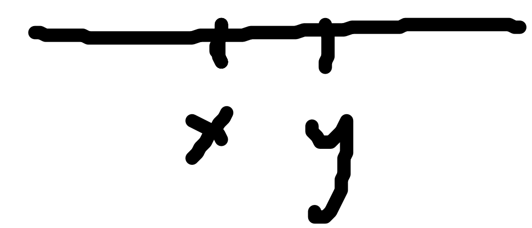
$\varphi_1 = \forall x\, P_a(x)$ $\quad L(\varphi_1) = a^*$

$succ(x,y) = (x<y) \wedge \neg \exists z \, (x<z \wedge z<y)$

$\varphi_2 = \exists x \exists y \; succ(x,y) \wedge P_a(x) \wedge P_b(y)$



$L(\varphi_2) = \Sigma^* a b \Sigma^*$

$min(x) = \forall y \, (x \leq y)$

$max(x) = \forall y \, (y \leq x)$

$L(\varphi_3) = (ab)^*$

$\varphi_3 = \forall x \, (min(x) \to P_a(x))$

$\wedge \forall x \, (max(x) \to P_b(x))$

$\wedge \forall x \forall y \, (succ(x,y) \to (P_a(x) \to P_b(y)) \wedge (P_b(x) \to P_a(y)))$

$$\exists R \, \exists G \left(\begin{array}{l} \forall x \; min(x) \rightarrow R(x) \\ \wedge \; \forall x \; max(x) \rightarrow G(x) \\ \wedge \; \forall x \forall y \left(succ(x,y) \rightarrow \left(\begin{array}{l} G(x) \rightarrow R(y) \\ R(x) \rightarrow G(y) \end{array} \right. \right. \\ \wedge \; \forall x \; R(x) \leftrightarrow \neg G(x) \end{array} \right.$$

$$\left(\forall x \; R(x) \vee G(x) \right) \wedge \neg \exists x \left(R(x) \wedge G(x) \right)$$

$L \subseteq \Sigma^*$ is definable in a logic (FO, MSO, ...)

if there is a sentence φ of the logic s.t.

$$L = L(\varphi)$$

$\varphi_A \equiv$

$\exists X_0 \ldots X_{n-1}$ (

1 $X_0 \ldots X_{n-1}$ encode a run

2 run starts in an initial state

3 run ends in a final state

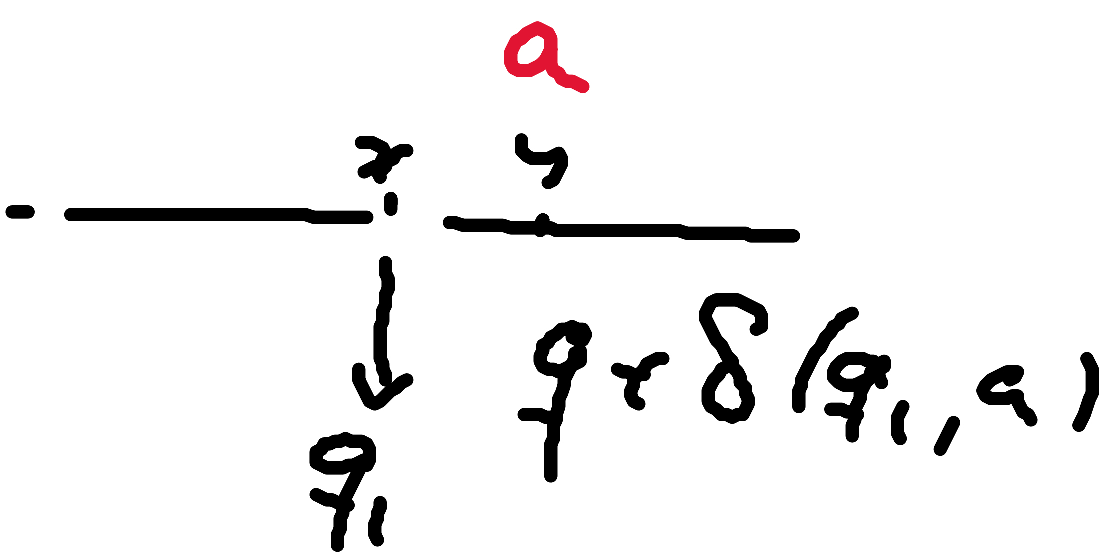
4 run obeys the transition δ)

1 $\forall x \left(\bigvee_i X_i(x) \right) \wedge \neg \exists x \left(\bigvee_{(\neq)} X_i(x) \wedge X_j(x) \right)$

2 $\bigwedge_{a \in \Sigma} \forall x \left(\min(x) \wedge P_a(x) \rightarrow \bigvee_{q_i \in \delta(q,a) \text{ for some } q \in Q_0} X_i(x) \right)$

3 $\forall x \left(\max(x) \rightarrow \bigvee_{i\, q_i \in F} X_i(x) \right)$

4. $$\bigwedge_{a\in\Sigma}\ \bigwedge_{i=0}^{n-1} \forall x \forall y \left(succ(x,y) \wedge X_i(x) \wedge P_a(y) \rightarrow \bigvee_{j \,\mid\, q_j \in \delta(q_i, a)} X_j(y) \right)$$



φ_A is true in $\mathfrak{M}_s$

iff A has an accepting run on s

$\iff$ A accepts s

Monadic Second-Order Logic (MSO)

extends FO

FO + quantification over sets

two kinds of variables

- first-order variables x
- second-order variables $X, Y,$

if X is a 2nd-order var, and x is a fo variable

then $X(x)$ is a fla meaning: $x \in X$

if φ is a fla, then $\exists X \varphi$ are flas
$\forall X \varphi$