Vocabulary (signature)

$(+, *)$

$(+, *, <)$

Structures

$(\mathbb{N}, +, *)$

$(\mathbb{R}, +, *)$

$(\mathbb{N}, +, *, <)$

$(\mathbb{R}, +, *, <)$

Syntax

$\exists x \exists y \, (x = y + y)$

$\exists x \forall y \; y > x$ ✓

$y = x$

φ

Semantics (satisfaction)

φ is true in $(\mathbb{N}, +, *, <)$

false in $(\mathbb{R}, +, *, <)$

Structures over a vocabulary σ

- a universe (set) U
- an interpretation of
  - each function symbol f_i of arity n_i as a function $U^{n_i} \to U$
  - each constant as an element of U
  - each predicate symbol P_i as a subset of U^{m_i}

Finite strings as structures

alphabet Σ

$\Sigma = \{a, b\}$

Vocabulary

$(\underbrace{P_a, P_b,}_{\text{arity}} \underbrace{<}_{\text{arity } 2})$

$s = abaab$

$\mathfrak{M}_s = (\{0,1,2,3,4\},\ P_a = \{0,2,3\},\ P_b = \{1,4\},\ <)$

First Order Logic (FO)

Terms

- each variable x is a term, $FV = \{x\}$
- each constant symbol c, is a term, $FV = \emptyset$
- If f is a funct. sym of arity n and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots t_n)$ is a term, and $FV = \bigcup_i FV(t_i)$

If no function symbols, all terms are variable and constants

atomic flas

Formulas

- If $t_1 \ldots t_n$ are terms and P a pred symb of arity n, then $P(t_1, \ldots t_n)$ is a fla, $FV(P(t_1, \ldots t_n)) = \bigcup_i FV(t_i)$
- If t_1 and t_2 are terms then $t_1 = t_2$ is a fla and $FV = FV(t_1) \cup FV(t_2)$

- If $\varphi, \varphi_1, \varphi_2$ are formulas then $\varphi_1 \vee \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\neg\varphi$ are formulas

$$FV(\varphi_1 \overset{\vee}{\wedge} \varphi_2) = FV(\varphi_1) \cup FV(\varphi_2)$$

$$FV(\neg\varphi) = FV(\varphi)$$

- If φ is a fla, then $\exists x\varphi$, $\forall x\varphi$ are flas and $FV(\exists x\varphi) = FV(\forall x\varphi) = FV(\varphi) - \{x\}$

Bound = not free

$\varphi(x_1 \; x_n)$ if $FV(\varphi) = \{x_1 \; x_n\}$

Fla w/o free variable = sentence

$$\psi = \exists x \underbrace{\forall y \, (x < y \lor x = y)}_{\varphi(x)}$$

$$(\mathbb{N}, <) \models \psi \iff (\mathbb{N}, <) \models \varphi(0)$$

for each $n \in \mathbb{N}$ $\quad 0 < n$ or $0 = n$ is true

$\lor, \land, \neg, \forall, \exists$ $\qquad \forall x \, \varphi(x) \iff \neg \exists x \neg \varphi(x)$

atomic, $\lor, \neg, \exists$, $\land, \neg, \exists$

$$f : U^n \to U$$

$$G_f \subseteq U^{n+1}$$

$$G_f = \{ (a_0, a_1, \ldots, a_n) \mid a_0 = f(a_1, \ldots, a_n) \}$$