An elementary proof that upper and lower powerdomain constructions commute

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1 Introduction

It was proved in [1] that lower and upper powerdomain constructions commute on all domains. In that proof, domains were represented as information systems. In [2] a rather complicated algebraic proof was given which relied on universality properties of powerdomains proved in the previous works of the author of [2]. Here we give an elementary algebraic proof that upper and lower powerdomain constructions commute. The proof is essentially a reduction of the problem to establishing a 1-1 correspondence between certain disjunctive and conjunctive normal forms.

2 Definitions

A subset $X$ of a partially ordered set is called directed if a common upper bound exists for any two elements of $X$, i.e. given $x_1, x_2 \in X$, there exists $x \in X$ such that $x \geq x_1, x_2$. A poset is called complete (abbreviated - cpo) if every directed subset has a least upper bound. An element of a cpo is called compact if it can not be below a least upper bound of a directed set $X$ without being below an element of $X$. A cpo is called algebraic if every element is the least upper bound of compact elements below it, see [3].

A domain in this paper is an algebraic cpo with bottom. Given a domain $D$, $\leq$ denotes its order and $KD$ is the set of its compact elements. Given $A, B \subseteq D$, lower and upper powerdomain orderings are given by

$$A \sqsubseteq^1 B \iff \forall a \in A \exists b \in B : \ a \leq b$$
$$A \sqsubseteq^1 B \iff \forall b \in B \exists a \in A : \ a \leq b$$

A subset of an ordered set is called an antichain if no two elements in it are comparable. If $(X, \leq)$ is an ordered set and $Y \subseteq X$, then $\max_{\leq} Y$ and $\min_{\leq} Y$ are sets of maximal and minimal elements of $Y$. We will use just $\max Y$ and $\min Y$ if the ordering is understood. $A_{\text{fin}}(X)$ stands for the

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set of all finite antichains of $X$. The lower and upper powerdomains are defined to be the ideal completions of $(\mathcal{A}_{fin}(KD), \sqsubseteq^1)$ and $(\mathcal{A}_{fin}(KD), \sqsubseteq^1)$ respectively. They are denoted by $\mathcal{P}'(D)$ and $\mathcal{P}^i(D)$. $(\mathcal{A}_{fin}(KD), \sqsubseteq^1)$ and $(\mathcal{A}_{fin}(KD), \sqsubseteq^1)$ are posets of compact elements of $\mathcal{P}'(D)$ and $\mathcal{P}^i(D)$ [3].

Remark: A traditional definition of the powerdomain construction is the ideal completion of $P_{fin}(KD)$, the set of all finite subsets of $KD$. The two can be easily shown to be equivalent. We prefer to work with antichains because $\sqsubseteq^1$ and $\sqsubseteq^1$ are partial orders on $A_{fin}(KD)$ but only preorders on $P_{fin}(KD)$.

Our goal is to prove

**Theorem** For any domain $D$, $\mathcal{P}'(\mathcal{P}'(D))$ and $\mathcal{P}^i(\mathcal{P}^i(D))$ are isomorphic.

The proof is constructive, i.e. an isomorphism and its inverse are explicitly described.

## 3 Proof

To prove that two domains $D_1$ and $D_2$ are isomorphic, i.e. that there exists a pair of continuos mutually inverse maps between $D_1$ and $D_2$, it is enough to prove that $KD_1$ and $KD_2$ are isomorphic as posets, i.e. that there exists a pair of monotone mutually inverse maps between $KD_1$ and $KD_2$.

A compact element of $\mathcal{P}'(\mathcal{P}'(D))$ is a finite antichain, w.r.t. $\sqsubseteq^1$, of finite antichains of compact elements of $D$, and a compact element of $\mathcal{P}^i(\mathcal{P}^i(D))$ is a finite antichain, w.r.t. $\sqsubseteq^1$, of finite antichains of compact elements of $D$. Given a finite set of finite sets $\mathcal{A} = \{A_1, \ldots, A_n\}$ where $A_i = \{a^i_1, \ldots, a^i_{k_i}\}$, let $F_{\mathcal{A}}$ be the set of functions $f : \{1, \ldots, n\} \to \mathbb{N}$ such that for any $i$: $1 \leq f(i) \leq k_i$. For $f \in F_{\mathcal{A}}$, let $f(\mathcal{A}) = \{a^i_{f(i)} \mid i = 1, \ldots, n\}$. If all $A_i$’s are subsets of $D$, define two maps $\phi$ and $\psi$ as follows:

$$\phi(\mathcal{A}) = \min_{f \in F_{\mathcal{A}}} (\max f(\mathcal{A}))$$

$$\psi(\mathcal{A}) = \max_{f \in F_{\mathcal{A}}} (\min f(\mathcal{A}))$$

Now, we claim that $\phi$ maps $KD_1$ to $KD_2$ and $\psi$ maps $KD_2$ to $KD_1$ and, moreover, these maps establish the desired isomorphism, i.e. they are mutually inverse and monotone. The first claim follows immediately from the definitions of $\phi$ and $\psi$. To complete the proof, it is enough to show that $\phi$ is monotone and $\phi \circ \psi = \text{id}$. The proof of monotonicity of $\psi$ and $\psi \circ \phi = \text{id}$ is dual. We start with two easy observations:

**Lemma** Let $Y_1, Y_2$ be finite subsets of an arbitrary poset $X$. Then

1) $Y_1 \sqsubseteq^1 Y_2$ iff $\max Y_1 \sqsubseteq \max Y_2$;
2) $Y_1 \sqsubseteq^1 Y_2$ iff $\min Y_1 \sqsubseteq \min Y_2$. \hfill $\square$

**Claim 1**: $\phi$ is monotone.

**Proof of claim 1**: Let $\mathcal{A}, \mathcal{B} = \{B_1, \ldots, B_m\} \in KD_1$ and $\mathcal{A} \sqsubseteq \mathcal{B}$. We must prove $\phi(\mathcal{A}) \sqsubseteq \phi(\mathcal{B})$. In view of lemma, it is enough to show that for any $f \in F_{\mathcal{B}}$ there exists $g \in F_{\mathcal{A}}$ such that $g(\mathcal{A}) \sqsubseteq \mathcal{B}$. Since for each $i = 1, \ldots, n$ there exists $j_i$ such that $A_i \sqsubseteq B_{j_i}$, there is an element $a^i_{p_i} \in A_i$ such that $a^i_{p_i} \leq b^j_{f(i)}$. Let $g(i) = p_i$. Then for this function $g$ one has $\{a^i_{g(i)} \mid i = 1, \ldots, n\} \sqsubseteq \{b^j_{f(i)} \mid i = 1, \ldots, m\}$, i.e. $g(\mathcal{A}) \sqsubseteq \mathcal{B}$. Claim 1 is proved.
Let $A \in \mathcal{K}P^i(\mathcal{P}^i(D))$ and $B = \{B_1, \ldots, B_m\} = \phi(A) \in \mathcal{K}P^i(\mathcal{P}^i(D))$. In view of lemma, to show that $\psi \circ \phi = \text{id}$, i.e. that $\psi(B) = A$, it suffices to prove

**Claim 2:** For any $f \in F_B$ there exists $A_i \in A$ such that $f(B) \sqsubseteq^i A_i$.

**Claim 3:** Every $A_i$ is in $\psi(B)$.

**Proof of claim 2:** Let $C$ be the collection of all sets $f(A)$ where $f \in F_A$; $C = \{C_1, \ldots, C_k\}$. Then for any $g \in F_C$, there exists $A_i \in A$ such that $A_i$ is contained in $g(C)$ because, if this is not the case, for any $A_i \in A$ there exists $j_i \leq k_i$ such that $A_{j_i} \in A$ and, for any $f \in F_A$, $g$ on $f(A)$ picks an element different from $a^{j_i}_{j_i}$. If we define $f_0$ such that $f_0(i) = j_i$, $g$ may pick only elements of form $a^{j_i}_{j_i}$ on $f_0(A)$, a contradiction. Therefore, $g(C) \sqsubseteq^i A_i$ for some $i$.

Let $f \in F_B$. Let $H$ be the set of functions in $F_A$ that correspond to elements of $B = \phi(A)$ or, in other words, max $h(A) \in B$ for $h \in H$. Then, for any $h' \in F_A - H$, there exists a function $h \in H$ such that max $h(A) \sqsubseteq^i$ max $h'(A)$, i.e. $h(A) \sqsubseteq^i h'(A)$. Since $h \in H$, max $h(A) \in B$, i.e. max $h(A) = B_i$. If $f(i) = j$, then there is an element in $h'(A)$ that is greater than $b^i_j$. Define a function $g \in F_C$ to coincide with $f$ on those $C_i$’s that are given by functions in $H$. On $C_i$ that corresponds to $f \in F_A - H$, let $g$ pick an element which is greater than some $b^i_j$ where $f(i) = j$ (we have just shown it can be done). Then $f(B) \sqsubseteq^i \{c_{g(i)}^{i,j} \mid i = 1, \ldots, k\} = g(C)$. We know that there exists $A_i \in A$ such that $g(C) \sqsubseteq^i A_i$. Thus, $f(B) \sqsubseteq^i A_i$. Claim 2 is proved.

**Proof of claim 3:** Prove that for any $a^i_j \in A_i$ there exists $B_i \in B$ such that $a^i_j \in B_i$. Consider the set $F_{A_i}^i$ of functions $f \in F_A$ such that $f(i) = j$. If for no $f \in F_{A_i}^i$, $a^i_j \in \text{max} f(A)$, then there exists $A_p \in A$ such that all elements of $A_p$ are greater than $a^i_j$, i.e. $A_i \sqsubseteq^i A_p$ which contradicts our assumption that $A$ is an antichain w.r.t. $\sqsubseteq^i$. Hence, $a^i_j \in \text{max} f(A)$ for at least one function in $F_{A_i}^i$. Since $A$ is an antichain, for any $p \neq i$ there exists $a^i_p \in A_p$ which is not greater than any element of $A_i$. Change $f$ to pick such an element for any $p \neq i$. Then $a^i_j$ is still in $\text{max} f(A)$. There exists a function $f' \in F_A$ such that $\text{max} f'(A) \sqsubseteq^i \text{max} f(A)$ and $\text{max} f'(A) \in \phi(A)$. If $f'(i) = j'$, then, since $f'(A) \sqsubseteq^i f(A)$ and $A_i$ is an antichain, $a^j_{j'} \leq a^i_p$ for some $p$ and $j'$, where $p \neq i$. But this contradicts the definition of $f$. Hence, $f'(i) = j$ and $a^i_j \in \text{max} f'(A)$ because $a^i_j \in \text{max} f(A)$. Since $\text{max} f'(A) = B_i$ for some index $l$, $a^i_j \in B_i$.

Let $B'$ be the collection of elements of $B$ that contain elements of $A_i$. Then we can define a function $f \in F_B$ on elements of $B'$ to pick all elements of $A_i$. Each $B_j \in B - B'$ either contains an element of $A_i$ or contains an element which is greater than some $a^i_p \in A_i$. Let $f$ pick any such element. Then $\min f(B) = A_i$. Suppose $A_i \notin \psi(B)$. Then $A_i \sqsubseteq^i \min g(B)$ for some function $g \in F_B$ such that $\min g(B) \in \psi(B)$. By claim 2, $g(B) \sqsubseteq^i A_j$ for some $A_j$. Hence, $\min g(B) \sqsubseteq^i A_j$ and since $A$ is an antichain w.r.t. $\sqsubseteq^i$, $A_i = A_j = \min g(B) \in \psi(B)$. This finishes the proof of claim 3 and the theorem.

**References**
