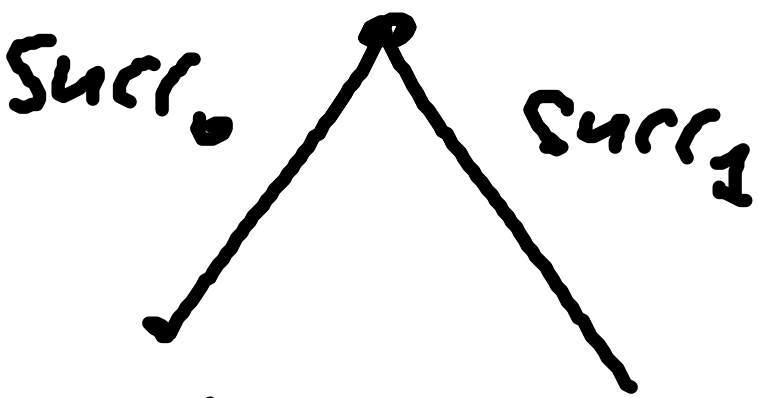
MSO on trees

$T = (D, \lambda)$, D - prefix closed subset of $\{0,1\}^*$

$\lambda : D \to \Sigma$

$\mathfrak{M}_T = (D, succ_0, succ_1, P_a)$

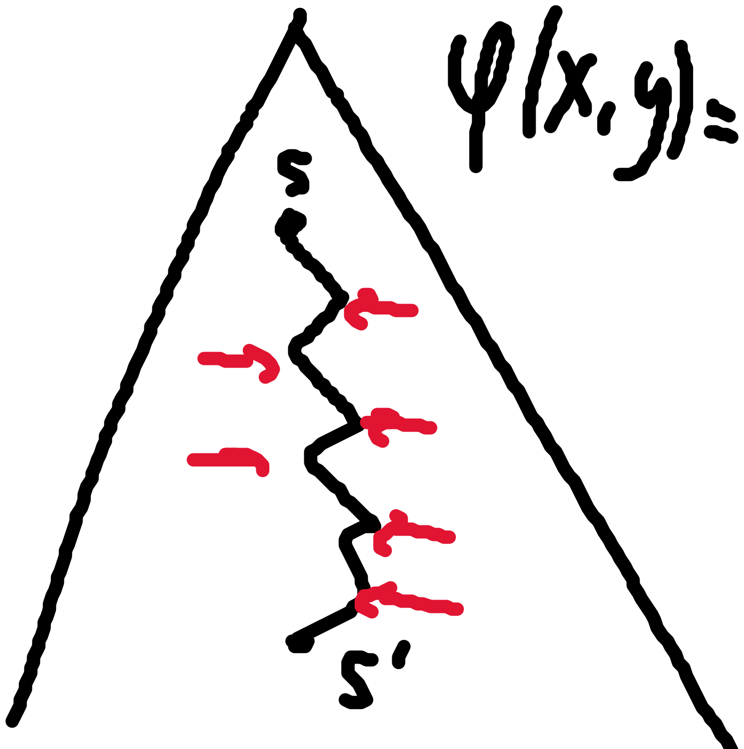
$P_a = \{ s \mid \lambda(s) = a \}$



$succ_0 = \{ (s, s0) \mid s, s0 \in D \}$ $\quad$ $succ_1 = \{ (s, s1) \mid s, s1 \in D \}$

$\preceq$ - descendant relation

$s \preceq s'$ $\quad$ s is a prefix of s'



$\psi(x,y) = \exists X \, (X \text{ is a path from } x \text{ to } y)$

↑ FO formula

$\pi(X, x, y)$

$$\pi(X,x,y) = X(x) \wedge X(y) \wedge \forall z,u \, (succ(u,z) \wedge z \neq x \wedge X(z) \rightarrow X(u)) \wedge$$

$$\forall z \, (succ(y,z) \xrightarrow[z \neq y]{} \neg X(z)) \wedge$$

$$\forall z,u,v \, (X(z) \wedge succ_0(z,u) \wedge succ_1(z,v) \rightarrow X(u) \leftrightarrow \neg X(v))$$

$$\forall z \, (succ(z,x) \rightarrow \neg X(z))$$

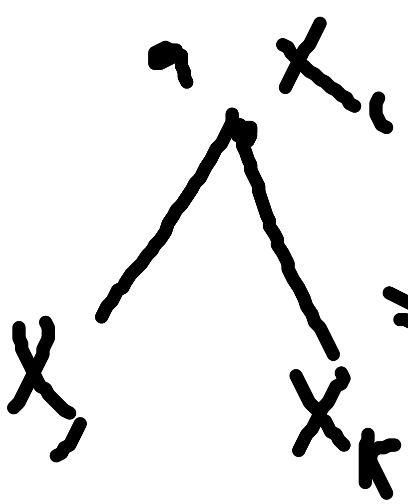
Theorem (Thatcher/Wright 1969-1970)

A set of trees is MSO-definable iff it is regular

Easy regular $\Rightarrow$ MSO (code automata)

$\mathcal{A} = (Q, Q_0, F, \delta)$ $\exists X_0 \;\; X_{n-1}$ (encode an accepting run)

$|Q| = n$



X_i = set of nodes assigned $q_i \in Q$

$\Rightarrow (q_j, q_k) \in \delta(q_i, a)$

MSO $\Rightarrow$ automata

$\hookrightarrow MSO^2_{pure}$

only 2nd order variables

$Sng(X)$

$X \subseteq Y \quad X \subseteq P_a,\ a \in \Sigma$

$Succ_1(X, Y)$

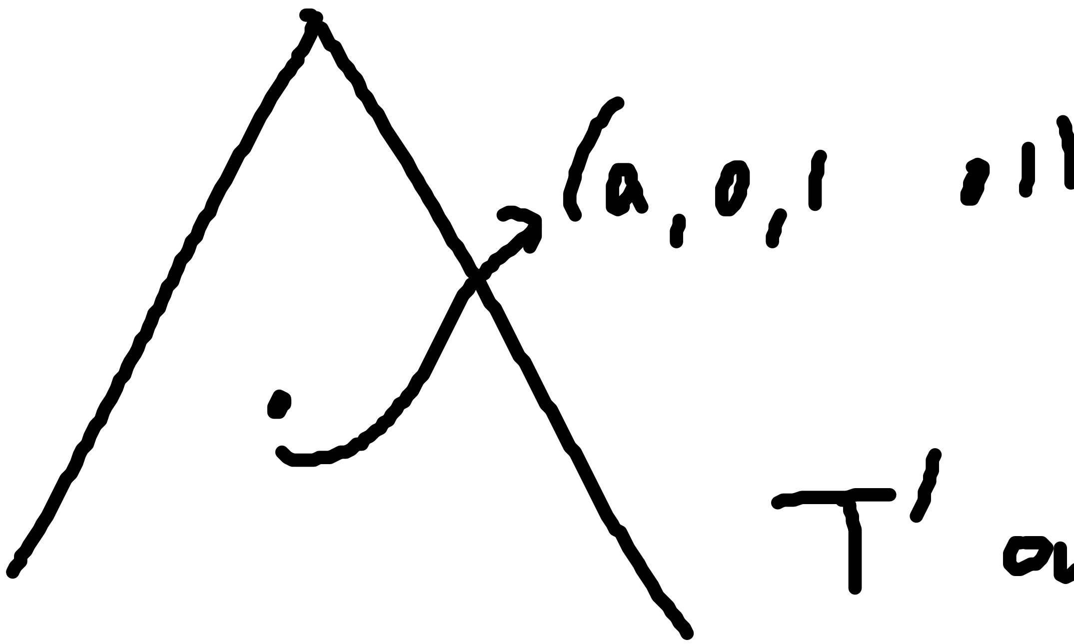
$X = \{x\} \quad Y = \{y\}$

$(x, y) \in succ_i\ ,\ i: 0, 1$

$\varphi(X_1, \ldots X_k)$

$\varphi(X_1 \ldots X_k) \hookrightarrow A_\varphi$ over $\Sigma \times \{0,1\}^k$

T



as $(T', A_1 \ldots A_k)$

T' only has Σ-labels

$A_i = \{ s \mid \lambda_T = (a, 0, 1 \ldots 1, \ldots) \}$

ith position

A_φ accepts $T \iff T' \models \varphi(A_1 \ldots A_k)$

Over strings/trees, MSO = regular

FO ?

<u>Thm</u> A language is definable in FO $\iff$ it is star-free

star-free given by a reg expr built from
$\emptyset$, $a, a \in \Sigma$, $e_1 \cdot e_2$, $e_1 \cup e_2$, $\overline{e}$

$\Sigma = \{a, b\}$ $\overline{\emptyset} = \{a,b\}^*$

start with a $a \cdot \{a,b\}^*$
end with b $\{a,b\}^* b$
no aa
no bb $\overline{\overline{\emptyset} \cdot a \cdot a \cdot \overline{\emptyset}}$

$(ab)^*$ is star-free.