Example 1

SkS - theory of k successors

$\Sigma = \{0, \ldots, k-1\}$

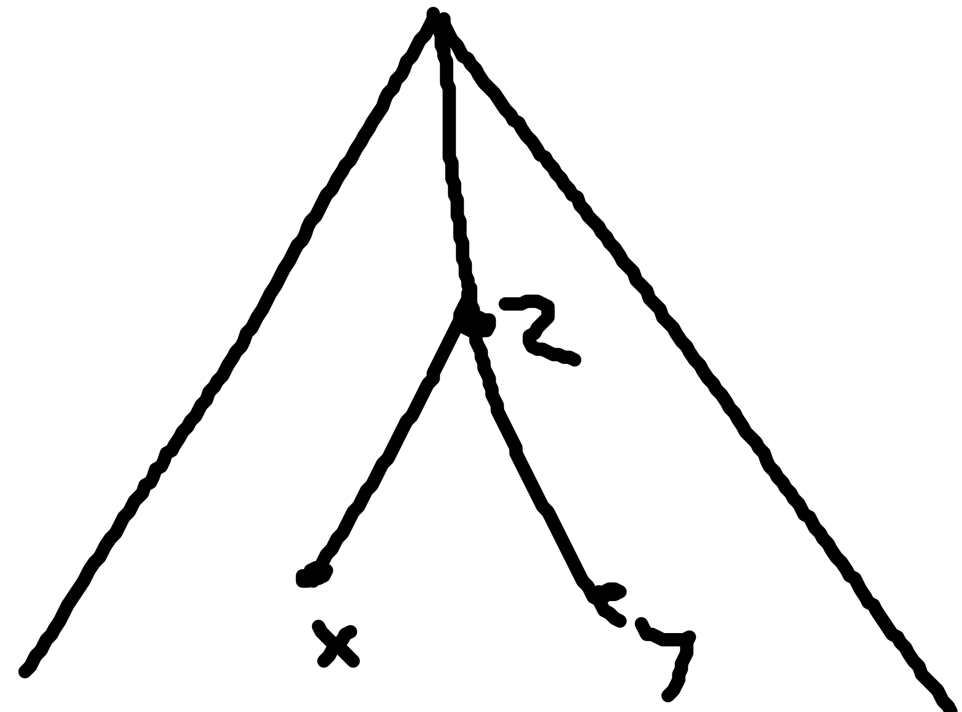
$T_k = (\Sigma^*, succ_0, \ldots, succ_{k-1})$

$succ_i = \{(s, si)\}$

SkS is decidable because it can be interpreted in S2S

Expressible over T_2

- $x \prec y$ (y is a descendant of x)
- $x \sqcap y = z$



$z \prec x \wedge$
$z \prec y \wedge$

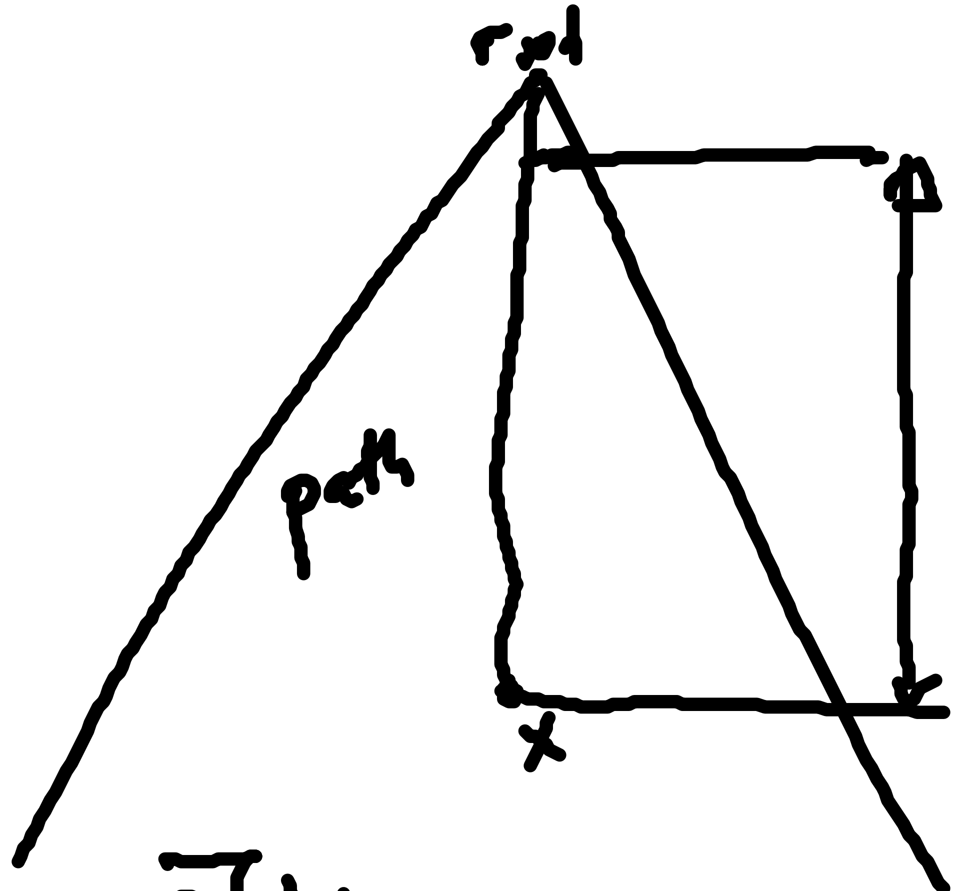
$$\begin{pmatrix} \mathrm{succ}_0(z) \prec x \wedge \\ \mathrm{succ}_1(z) \prec y \end{pmatrix} \vee \begin{pmatrix} \mathrm{succ}_0(z) \prec y \wedge \\ \mathrm{succ}_1(z) \prec x \end{pmatrix}$$

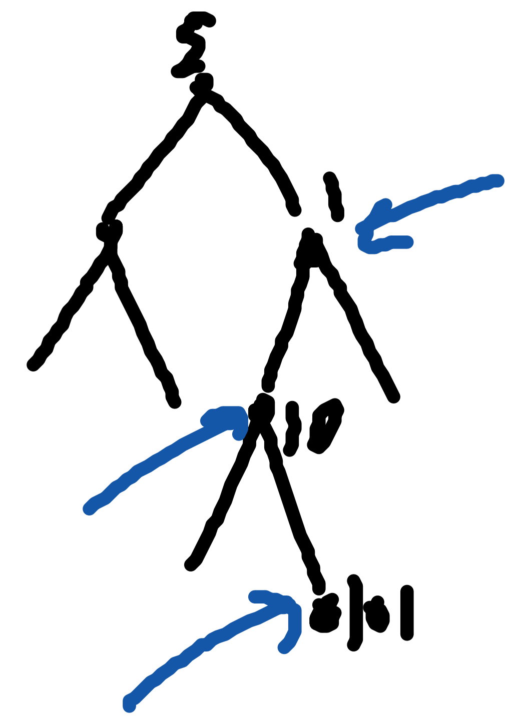
- $x <_{lex} y \iff x \prec y \vee \exists z \begin{pmatrix} z \prec x \wedge z \prec y \wedge \\ \mathrm{succ}_0(z) \prec x \wedge \\ \mathrm{succ}_1(z) \prec y \end{pmatrix}$

L is a regular language over $\{0, 1\}^*$

$\alpha_L(x)$

$x \in L$





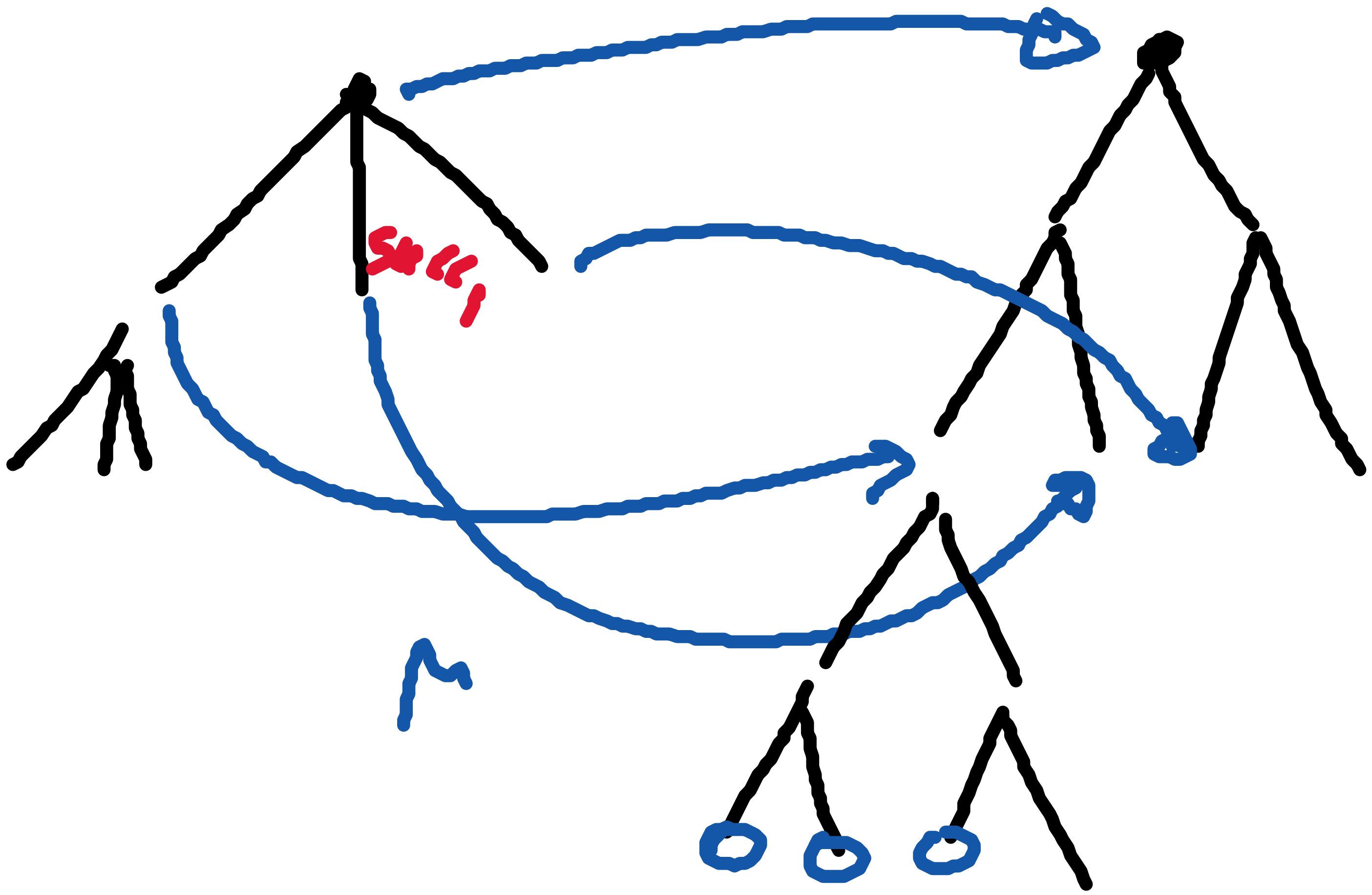
L

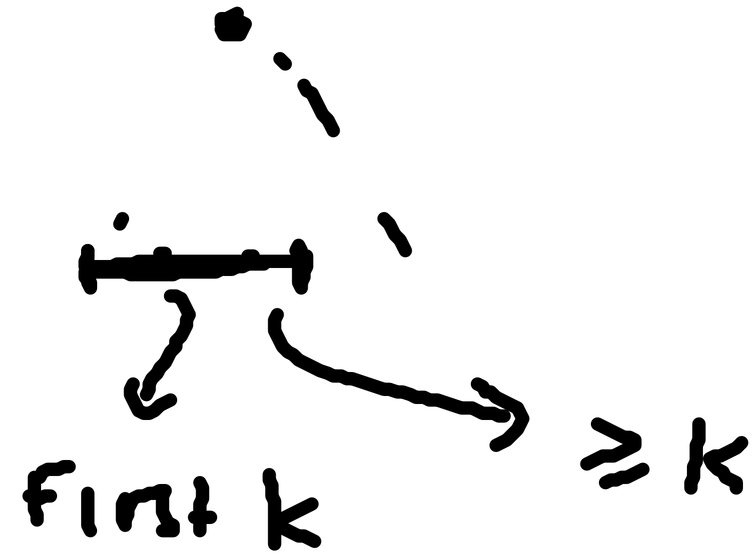
$\exists X_1 \quad X_\ell \quad \beta$

$\exists X_1 \quad X_\ell \subseteq \text{path} \quad \beta$

in place of P_0 left succ, in place of P_1 right successor

Interpreting S3S in S2S





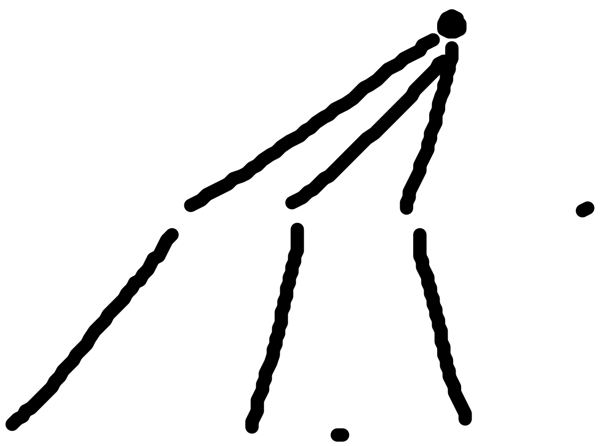
$$\mathrm{Im}(\mu).\quad \alpha_{(00)^*}(x) \wedge \neg\exists y\, z \quad y = \mathrm{succ}_1(z) \wedge x = \mathrm{succ}_1(y)$$

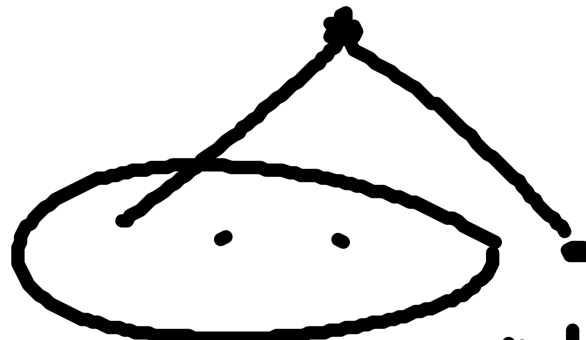
$$\psi_{\mathrm{succ}_1}(x,y) = \exists z \quad z = \mathrm{succ}_0(x) \wedge y = \mathrm{succ}_1(z)$$

SwS

$(\mathbb{N}^*, \prec, <_{lex})$

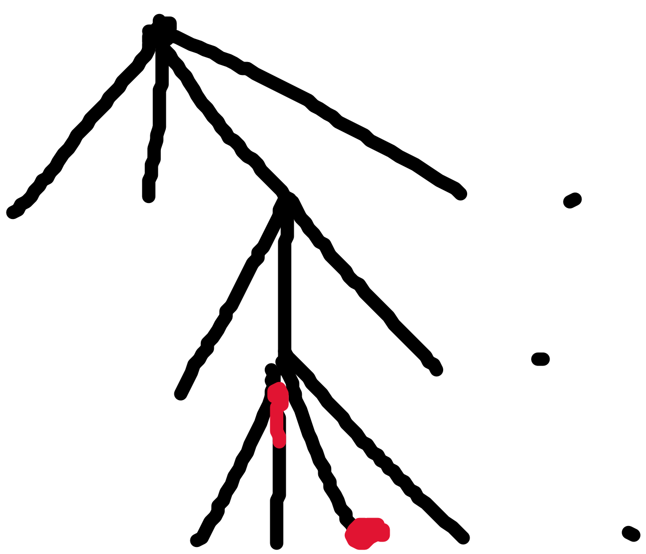
SwS is interpretable in S2S

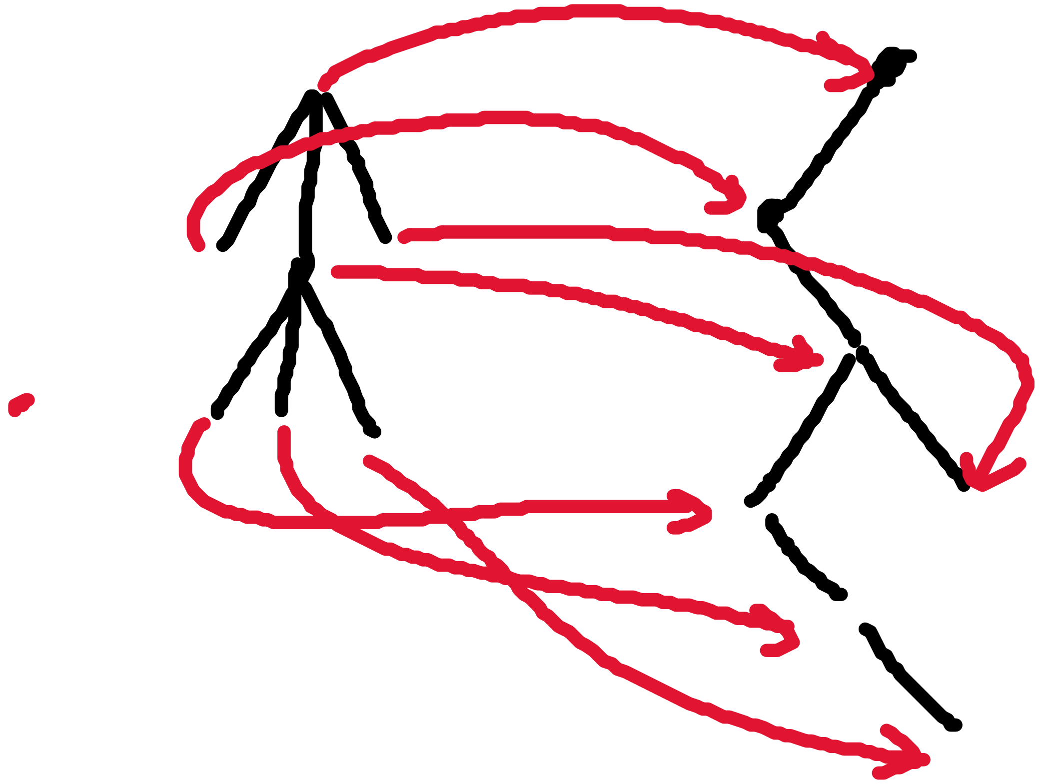




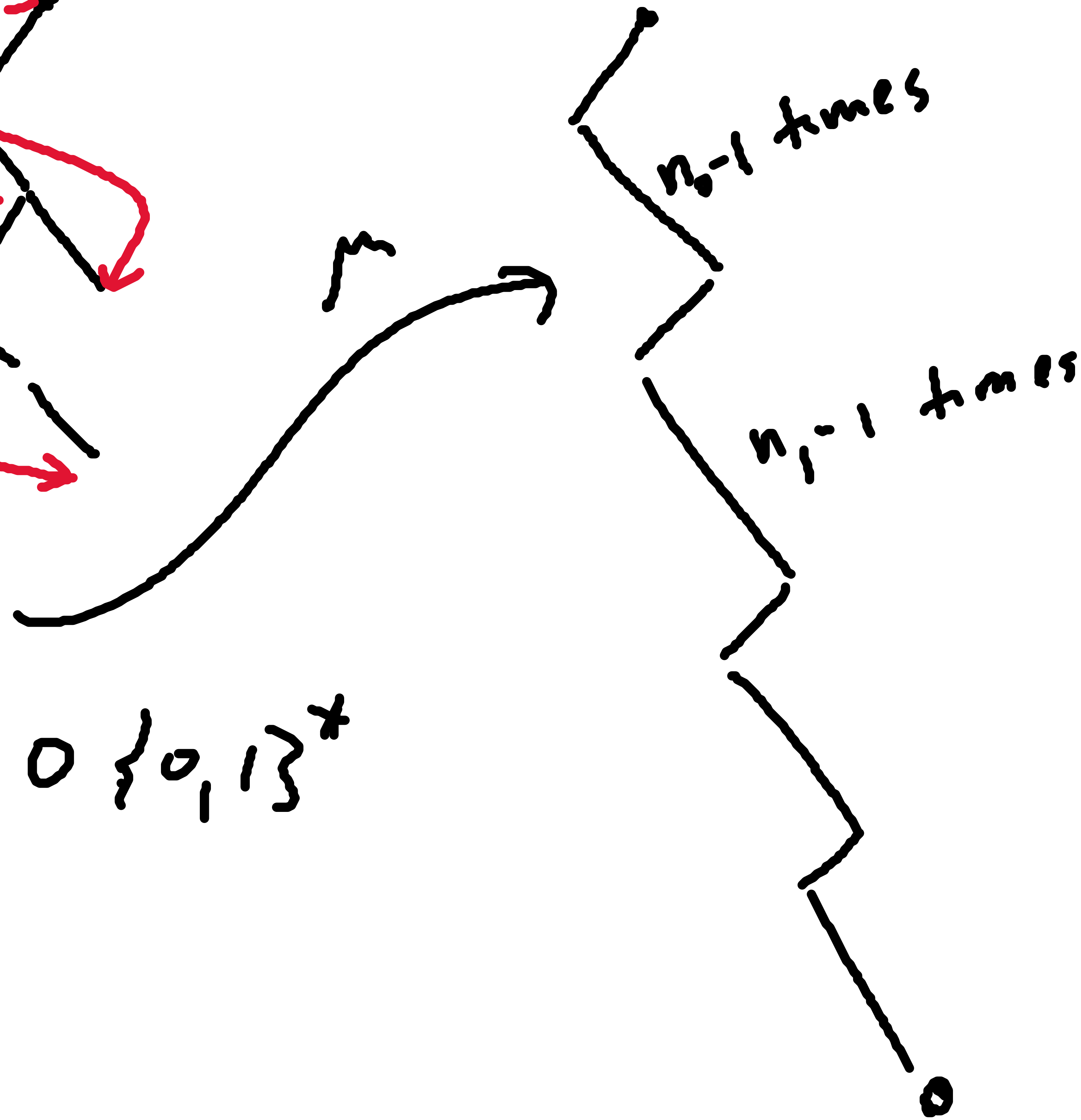
$y <_{lex} x$

$\exists$ exactly n-1 of them





$n_0 n_1 n_2 \ \ n_k \in \mathbb{N}^*$



$\mathrm{Im}(\mu) = \varepsilon \cup 0\{0,1\}^*$

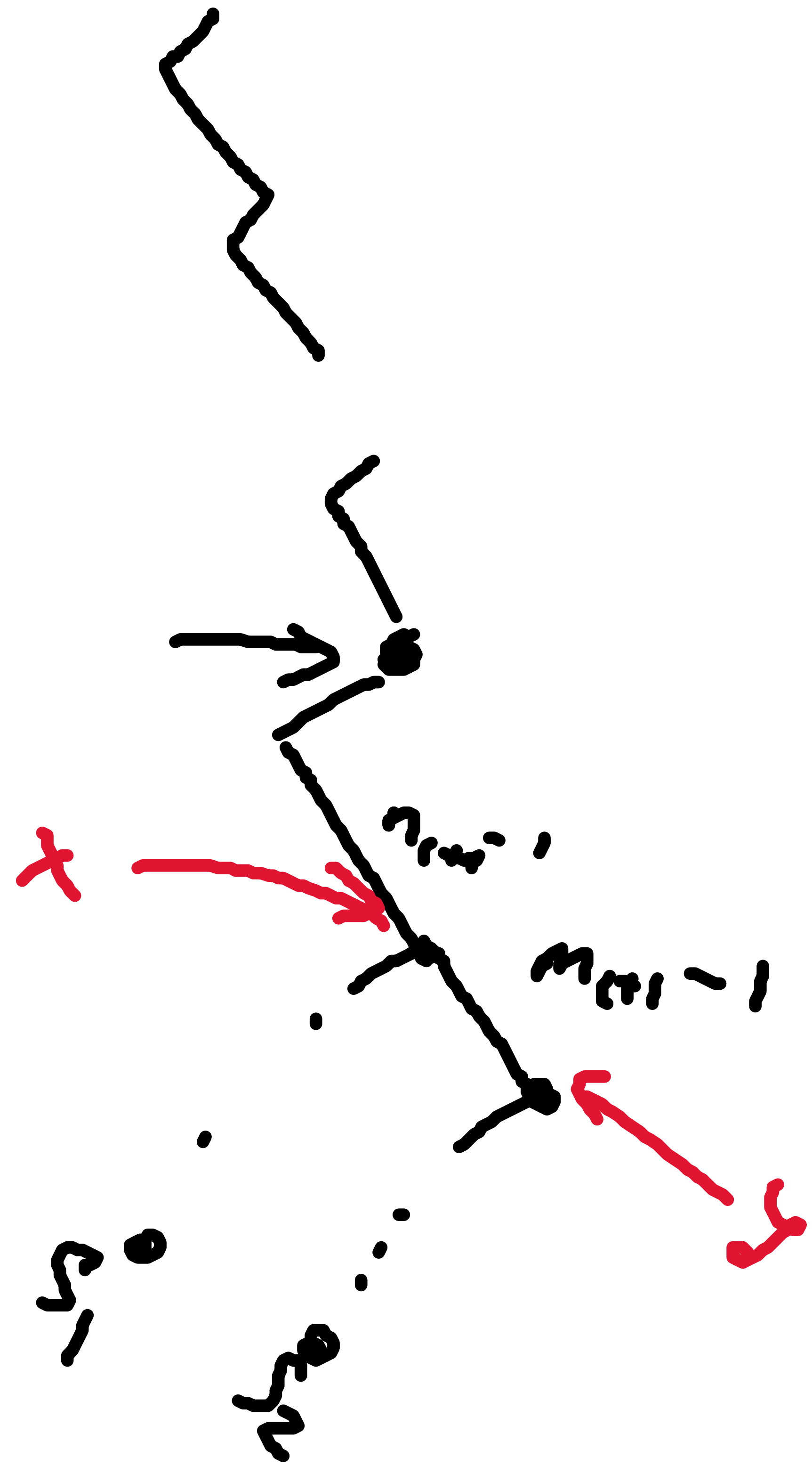
$<_{lex}$

$<_{lex}$

$\underline{n_0 \ldots n_l}\, n_{l+1} \ldots n_k = s_1$

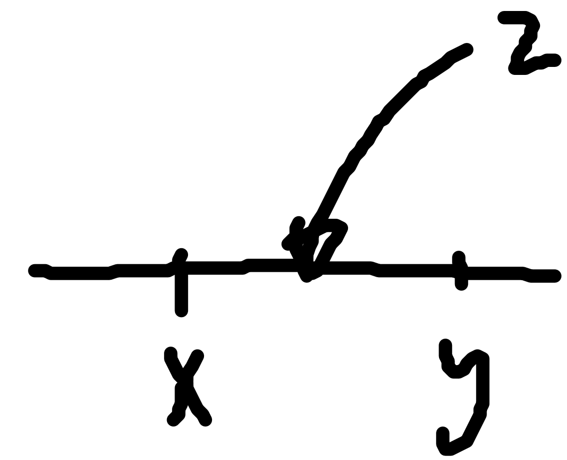
$\, m_{l+1} \ldots m_\ell = s_2$

$m_{l+1} > n_{l+1}$



$Th_{FO}(\mathbb{Q}, <)$

$Th_{MSO}(\mathbb{Q}, <)$



$(\mathbb{Q}, <) \xrightarrow{\text{interpret}} S2S$

$(\{0,1\}^*1, <_{lex}) \cong (\mathbb{Q}, <)$

There is exactly one countable dense linear ordering without endpoints

$\{0,1\}^*1$, $<_{lex}$ is dense and has no min and no max elt

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$s_0 1 <_{lex} s_1 1$