The Finite Model Theory Toolbox of a Database Theoretician

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Finite Model Theory (FMT)

- The main object of study: logics over finite structures
- Foundational role in the study of the theory of relational databases:
  - relational database = finite relational structure
  - query languages are logic based
- Finite model theory was described as the *"backbone of database theory"* (Vianu, 1995)
- Connections work both ways: much of the motivation for finite model theory research came from databases.
- Many other applications in CS: verification, AI, constraint satisfaction, algorithms, complexity ...

Finite Model Theory (FMT)

- Mature field; 4 textbooks:
  1. Ebbinghaus-Flum *"Finite Model Theory"*, 1994
  2. Immerman *"Descriptive Complexity"*, 1999
  3. Libkin *"Elements of 'Finite Model Theory"*, 2004
  4. Grädel-Kolatis-Libkin-Marx-Spencer-Vardi-Venema-Weinstein *"Finite Model Theory and its Applications"*, 2007
- Several 'personal perspective' surveys:
  1. Fagin (ICDT 1990, LICS 2000)
  2. Kolaitis (LICS 2007)
- Well-developed subfields:
  1. Combinatorial games (Kolaitis PODS 1995)
  2. Logic for PTIME (Kolaitis ICDT 1995, Grohe LICS 2008)
  3. Descriptive complexity, algorithmic model theory, embedded finite models, etc etc
- Surveys they tell you how to prove results, what the main open problems are, etc.

FMT for a database theoretician: Key problems

1. Expressiveness of query languages
   - Limited expressiveness (e.g., relational calculus = first-order logic)
   - What is not expressible? When to add new language constructs?
   - Adding new constructs is not free — optimizations!
2. Complexity of query languages
   - Do we know the complexity of query evaluation from the logical formalism?
3. Equivalence of query languages
   - Can we lower the complexity by changing the syntax?
   - Important in the study of languages for XML.
4. Satisfiability (usually, finite satisfiability)
   - Used in: static analysis, incomplete information.

FMT for a database theoretician cont'd

- Many database results involving FMT techniques were shown by people actively working in FMT.
- This was perhaps necessary in the early days of FMT.
- Now the field has built a large arsenal of tools.
- These tools can be used without knowing how they are proved!
- Most of them are actually quite easy to apply.
- They should be a part of the toolbox of every database theoretician.
- Our goal: present finite-model theory as such a toolbox.

Plan

- Expressiveness: Tools for first-order logic
- Expressiveness: Tools that work beyond first-order logic
- Language equivalence (better query evaluation via the composition method)
- Descriptive Complexity
- Satisfiability

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First-Order (FO)

- The most fundamental query language — first-order logic (FO)
- Database people often refer to it as relational calculus.
- The core of SQL (minus aggregation – we'll address it later).
- A basic question: what are the limitations of FO?
- Intuition: FO cannot express:
    1. nontrivial counting properties, and
    2. queries requiring recursion
- We'll now see some *"canonical"* examples of inexpressible queries.

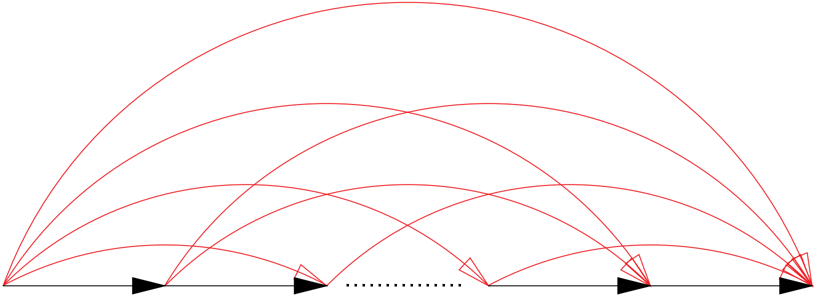
Even cardinality

- Active domain = set of all elements stored in a database
- A Boolean query

$$\text{EVEN}(D) = \text{true} \quad \Leftrightarrow \quad |\text{ActiveDomain(D)}| = 0 \ (mod 2)$$

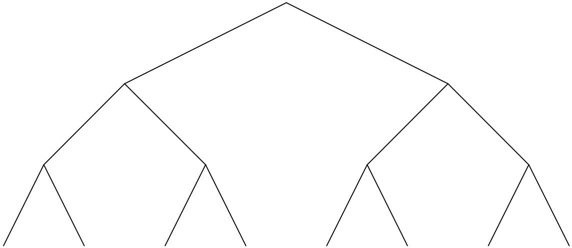
Transitive closure

Transitive closure



$$\begin{array}{lcl} trcl(x,y) & :- & e(x,y) \\ trcl(x,y) & :- & e(x,z), trcl(z,y) \end{array}$$

Same Generation



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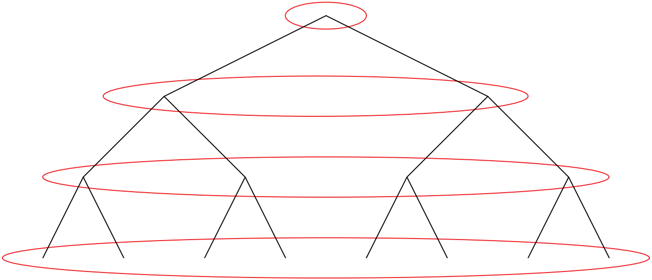

$$\begin{array}{lll} sg(x,x) & :- & \\ sg(x,y) & :- & e(x',x), e(y',y), sg(x',y') \end{array}$$

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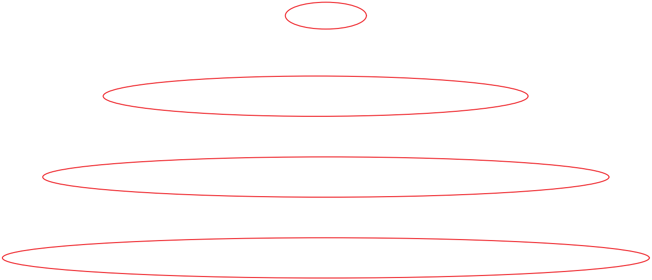


$$\begin{aligned} sg(x,x) &\ :- \\ sg(x,y) &\ :- \quad e(x',x), e(y',y), sg(x',y') \end{aligned}$$

A bit of history

- How to prove that these queries are not expressible in FO?
- Classical model theory offers us powerful tools, like compactness.
- They can be used to prove that EVEN is not FO-definable (also shown in the paper).
- They can also be used to prove that graph connectivity (and hence transitive closure) are not definable over arbitrary graphs.
- But the proof does not work for finite graphs: compactness fails in the finite.
- However, databases are finite!

A bit of history cont'd

- Fagin 1975: Transitive closure is not FO-expressible over finite graphs. Technique: games.
- Afterwards (1970s, 1980s): more and more advanced game proofs.
- Require nontrivial combinatorial arguments.
- A notable exception: 0-1 laws (Fagin 1976) – an easily applicable tool.
- 1990s: proper tools are being developed. They do not require complicated combinatorial proofs.

First-Order Logic (FO)

- Assumption: databases are graphs – one binary relation $E(\cdot,\cdot)$.

  Just to make things simple for the talk.
- Syntax of FO:
  - Atomic formulae: $E(x, y)$, $x = y$
  - Boolean combinations: $\varphi \vee \psi$, $\varphi \wedge \psi$, $\neg\varphi$
  - Quantification: $\exists x\ \varphi$, $\forall x\ \varphi$
  - $\varphi(\bar{x})$ means that $\bar{x}$ is the tuple of free variables of φ
- Quantifier rank $\mathrm{qr}(\varphi)$:
  - The depth of quantifier nesting in φ.
  - Example: the quantifier rank of $\exists x\ \big(\forall y\ E(x, y)\ \vee\ \forall z\ \neg E(x, z)\big)$ is 2 and not 3.

First-Order Logic – Examples

- There are at least k elements

$$\exists x_1 \ldots \exists x_k \bigwedge_{i,j \leq k,\ i \neq j} \neg(x_i = x_j)$$

- There is a path of length k from x_0 to x_k

$$\exists x_1 \ldots \exists x_{k-1} \bigwedge_{0 \leq i < k} E(x_i, x_{i+1})$$

- There is no cycle of length k

$$\neg \exists x_1 \ldots \exists x_k\ E(x_k, x_1) \wedge \bigwedge_{i<k} E(x_i, x_{i+1})$$

First-Order Logic – Examples

- There are at least k elements

  The number of elements is even – inexpressible.

- There is a path of length k from x_0 to x_k

  There is a path from x_0 to x_k – inexpressible.

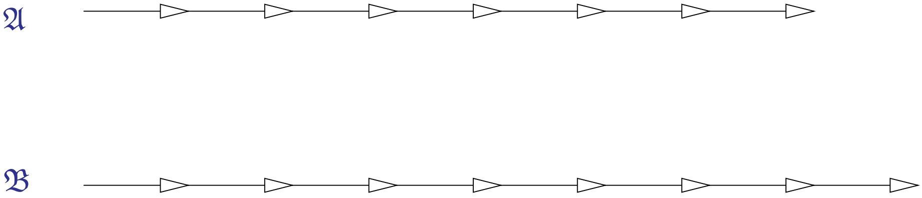
- There is no cycle of length k

  There is no cycle – inexpressible.

Ehrenfeucht-Fraï ssé games

- The tool of choice for the neolithic period of FMT.
- Played on two databases (graphs) $\mathfrak{A}$ and $\mathfrak{B}$.
- Two players:
  - Spoiler (the bad guy): tries to show that $\mathfrak{A}$ and $\mathfrak{B}$ are different.
  - Duplicator (the good guy): tries to show that $\mathfrak{A}$ and $\mathfrak{B}$ are the same.
- Play for k rounds.
- In each round:
  - the spoiler moves first: selects a database and an element there.
  - the duplicator responds by an element in the other database.
- The duplicator wins if at the end, the played elements form a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

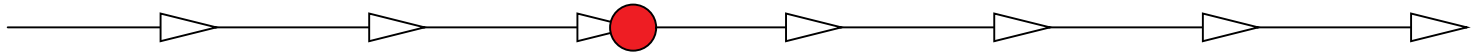
Ehrenfeucht-Fraï ssé game - example 1



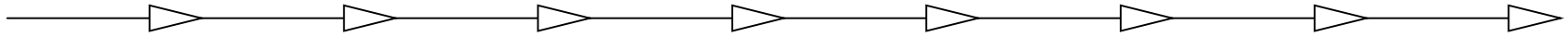
Spoiler and Duplicator play for 3 rounds.

Ehrenfeucht-Fraï ssé game - example 1

$\mathfrak{A}$



$\mathfrak{B}$



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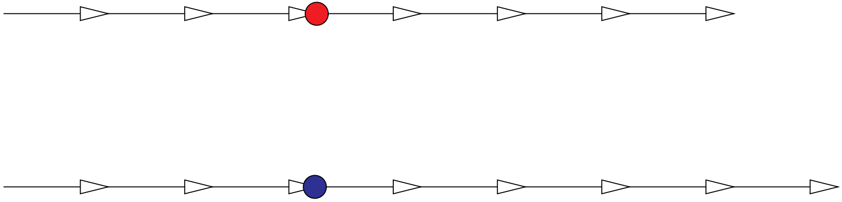

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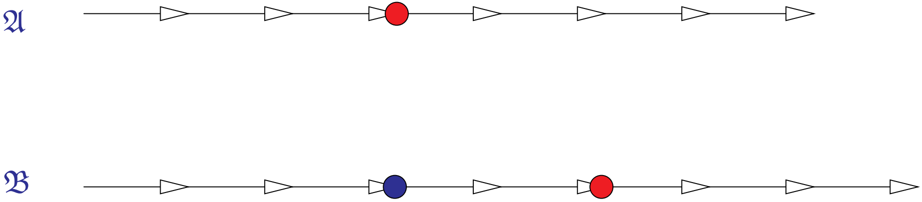

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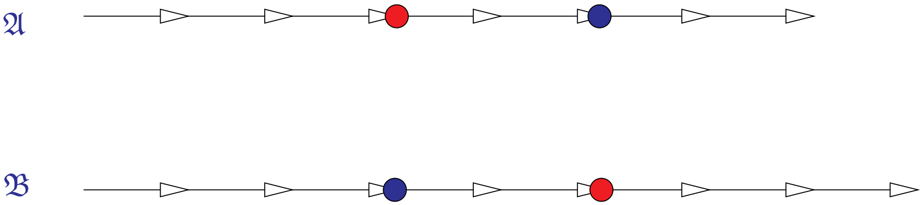

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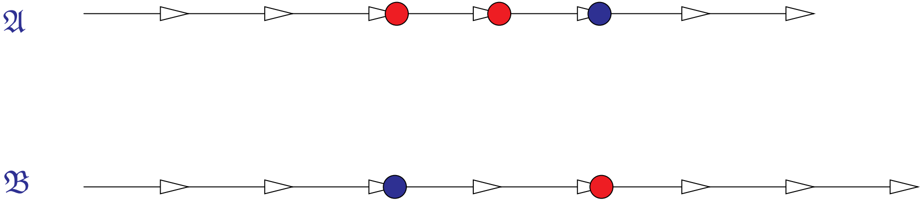

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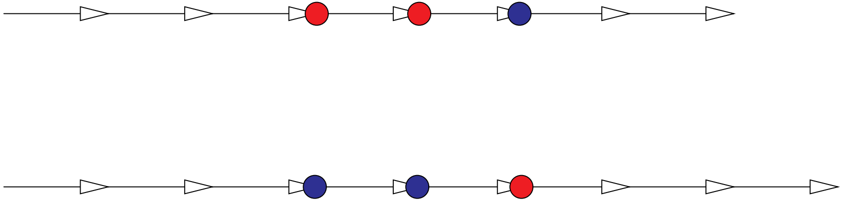

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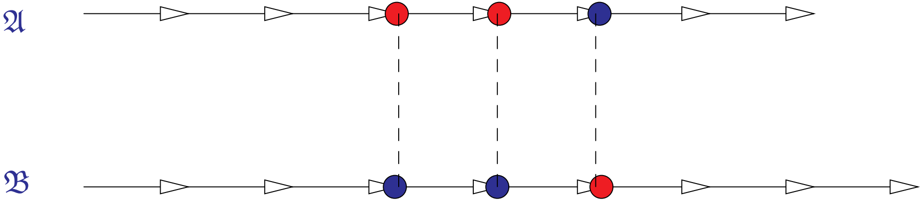

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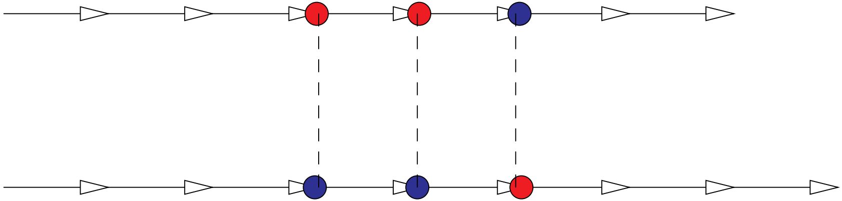
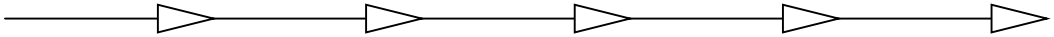


Spoiler and Duplicator play for 3 rounds.

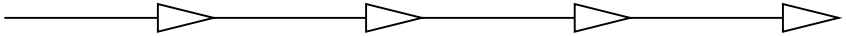
The duplicator wins in 3 rounds.

Ehrenfeucht-Fraï ssé game - example 2

$\mathfrak{A}$



$\mathfrak{B}$



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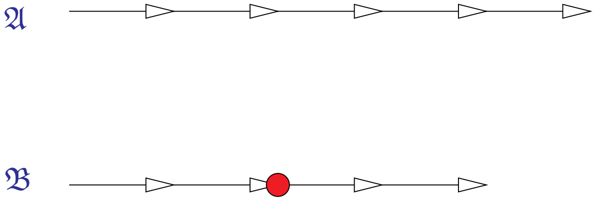

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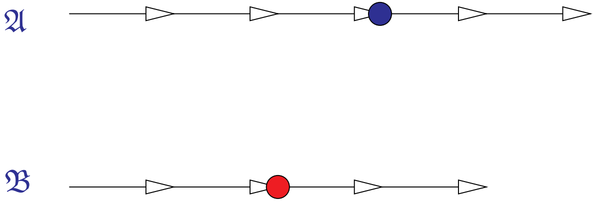

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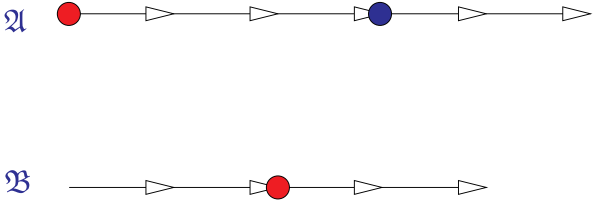

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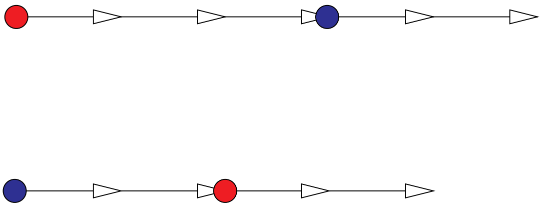

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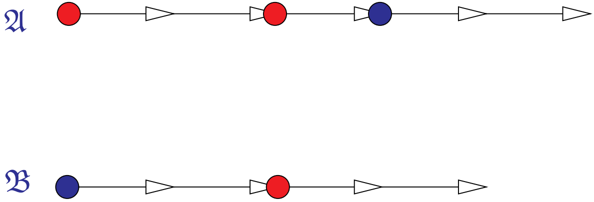

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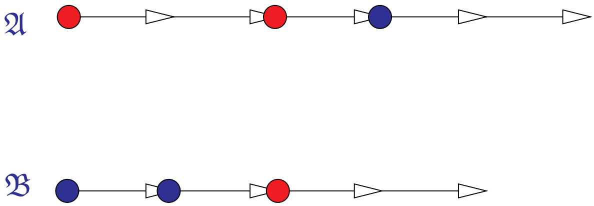

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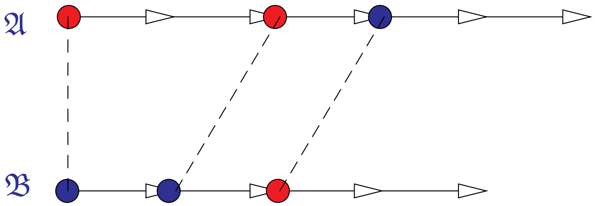

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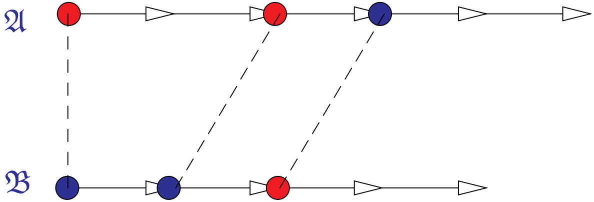


The spoiler wins in 3 rounds.

Ehrenfeucht-Fraï ssé games and FO

The duplicator has a winning strategy in the k-round Ehrenfeucht-Fraï ssé game if he can win in k rounds no matter how the spoiler plays.

Theorem

The duplicator has a winning strategy in the k-round Ehrenfeucht-Fraï ssé game on $\mathfrak{A}$ and $\mathfrak{B}$

$$\Leftrightarrow$$

$\mathfrak{A}$ and $\mathfrak{B}$ cannot be distinguished by FO sentences of quantifier rank up to k.

How to prove that a property $\mathcal{P}$ is not expressible in FO?

Find families of graphs $\mathfrak{A}_k$ and $\mathfrak{B}_k$, for $k \in \mathbb{N}$ so that:

1. All $\mathfrak{A}_k$ have property $\mathcal{P}$;
2. None of $\mathfrak{B}_k$ has property $\mathcal{P}$;
3. The duplicator has a winning strategy in the k-round Ehrenfeucht-Fraï ssé game on $\mathfrak{A}_k$ and $\mathfrak{B}_k$

If $\mathcal{P}$ were expressible by a sentence φ of quantifier rank k,

- $\mathfrak{A}_k$ and $\mathfrak{B}_k$ must agree on φ — by 3,
- but by 1 and 2 they disagree on φ.

A surprisingly powerful example

Let L_n be a linear ordering of length n.

> *Theorem*
>
> *If $m, n \geq 2^k$, then the duplicator has a winning strategy in the k-round Ehrenfeucht-Fraï ssé game on L_n and L_m.*

A surprisingly powerful example

Let L_n be a linear ordering of length n.

Theorem

If $m, n \geq 2^k$, then the duplicator has a winning strategy in the k-round Ehrenfeucht-Fraï ssé game on L_n and L_m.

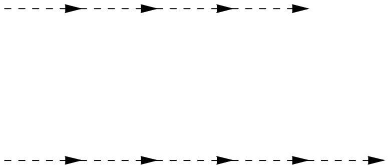
Corollary

Query EVEN *is not expressible over linear orderings.*

Because: take $\mathfrak{A}_k$ to be L_{2^k+1} and $\mathfrak{B}_k$ to be L_{2^k}.

The early 1980s: the era of tricks

The result about EVEN shows that none of the following is definable in FO: Graph connectivity; graph acyclicity; transitive closure.



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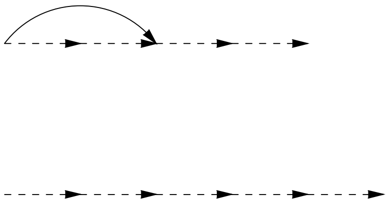

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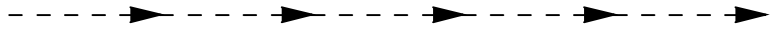

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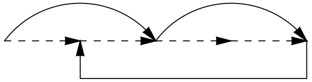

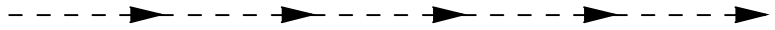

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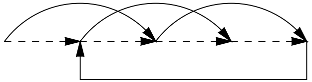

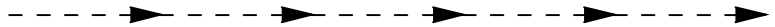

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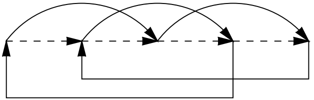

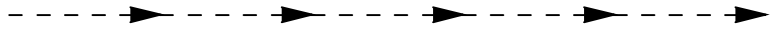

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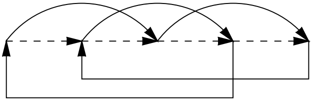

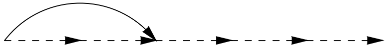

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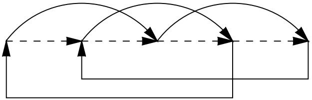

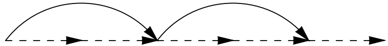

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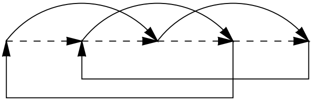

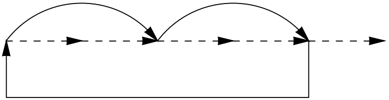

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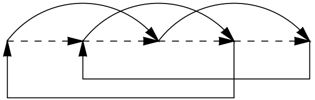

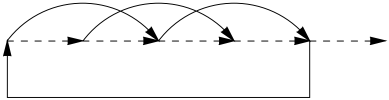

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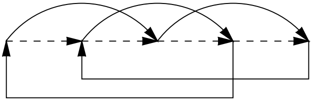

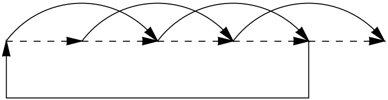

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The early 1980s: the era of tricks

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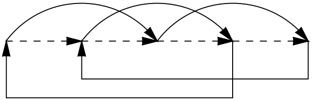

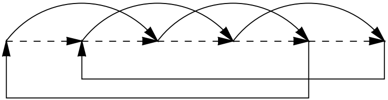

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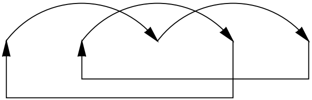

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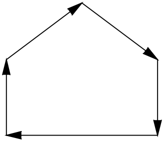

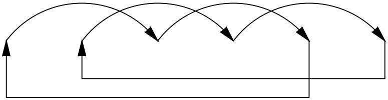

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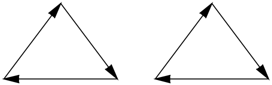

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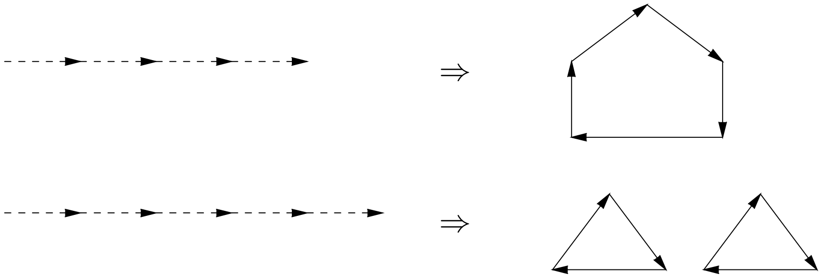


Connected if Even is false; disconnected if Even is true.

The era of tricks cont'd

Acyclicity is not FO-expressible:

- a similar trick – with one backedge instead of two.

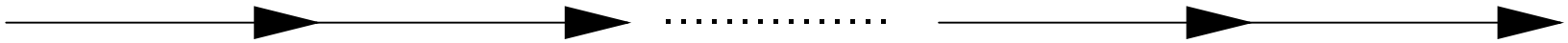
Transitive closure is not FO-expressible:

- the symmetric-transitive closure of G is a complete graph iff $G$is connected.

The 1990s: the era of tools

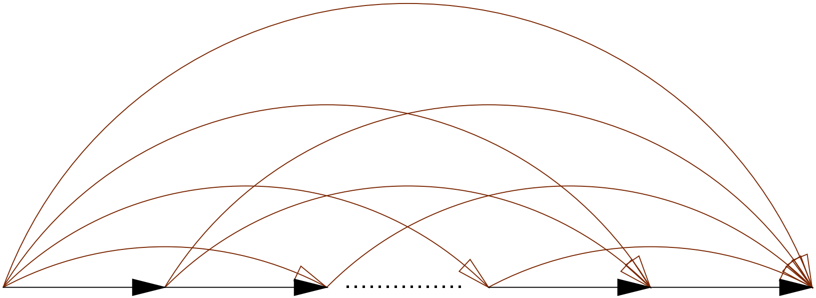
- The iron age of FMT.
- For more complicated problems, stone (pebble) tools – games – become very hard to use.
- More and more complicated winning conditions are used:
  - Fagin, Ajtai, Vardi, Stockmeyer, Schwentick, Kolaitis, Väänänen, etc
- Fagin, Stockmeyer, Vardi, 1993: Let's build a library of winning strategies for the duplicator.
- Key idea: locality.
  - Already present in earlier work by Gaifman 1980 and Hanf 1965.

Transitive closure revisited



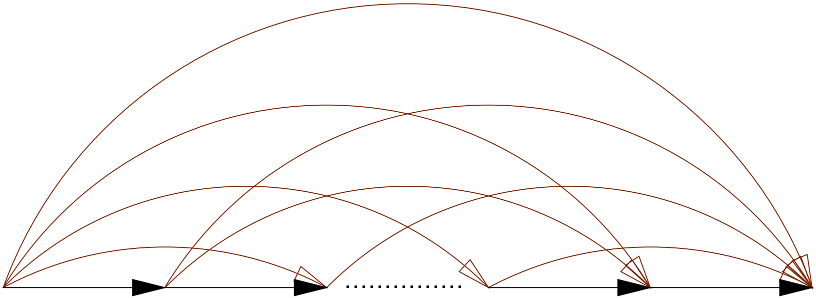
Degrees of nodes: $0, 1$

Transitive closure revisited



Degrees of nodes: $0, 1, \ldots, n$ – depends on the input.

Transitive closure revisited



Degrees of nodes: $0, 1, \ldots, n$ – depends on the input.

This cannot happen for FO queries!

A useful property: BNDP

- A query Q from graphs to graphs has the Bounded Number of Degrees Property if there is a function $f_Q : \mathbb{N} \to \mathbb{N}$ such that:

$$\boxed{\begin{array}{c}\text{all degrees in } G \text{ are bounded by } k \\ \Downarrow \\ \text{the number of different degrees in } Q(G) \text{ is at most } f_Q(k)\end{array}}$$

- We've just seen that transitive closure violates the BNDP.

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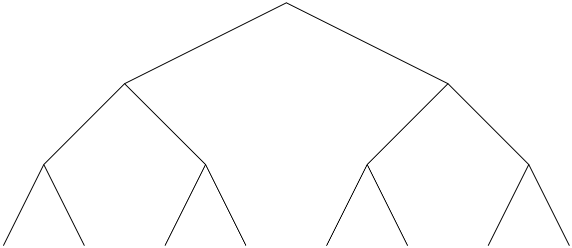
- We've just seen that transitive closure violates the BNDP.

> *Theorem (Dong, L., Wong'95, L. '97)*
>
> *Every FO query has the BNDP.*

- Corollary: transitive closure is not FO-definable.

Another application of BNDP – Same Generation



Degrees: $0, 1, 2$

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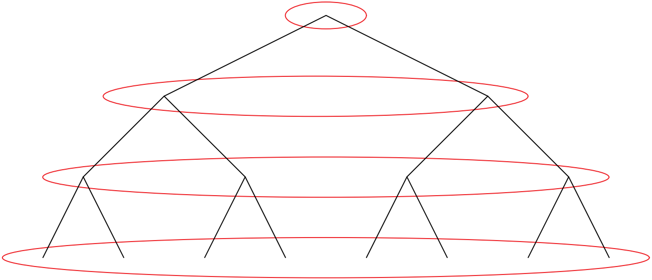

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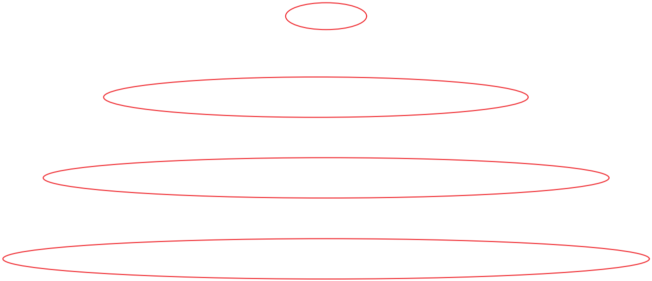
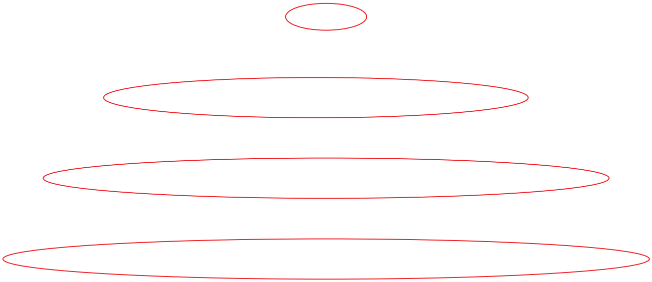


Degrees: $1, 2, 4, 8, \ldots, 2^{\mathsf{depth}(G)-1}$ Number of degrees: $\mathsf{depth}(G)$

Another application of BNDP – Same Generation



Violates the BNDP — Hence same-generation is not FO-definable

What makes the BNDP work?

- Locality of FO.
- There are two tools based on locality:
  1. Gaifman-locality (Gaifman 1982)
  2. Hanf-locality (Hanf 1965, Fagin-Stockmeyer-Vardi 1993)
- Key concept: neighborhood.
- A neighborhood of radius r of $\bar{a}$ in a graph G is denoted by $N_r^G(\bar{a})$.
- It is the subgraph induced by all the nodes of distance $\leq r$ from one of the nodes in $\bar{a}$.
- Nodes $\bar{a}$ are distinguished:
  - if we have an isomorphism $h : N_r^G(a_1, \ldots, a_n) \to N_r^{G'}(b_1, \ldots, b_n)$ then $h(a_1) = b_1, \ \ldots, \ h(a_n) = b_n$.

Gaifman-locality

Theorem (Gaifman 1982, bound from L., '98)

For every FO formula $\varphi(\bar{x})$ of quantifier rank k and for every graph G:

$$N_{2^k}^{G}(\bar{a}) \text{ and } N_{2^k}^{G}(\bar{b}) \text{ are isomorphic } \Rightarrow \; G \models \varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$$

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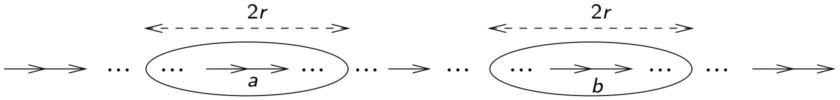
Application: Transitive closure is not definable in FO.



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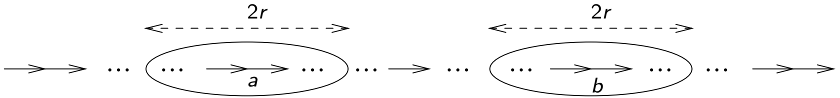


If $\varphi(x, y)$ is of quantifier rank k and $r = 2^k$ then both $\varphi(a, b)$ and $\varphi(b, a)$ are true, or both are false.

Hanf-locality

- Write $G \leftrightarrows_r G'$ if there exists a bijection $f : G \rightarrow G'$ such that $N_r^G(a)$ and $N_r^{G'}(f(a))$ are isomorphic for every node a.
- Locally two graphs look the same, up to a bijection f.

Hanf-locality

- Write $G \leftrightarrows_r G'$ if there exists a bijection $f : G \to G'$ such that $N_r^G(a)$ and $N_r^{G'}(f(a))$ are isomorphic for every node a.
- Locally two graphs look the same, up to a bijection f.

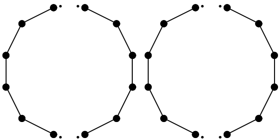
Theorem (Fagin, Stockmeyer, Vardi, '93, bound from L.'98)

For every FO sentence φ of quantifier rank k,

$$\text{if } G \leftrightarrows_{2^k} G' \text{ then } G \models \varphi \;\Leftrightarrow\; G' \models \varphi$$

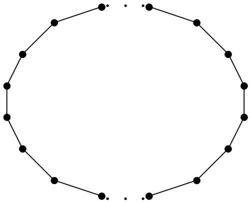
- Can be extended to arbitrary queries, but most often this notion is used for Boolean queries.

Hanf-locality: application



G

two cycles of length m



G'

one cycle of length $2m$

- If $m > 2r + 1$ then $G \leftrightarrows_r G'$ (all r-neighborhoods are the same).
- Hence no FO sentence φ defines connectivity: as long as $m > 2^{\mathsf{qr}(\varphi)+1} + 1$, graphs G and G' cannot be distinguished by φ.

Summary: locality notions

A query Q is:

- Hanf-local if there is $r \geq 0$ so that $G \leftrightarrows_r G'$ implies $Q(G) = Q(G')$.
  - for Boolean queries; a natural extension to non-Boolean queries exists.
- Gaifman-local if there is $r \geq 0$ such that if $N_r^G(\bar{a})$ and $N_r^G(\bar{b})$ are isomorphic, then $\bar{a} \in Q(G) \Leftrightarrow \bar{b} \in Q(G)$.
- has the BNDP if there is function $f_Q : \mathbb{N} \to \mathbb{N}$ so that for G with all degrees $\leq k$, the number of different degrees in $Q(G)$ is $\leq f_Q(k)$.

Theorem (L. '97)

Hanf-local $\Rightarrow$ *Gaifman-local* $\Rightarrow$ *BNDP.*

Corollary

FO queries are Hanf-local, Gaifman-local, and have the BNDP.

Counting: towards 0-1 laws

- How to prove that nontrivial counting properties are not expressible?
- Even: roughly half of databases have the property, and half don't.
- FO cannot exhibit such a behavior.

Counting: towards 0-1 laws

- How to prove that nontrivial counting properties are not expressible?
- EVEN: roughly half of databases have the property, and half don't.
- FO cannot exhibit such a behavior.

- Pick a database "at random".
- Check if it satisfies a property $\mathcal{P}$.
- What's the probability of that?
- If $\mathcal{P}$ is FO-definable, it is 0 or 1: 0-1 law.
- Need to formalize: 'pick a database at random".

Towards 0-1 laws

- For each n look at graphs with nodes $1, \ldots, n$.
- For a property $\mathcal{P}$, let

$$\mu_n(\mathcal{P}) = \frac{|\{\text{graphs on } 1, \ldots, n \text{ that satisfy } \mathcal{P}\}|}{|\{\text{graphs on } 1, \ldots, n\}|}$$

- Proportion of graphs on $1, \ldots, n$ satisfy $\mathcal{P}$, or
- Probability that a randomly picked graph on $1, \ldots, n$ — with respect to the uniform distribution — satisfies $\mathcal{P}$.
- Asymptotic probabilities:

$$\mu(\mathcal{P}) = \lim_{n \to \infty} \mu_n(\mathcal{P})$$

Asymptotic probabilities: examples

- $\mu(\text{Even})$ – does not exist: $\mu_n(\text{Even}) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$

- $\mu(\text{exists isolated node}) = 0.$ $\exists x \forall y\ \neg E(x, y)$
- $\mu(\text{diameter} \leq 2) = 1.$ $\forall x \forall y \exists z\ E(x, z) \wedge E(y, z)$
- $\mu(\text{graph is connected}) = 1.$

- Two sets A and B with $B \subseteq A$.
  Parity is true iff $|B|$ is even.
  $\mu(\text{Parity}) = \frac{1}{2}.$

0-1 law

Theorem (Fagin 1976)

If $\mathcal{P}$ is FO-definable, then $\mu(\mathcal{P})$ exists and equals 0 or 1.

0-1 law

> *Theorem (Fagin 1976)*
>
> *If $\mathcal{P}$ is FO-definable, then $\mu(\mathcal{P})$ exists and equals 0 or 1.*

- If you like truly beautiful proofs, this is the one for you!
- Immediate corollaries: EVEN and PARITY are not FO-definable.
- Warning: the result does not hold when we consider specific classes of structures.
- For example, 0-1 law fails over ordered graphs:
- μ(there is an edge between the first and the last element) $= \frac{1}{2}$.

Plan

- Expressiveness: Tools for first-order logic
- Expressiveness: Tools that work beyond first-order logic
- Language equivalence (better query evaluation via the composition method)
- Descriptive Complexity
- Satisfiability

FO extensions

- Ordering: elements stored in a database are typically ordered, and order comparisons can be used in queries.
- Counting and aggregation: we all know it from SQL; a very common feature in database queries.
- Fixed points: for many years a popular topic in database research (Datalog). Now also part of SQL-3.
- Interpreted operations: e.g., arithmetic operations such as $x^2 + y \leq x \cdot z$ in queries.

Ordering on the domain

- Can transitive closure be expressed over ordered graphs? What about connectivity? acyclicity? etc.
- We know that Even is not expressible.
- Queries such as transitive closure do not refer to ordering.
- Order-invariant queries: can use an ordering, but it does not matter which ordering is used.
- Order-invariant formulae over graphs: $\varphi(\bar{x})$ over $E(\cdot,\cdot),<$ so that

$$(G,<_1) \models \varphi(\bar{a}) \quad \Leftrightarrow \quad (G,<_2) \models \varphi(\bar{a})$$

  for every two orderings $<_1$ and $<_2$ on the nodes.
- Defines an order-invariant query Q_φ:

$$\bar{a} \in Q_\varphi(G) \quad \Leftrightarrow \quad (G,<) \models \varphi(\bar{a}) \text{ for some ordering } <$$

Order-invariant queries

- A mysterious class:
  - only makes sense in the finite;
  - a non-r.e. class of queries;
  - locality techniques do not seem to help: with $<$ everything is a neighborhood of radius 1.
- But quite remarkably:

Theorem (Grohe, Schwentick, 2000)

Order-invariant queries are Gaifman-local and have the BNDP.

- Corollary: Transitive closure, connectivity, etc are not expressible even with order.
- Hanf-locality for order-invariance: open (partial results by Niemistö).

Adding counting and aggregation to the language

- Standard SQL feature.
- Assume domain of 2 sorts:
  - usual database entries (graph nodes);
  - numbers (for examples, $\mathbb{Q}$).
- Add counting terms and operations:
  - $\#\bar{x}.\varphi$ – how many $\bar{x}$ satisfy φ.
  - $P_{\text{property}}(\cdot)$ testing the property of numbers.
- Examples:
  - $\exists x\ P_{\text{even}}(\#y.E(x,y))$ – there is a node of even degree.
  - Degree of x is (degree of y)^2:
    $P_{n=m^2}(\#z.E(x,y), \#z.E(y,z))$

Adding counting and aggregation to the language

- aggregates and grouping by example: sum up all even degrees in a graph
  - in SQL:
    ```
    SELECT SUM(R.C)
    FROM (SELECT E.A, COUNT(E.B) AS C
          FROM E
          GROUPBY E.A
          HAVING MOD(COUNT(E.B),2) = 0) R
    ```
  - in logic: $\mathrm{Aggr}_{SUM}\; x\; \big(P_{even}(\#y.E(x,y)),\;\; \#y.E(x,y)\big)$

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    ```
  - in logic: $\mathrm{Aggr}_{SUM}\ x\ \big(P_{even}(\#y.E(x,y)),\ \ \#y.E(x,y)\big)$
- Formally: $\mathcal{F}$ is an aggregate (e.g., SUM, COUNT...)
- $\mathrm{aggr\_term}(\bar{x}) \ = \ \mathrm{Aggr}_{\mathcal{F}}\bar{y}\ \big(\varphi(\bar{x},\bar{y}), t(\bar{x},\bar{y})\big)$
- Semantics:
  - Find all $\bar{y}_1,\ldots,\bar{y}_k$ so that $\varphi(\bar{x},\bar{y}_i)$ holds
  - Calculate $v_i = t(\bar{x},\bar{y}_i)$
  - $\mathrm{aggr\_term}(\bar{x})$ is $\mathcal{F}(\{\!|v_1,\ldots,v_k|\!\})$

Expressiveness of aggregation

- Question: which arithmetic predicates and which aggregate functions to add?
- Let's be generous: add them all.
- But still look at queries over graph nodes (e.g., transitive closure).

> *Theorem (Hella, L., Nurmonen, Wong'99, improved L.'01)*
>
> *Queries expressed in the aggregate language with arbitrary arithmetic and aggregates are local:*
> *i.e., Hanf-local, Gaifman-local, and have the BNDP.*

- In particular, the usual SQL (select-from-where-groupby-having) cannot express transitive closure.

Aggregation and order

- What if we have an order on graph nodes? Can we recover locality?
- No, even in a minimalistic setting:
  - Arithmetic: $<, +, \times$
  - Aggregation: SUM
- If such an aggregate language cannot express transitive closure over ordered graphs, then some complexity classes are separated:
  - TC^0 and NLOGSPACE
  - big open problem in complexity theory

Recursion and Datalog

- Have seen it already:
    - transitive closure:

$$\begin{array}{rcl} trcl(x,y) & :- & e(x,y) \\ trcl(x,y) & :- & e(x,z), trcl(z,y) \end{array}$$

    - same-generation:

$$\begin{array}{rcl} sg(x,x) & :- & \\ sg(x,y) & :- & e(x',x), e(y',y), sg(x',y') \end{array}$$

- Now available in the latest SQL standard: WITH RECURSIVE.
    - But without negation.
    - With negation, several semantics exist.

Datalog: expressive power

- Without negation, queries are monotone.
- Even with negation and inflationary semantics:

> *Theorem (Blass, Kozen, Gurevich, 1985)*
>
> *Datalog has the 0-1 law.*

- This is without order. What if order is added?
- Then Datalog (with negation) captures PTIME.
- To prove bounds, one needs to separate complexity classes again.
- But without order, it can be separated from NP: 3-colorability is not expressible in Datalog with negation (Dawar, '98).
  - A useful result (recent application in the work on schema mappings)

Extensions: summary

First-Order:

- cannot do recursive (fixed-point) queries;
- cannot count; this continues to hold with the order on the domain.

Extensions:

- with Counting/aggregation:
  - cannot do fixed-point queries
- with fixed-points:
  - cannot count
- But only without ordering on the domain:
  - with ordering, bounds on the as hard as separating complexity classes

Complex constraints

- Graph nodes: numbers. Query "does a graph lie on a circle?":

$$\exists r \exists a \exists b\, \forall x \forall y\ E(x, y) \rightarrow (x-a)^2 + (y-b)^2 = r^2$$

- What is the power of such extensions? Can graph connectivity be expressed?
- Look at queries that talk about proper graph properties (formally, isomorphism types of graphs). Known as generic queries.
- The answer depends on the class of numbers and arithmetic operations.
- Graph connectivity is expressible over $\mathbb{N}$ with arithmetic $0, 1, +, \times, <$ (easy) but is not expressible over $\mathbb{R}$ with arithmetic (much harder; Benedikt, L., '98).
- Results come from the field of embedded FMT: finite structures (graphs) living inside infinite ones (e.g., the real ordered field).
    - Survey: Chap. 5 of "Finite Model Theory and its Applications".

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- Expressiveness: Tools for first-order logic
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Language equivalence: games come back

- The focus of FMT applications in databases switches in the 21st century from inexpressibility results to proving language equivalence.
- Goal: start with a benchmark of expressiveness, and find a language with good complexity of query evaluation.
- Usually in the context of data that comes with a nice structure.
- XML: labeled trees with some nodes carrying data values.

For the talk, use words:

- to keep pictures and notations simple;
- the paper deals with trees as well.

Language equivalence: games come back

For words/trees, benchmark expressiveness is typically MSO.

- Monadic Second Order Logic — adds quantification over sets to FO: $\exists X_1 \forall X_2 \ldots \ \varphi(X_1, X_2, \ldots)$ where φ is FO.
- Same expressiveness as automata.
- But problematic complexity of query evaluation.

Question: is it possible to achieve better complexity simply by syntactic manipulations?

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Yes! Key technique: composing Ehrenfeucht-Fraï ssé games.

Such composition tells us how queries on substructures combine for evaluating queries on the whole structure.

Changing syntax to lower complexity: LTL

Syntax: $\varphi \;:=\; a \;\; (\in \Sigma) \;\mid\; \varphi \vee \varphi' \;\mid\; \neg\varphi \;\mid\; \mathbf{X}\varphi \;\mid\; \varphi \mathbf{U} \varphi'$

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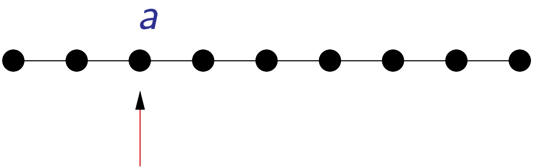

$a, \quad a \in \Sigma$

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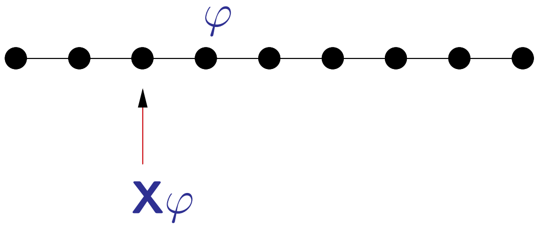

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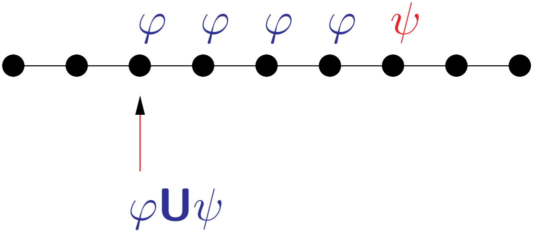
Syntax: $\varphi \;:=\; a \;\; (\in \Sigma) \;\mid\; \varphi \vee \varphi' \;\mid\; \neg\varphi \;\mid\; \mathbf{X}\varphi \;\mid\; \varphi\mathbf{U}\varphi'$

Semantics:



Changing syntax to lower complexity

FO or MSO evaluation over trees and words with linear data complexity implies non-elementary query complexity.

Need another – but equivalent – logic!

Over words, LTL=FO (Kamp, 1969)

What is the query complexity of evaluating FO and LTL over words? With linear data complexity, it is:

- non-elementary for FO (a stack of exponentials, Frick, Grohe, '03)
- linear for LTL

Words as databases

A word w over $\Sigma = \{a_1, \ldots, a_m\}$ is a database with relations $E(\cdot,\cdot),\ L_1(\cdot), \ldots, L_m(\cdot)$:

- E is the ordering of positions;
- L_i's define labelings.

$w = a_1 a_2 a_1 a_2$:

positions $0, 1, 2, 3$; positions 0,2,3 labeled a_1; position 1 labeled a_2

$E =$

0	1
1	2
2	3
0	2
1	3
0	3

$L_1 =$

0
2
3

$L_2 =$

1

MSO over words

Each MSO sentence φ defines a language

$$\mathcal{L}(\varphi) \;=\; \{w \in \Sigma^* \mid w \models \varphi\}$$

Theorem (Büchi, Elgot, Trakhtenbrot 1960)

MSO-definability $=$ *Regular languages*

A similar result holds for trees as well – both binary and unranked.

We now show how to go from MSO to automata.

Types

Each FO sentence is a disjunction of types.

- Rank-k type $\mathrm{tp}_k(D)$: set of all sentences of quantifier rank k true in a database D.
- Types are finite objects, definable in the logic: finitely many distinct FO sentences of quantifier rank k, up to logical equivalence.

Another way of looking at Ehrenfeucht-Fraï ssé games:

$$\boxed{\begin{array}{c}\mathrm{tp}_k(D) = \mathrm{tp}_k(D') \\ \Updownarrow \\ \text{Duplicator has a winning strategy in the } k\text{-round game on } D \text{ and } D'.\end{array}}$$

For MSO, the same is true, but the game is slightly more complex: players can play both points and sets.

From MSO to automata via composition

Rank-k type of w uniquely determines rank-k type of $w \cdot a$.

If $\mathrm{tp}_k(w) = \mathrm{tp}_k(w')$, then $\mathrm{tp}_k(w \cdot a) = \mathrm{tp}_k(w' \cdot a)$: compose games!

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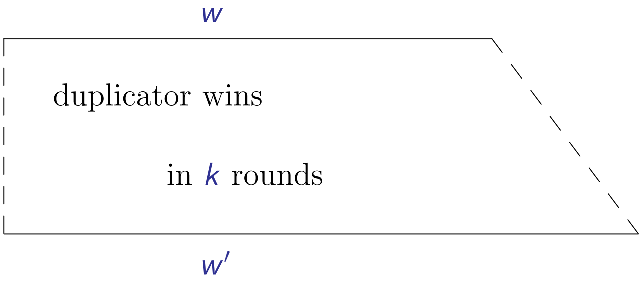

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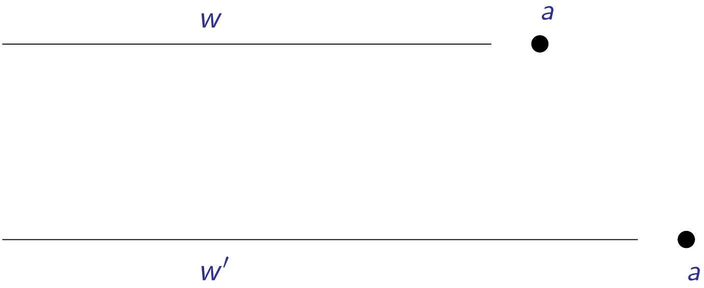

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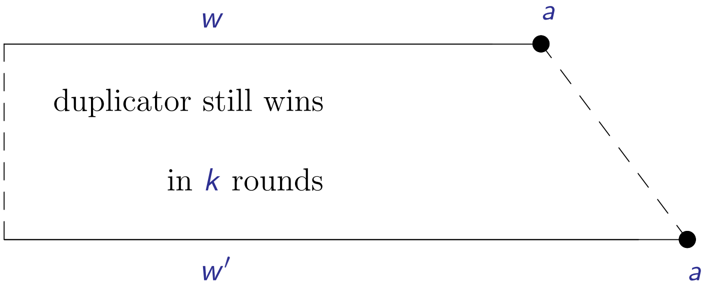

From MSO to automata: automata compute types

The rank-k type of w uniquely determines the rank-k type of $w \cdot a$.

Deterministic Automaton for sentence φ:

- States are rank-k types;
- Initial state: the type of the empty word;
- Final states: those types whose disjunction forms φ.
- Transition $\delta(\tau, a)$: the type of $w \cdot a$ if the type of w is τ.

After reading w, the state of the automaton is $\mathrm{tp}_k(w)$.

Language for extracting positions in words?

We need a language for extracting positions in trees:

- Information extraction from XML document;
- Work by Gottlob, Koch, and colleagues; Lixto system

We demonstrate the idea on words; it works for trees as well.

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Use the composition method:

$$1) \quad \begin{array}{l} \mathrm{tp}_k(w) = \mathrm{tp}_k(w') \\ \mathrm{tp}_k(u) = \mathrm{tp}_k(u') \end{array} \quad \Rightarrow \quad \mathrm{tp}_k(w \cdot a \cdot u) = \mathrm{tp}_k(w' \cdot a \cdot u')$$

$$2) \quad \mathrm{tp}_k(u) = \mathrm{tp}_k(w) \quad \Rightarrow \quad \mathrm{tp}_k(w^{-1}) = tp_k(u^{-1})$$

Language for extracting positions in words

How to express MSO (or FO) $\varphi(x)$ over words?

Idea: for $w \cdot a \cdot u$, compute

1. $\mathrm{tp}_k(w)$ going forward from the first position;
2. $\mathrm{tp}_k(u^{-1})$ going backwards from the last position;
3. These types tell us whether the a position is selected.

Express this in Datalog. Compute $\mathrm{tp}_k(w)$ going forward – use predicates U_τ for types:

$$
\begin{array}{lcll}
U_{\tau_a}(x) & :- & \mathrm{First}(x), L_a(x); & a \in \Sigma \\
U_{\tau'}(x) & :- & Succ(y,x), L_a(x), U_\tau(y); & a \in \Sigma,\ \delta(\tau, a) = \tau'
\end{array}
$$

Datalog program cont'd

- Types going forward:

$$\begin{array}{lcll} U_{\tau_a}(x) & \text{:-} & \mathrm{First}(x), L_a(x); & a \in \Sigma \\ U_{\tau'}(x) & \text{:-} & Succ(y,x), L_a(x), U_\tau(y); & a \in \Sigma,\ \delta(\tau, a) = \tau' \end{array}$$

- Types going backwards V_τ: symmetric.
- Answer – for all triples (τ, a, τ') saying that a is selected, add:

$$\mathsf{Answer}(x) \text{ :- } U_\tau(y), Succ(y,x), P_a(x), Succ(x,z), V_{\tau'}(z)$$

- We used Monadic Datalog: all idb predicates are monadic.
  - Edb predicates: successor, labelings, First and Last.
- It captures MSO over words.
- Complexity of evaluating program P on w:

$$O(\|P\| \cdot |w|)$$

Review of the journey

Composition technique suggested using monadic datalog:

- captures MSO;
- has very good complexity bounds.

The approach works for trees and yields many XML languages:

- Monadic datalog captures MSO for trees, with the same complexity – one needs to add predicates for the root, leaves, first and last children of nodes (Gottlob, Koch, '01)
- Other approaches:
  - ETL – Efficient tree logic (Neven, Schwentick, '00)
  - Temporal logics with good query evaluation properties (Schlingloff '92, Marx '04, Barceló, L., '05)
  - Dialects of XPath (Marx '04)

Plan

- Expressiveness: Tools for first-order logic
- Expressiveness: Tools that work beyond first-order logic
- Language equivalence (better query evaluation via the composition method)
- Descriptive Complexity
- Satisfiability

Descriptive complexity

- Machine-independent characterization of complexity classes.
- A query language tells you a lot about the complexity.
- If your logic captures a complexity class, you have even more information:
  - complexity cannot be lowered.
- First result:

Theorem (Fagin 1974)

NP = Existential Second-Order Logic (ESO)

- ESO $= \exists R_1 \ldots \exists R_k \ \varphi(R_1, \ldots, R_k, E)$
- 3-colorability: $\exists R \exists G \exists B \ \varphi$
  - φ says that R, G, B partition the set of nodes and endpoints of an edge cannot be in the same set.

Descriptive complexity – other classes

- FO is contained in AC^0
  - AC^0 – constant parallel time: constant time with polynomially many processors. Suggests very efficient parallel algorithms.
  - Complexity of the relational calculus.
  - Uniform version of AC^0 is captured by FO with $<, +, \times$ on the finite universe.
- Basic SQL (FO+simple arithmetic+aggregation) is contained in TC^0
  - TC^0 – constant parallel time with more complex gates (including majority gates). Probably a small subset of DLOGSPACE but not yet separated from NP!
- FO+transitive closure is contained in NLOGSPACE.
- FO+fixed-points (including Datalog) is contained in PTIME
  - Over ordered databases, captures PTIME.

Descriptive complexity – other classes

- NP = ESO.
- Second-order logic = Polynomial hierarchy (between PTIME and PSPACE)
- FO+partial fixed-point (need not converge: if it does not converge, the result is empty) – contained in PSPACE
  - Captures PSPACE over ordered databases
- These classes typically appear when deals with schemas, queries, etc
  - small objects compared to databases
  - e.g., conjunctive query containment is NP-complete;
  - e.g., composing schema mappings in data exchange is NP-complete;
  - e.g., equivalence of DTDs is PSPACE-complete

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Satisfiability

- Satisfiability problem:

INPUT:	FO sentence φ
QUESTION:	does it it have a model ($\mathfrak{A} \models \varphi$)?

- Undecidable, but the complement is r.e. (recursively enumerable):
  - The complement is validity: Given φ, is it true in every model $\mathfrak{A}$?
  - Because φ is satisfiable iff $\neg\varphi$ is not valid
  - Validity is r.e. due to the completeness of FO.

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  - Validity is r.e. due to the completeness of FO.
- Finite satisfiability: Given φ, does it have a finite model?
- Finite validity: Given φ, is it true in all finite models?
- Completeness fails in the finite, so we cannot get r.e. as before.

Theorem (Trakhtenbrot 1950)

Finite validity is not r.e.

Satisfiability: application

- Static analysis problems (schemas, queries, etc)
- certain answers: key concept for
  - incomplete information;
  - query answering using views;
  - data integration and exchange
- Databases with incomplete information:

student	course	grade
Jones	x	A
y	x	B
y	CS1	A-
Jones	CS2	z

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y	x	B
y	CS1	A-
Jones	CS2	z

$\Rightarrow$

student	course	grade
Jones	CS4	A
Smith	CS4	B
Smith	CS1	A-
Jones	CS2	B+
Jones	CS3	B-

- All such databases representing a table T: $\text{Rep}(T)$

Certain answers

- Answers to a query that do not depend on the interpretation of missing information:

$$\text{certain}(Q, T) \;=\; \bigcap_{R \in \text{Rep}(T)} Q(R)$$

- What if Q is an FO sentence and $T = \emptyset$?
- $\text{certain}(Q, T) = \text{true}$ iff Q is valid! Hence:

> *Corollary*
>
> *For FO queries, computing certain answers is undecidable.*

- Hence we (database people) are interested in classes with decidable satisfiability.

Decidable classes

- Conjunctive queries and their unions
    - finding certain answers - PTIME
    - plays very important in data integration/exchange
- Other classes go via finite model property: if there is a model, there must be a finite one.
- Two important classes:
    - Bernays-Schönfinkel class: $\exists x_1 \ldots \exists x_m \forall y_1 \ldots \forall y_k \ \alpha$
    - FO^2 – formulae with 2 variables x and y.
- In both cases, satisfiability is NEXPTIME-complete.
- Both results have been used in database theory (e.g. recently in the XML context).

Conclusion

- Quite likely, if you need to use FMT in database research, you need one of the techniques described in this tutorial (paper).
- No need to read textbooks/proofs – just use the toolbox!
    - Unless you want to work in FMT.

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- No need to read textbooks/proofs – just use the toolbox!
    - Unless you want to work in FMT.
- But if you want to read an FMT book, ask me after the talk and I'll tell you which one to buy.

The future of FMT

- We've learned how to work with both stone and iron age tools.
- It's time to go back to games and start putting stones together.
- Middle Ages tools: actively developed by Ben Rossman (the rest of the world is trying to catch up). Solved 3 long-standing open problems:
    1. successor-invariance
    2. preservation under homomorphisms in the finite
    3. strictness of the FO^k hierarchy over ordered structures.
- Next step, N years away: modern era tools.
    - Main application: separating complexity classes.

Good news – most of the database theory tasks are easily doable with stone and iron age tools.