$\varphi(X_1 \ldots X_n)$ of MSO^2_{pure}, invol $<$, $P_{a_1} \ldots P_{a_k}$

$$
\begin{array}{lcccc}
s = & \boxed{b_0} & \ldots & \boxed{b_{\ell-1}} \\
A_1 & 1 \; 0 & & 1 \\
\vdots & & & \\
A_n & 0 \; 1 \; 0 \; 1 & & 1
\end{array}
$$

$\mathfrak{M}_s \models \varphi(A_1 \ldots A_n)$

$A_1 \ldots A_n \subseteq \{0, \ldots \ell-1\}$

Code φ as an automaton over $\Sigma \times \underbrace{\{0,1\} \times \ldots \times \{0,1\}}_{n \text{ times}}$

If $s = b_0 \ldots b_{\ell-1}$ is a string of length ℓ, and $A_1 \ldots A_n$ are subsets of $\{0, \ldots, \ell-1\}$

then $[s, \bar{A}]$ is a string over $\Sigma \times \{0,1\}^n$ of length ℓ where the ith letter is

$$\begin{bmatrix} b_i \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \alpha_j = \begin{cases} 0 & \text{if } i \notin A_j \\ 1 & \text{if } i \in A_j \end{cases}$$

ith letter

$\varphi(x_1 \ldots x_n)$

$\rightsquigarrow A_\varphi$ over $\Sigma \times \{0,1\}^n$ s.t.

$M_s \models \varphi(A_1 \ldots A_n)$

$\Updownarrow$

A_φ accepts $[s, \bar{A}]$

Conjunction

$$\varphi(x_1 \quad x_n) \longrightarrow A_\varphi$$
$$\psi(x_1 \quad x_n) \hookrightarrow A_\psi$$

$$\varphi \wedge \psi(X_1 . X_n) \quad \cdot \quad A_\varphi \times A_\psi$$

$\varphi(X\,Y)$ $\hookrightarrow$ $\varphi(X, Y, Z)$

$\psi(X\,Z)$ $\hookrightarrow$ $\psi(X, Y, Z)$

A_φ $\Sigma \times \{0,1\} \times \{0,1\}$ $\hookrightarrow$ A'_φ over $\Sigma \times \{0,1\} \times \{0,1\} \times \{0,1\}$

$\psi(X_1 \ldots X_n) = \exists X_{n+1}\, \varphi(X_1 \ldots X_{n+1})$

A_φ over $\Sigma \times \{0,1\}^n \times \{0,1\}$

need A_ψ over $\Sigma \times \{0,1\}^n$

A_φ has transition function $\delta\ Q \times \Sigma \times \{0,1\}^{n+1} \to Q$

then in A_ψ states, initial states
final states $\to$ as in A_φ

$$\delta'\left(q, \begin{bmatrix} a \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}\right) = \delta\left(q, \begin{bmatrix} a \\ \alpha_1 \\ \vdots \\ \alpha_n \\ 0 \end{bmatrix}\right) \cup \delta\left(q, \begin{bmatrix} a \\ \alpha_1 \\ \vdots \\ \alpha_n \\ 1 \end{bmatrix}\right)$$

Proof - Induction

Base Case $\longrightarrow$ explicit construction (exercise)

$\wedge$ $\longrightarrow$ product construction (relatively easy)

$\neg$ $\longrightarrow$ complementation (!!! hard)

$\exists$ $\longrightarrow$ nondet guess (easy)

Existential MSO (EMSO)

$\exists X_1 \ \exists X_n \ \varphi$, φ is a FO formula

<u>Corollary</u> Over finite strings,

MSO = EMSO

MSO $\varphi \hookrightarrow A_\varphi \xhookrightarrow[\text{code } A_\varphi]{} \psi$

quant free fla

$\exists \,.\, \forall \ \exists \ \forall \ .\exists \,(\qquad)$

alternation