Multi-Büchi Automaton

\((Q, Q_0, S, F_1, \ldots, F_k)\)

An \( \mu \) is accepting if \( \text{Inf}(\mu) \cap F_i \neq \emptyset \).  

\[ \text{Accepting} \leftarrow \text{Accepting} \]

\[ \text{Accepting} \]

(like before: product construction)

\[ \# \text{States} \leq 2 \]

Büchi Automaton

\[ \# \text{States} \leq 2 \cdot 0(141) = 2 \]

The counter each subformula of \( \mu \).

Infinite Trees (Binary)

Tree Domain: \( \{0, 1\}^* \)

Infinite tree over \( \Sigma \)

\[ \text{Cau} \text{sed by a labelling } \lambda : \{0,1\}^* \rightarrow \Sigma \]

(Goal: look at S2S: Theory of Two Successors)

\( S2S = \text{Th}_{\text{MSO}}(\{0, 1\}^*, \text{Succ}_0, \text{Succ}_1) \)

\[ \text{Succ}_0 = \{(s, s) \mid s \in \{0, 1\}^*\} \]

\[ \text{Succ}_1 = \{(s, s) \mid s \in \{0, 1\}^*\} \]

\( S1S = \text{Th}_{\text{MSO}}(\text{IN}, \text{Succ}) \cdot \text{Th}_{\text{MSO}}(\omega^+, \text{Succ}_0) \)

Decidable

\[ f : \omega^+ \rightarrow \text{IN} \quad f(0^i) = i \quad f(0^i\omega) = i+1 \]

(Rabin Theorem) \( S2S \) is Decidable.
Automata

Top-down = non-deterministic

$$A = (Q, q_0, S: Q \times \Sigma \rightarrow 2^{Q \times Q}, \text{Acc})$$

$$q_0$$

$$1 \searrow q_1$$

$$q_2 \leftarrow (q_1, q_2) \in S(q_0, a)$$

Run $$\rho : \{0, 1\}^* \rightarrow Q$$

1) $$\rho(\varepsilon) \in Q_0$$

2) $$(\rho(s.0), \rho(s.1)) \in S(\rho(s), \lambda(s))$$

Let $$t$$ be an infinite branch of a tree $$T$$. Then $$\rho_t$$ can be viewed as an infinite word over $$Q$$.

$$\rho$$ is accepting if for every infinite branch $$t$$, $$\rho_t$$ satisfies $$\text{Acc}$$.

1. Acc: Muller Condition $$F \subseteq 2^Q$$

$$\forall t : \text{Inf}(\rho_t) \in F$$

$$w$$-regular tree languages are those accepted by a tree automaton with a Muller acceptance condition.

2. Acc: B"{u}chi Condition $$F \subseteq Q$$

$$\forall t : \text{Inf}(\rho_t) \cap F \neq \emptyset$$

B"{u}chi-tree languages are those accepted by a tree automaton with a B"{u}chi condition. Are strictly weaker than $$w$$-regular tree languages.
Example

$L = \{ T \mid \exists \text{branch } t \text{ with finitely many } a's \}$

(Few det choices. (guess)

guess: we have $a$'s!

$F = \{ q', q_f \}$

Büchi tree automaton

$L' = \{ T \mid \forall \text{ branch } t \text{ there are only finitely many } a's \}$

Not accepted by any non-deterministic tree automaton with Büchi acceptance condition.

(So we work with Muller).

**Closure Properties**
- Union: $\checkmark$
- Intersection, Product Construction (just like a-string)
- Complementation: Very hard!

(Rabin) For every regular tree language $L$ accepted by an automaton $A$, one can construct an automaton $T$ which accepts $L$.
- Different automata (parity acceptance condition)
- Games (memoryless finite)
- Complementation: Path finder wins.
Theorem

Every MSO sentence over \((\exists, \exists^*, \text{Succ}, \text{Succ})\) can be converted (by an algorithm) to a tree automaton \(A\) with a Muller acceptance condition such that:

\[ T \models \varphi \iff T \text{ is accepted by } A \upharpoonright \varphi. \]

\((\Leftarrow)\) Code the run

- \(\subseteq^*\) - descendant relation
- \(a(x) = \forall y \exists z \forall y (x(z) \land x(y) \rightarrow x^* z \lor y \subseteq^* x)\)
- \(x\) a branch
- \(\forall x \exists y (x(y) \land (\text{Succ}(x,y) \land \text{Succ}(y,x)))\)

\((\Rightarrow)\) Induction

\(\forall \varphi, \exists, T, E\) non-det guess

\(\forall\) MSO sentence over \((\exists, \exists^*, \text{Succ}, \text{Succ})\)

- \(A\upharpoonright \varphi\) Boolean combination \(\exists \forall \varphi(x)\)

over \((\exists, \exists^*, \text{Succ}, \text{Succ})\)

- \(s. \quad \exists q \exists x \exists x \quad T_x \quad \overrightarrow{A} \) accepts \(T_x\)

- \(T_x \uparrow \varphi(x)\)

- \(L(A \upharpoonright \varphi) \neq \emptyset?\)

- Is decidable, and NP-complete.