not:

Well-Formed Logical Graphs, Graphically

- $\nu$ singleton
- $G_1 \ldots G_n$ tensor
- $G_1 \ldots G_n$ empty
- $G \rightarrow H$ implies

- $G \rightarrow H$ = draw edges from every conclusion of $G$ to every conclusion of $H$.
- Does not compare too badly to BNF: $F ::= \nu \mid F \otimes F \mid F \rightarrow F$

Example Construction

\[ a \rightarrow b \]

\[ p_1 \rightarrow q_1 \rightarrow r_1 \]
\[ q_2 \rightarrow r_2 \]
\[ p_2 \rightarrow q_3 \rightarrow r_3 \]
More Graphical Notation

A is an assumption subgraph: $G = A \rightarrow P$

$\iff (A \cap P = \emptyset) \land (A \subset G) \land (P \subset G) \land (\exists s \subset V_p. \ (A \rightarrow s) \cup P = G)$

$A \rightarrow P \approx \frac{A}{\rightarrow} p1 \rightarrow \ldots \rightarrow p2 \rightarrow \ldots$

A is a direct assumption subgraph: $G = A \rightarrow P$

$\iff (A \cap P = \emptyset) \land (A \subset G) \land (P \subset G) \land (\exists s \subset C(P). \ (A \rightarrow s) \cup P = G)$

$A \rightarrow P \approx \frac{A}{\rightarrow} p1 \rightarrow \ldots \rightarrow p2$
Proof Terms?

- Proof terms are symmetric monoidal categories
  \[ \begin{array}{c}
  a_1 \rightarrow b_1 \\
  a_2 \rightarrow b_2 \\
  f_1 \rightarrow b_1 \\
  f_2 \rightarrow b_2
  \end{array} \]

- \text{plug} = \text{composition}; \text{id} = \text{id}; \text{tensor} = \text{tensor}

Properties

Sound and complete: using basic schemes w.r.t. MILL
\[
\frac{A}{B} \iff \begin{array}{c}
  A \\
  A \rightarrow B
  \end{array}
\]

Symmetric monoidal category have equivalences by graph-isomorphisms.

Higher-order graphs:
\[
\forall X = X \rightarrow X
\]

Summary

- **Logical graphs** absorbing $\otimes/\Rightarrow$ symmetry; $\approx$ MILL, allows deep inference.

- **Related work**: Games and Proof-nets for ILL (polarity): Lamarche’08;
  Categorial DP for MILL: Soloviev’02; but nothing higher-order

- ... **lots lots more to do**... schemes/definitions with higher-order graphs:
  
  *Contraction*
  
  \[
  \begin{array}{c}
  X \\
  X
  \end{array}
  \]

  *Weakening*
  
  \[
  \begin{array}{c}
  X \\
  A \rightarrow X \\
  X \\
  B \rightarrow X
  \end{array}
  \]

  \[
  A \oplus B
  \]

  \[
  A \rightarrow X
  \]

  \[
  \ldots
  \]