# Spanning subgraphs of randomly perturbed graphs

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# Randomly perturbed graphs

The model: $G_\alpha \cup G(n,p)$

- $G_\alpha = n$-vertex graph with minimum degree $\delta(G) \geq \alpha n$
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For which $\alpha$ and which $p$ do we a.a.s. find $H = (H_n)$ in $G_\alpha \cup G(n,p)$?

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Why?

Combining two worlds: extremal graph theory, random graphs

- In $G_\alpha$: often large $\alpha$ needed for forcing $H$, because of e.g. connectivity
- In $G(n,p)$: the threshold for $H$ is often influenced by local phenomena, e.g. enough copies of certain graphs at every vertex

# Example: Hamilton cycles

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- $p \ll \frac{1}{n}$ is not possible:

  Adding less than $(1-2\alpha)n$ (random) edges does not suffice if $G_\alpha = K_{\alpha n,(1-\alpha)n}$.

# Example: $K_r$ factors

- Every $n$-vertex graph $G$ with $\delta(G) \geq (1 - \frac{1}{r})n$ has a $K_r$ factor.

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For every $2 \leq k \leq r$, and $1 - \frac{k}{r} < \alpha < 1 - \frac{k-1}{r}$, there is $C$ such that $G_\alpha \cup G(n, \frac{C}{n^{2/k}})$ a.a.s. has a $K_r$-factor.

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- So, if $\delta(G) \geq (1 - \frac{2}{r} + \varepsilon)n$

  then adding linearly many random edges gives a $K_r$-factor.

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For each $\alpha > 0$, $k \geq 2$ there is $\eta > 0$ s.t. for $p \geq n^{-1/k-\eta}$

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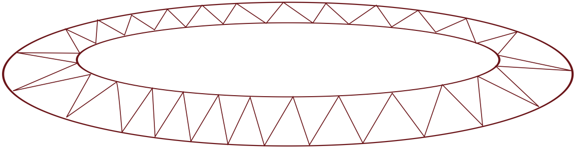
- We have a universality version of the KKS result.

# Proof idea: Hamilton cycle powers

We want to find a copy of $C_n^k$ in $G_\alpha \cup G(n,p)$ for $p \geq n^{-1/k-\eta}$.

Absorbing method: Step 1



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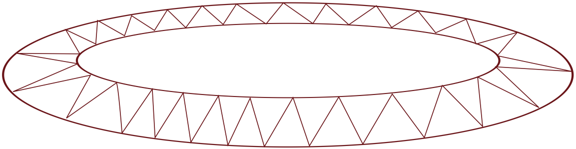

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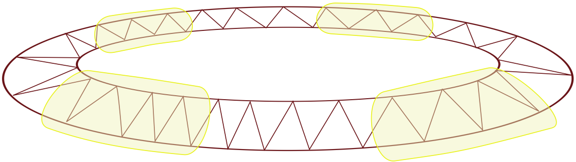

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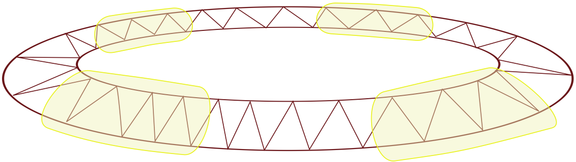
  - $G(n,p)$ contains $F^*$ for $p \geq Cn^{-1/k-\eta}$,
  - choose a random $F^*$-copy $\hat{F}$.



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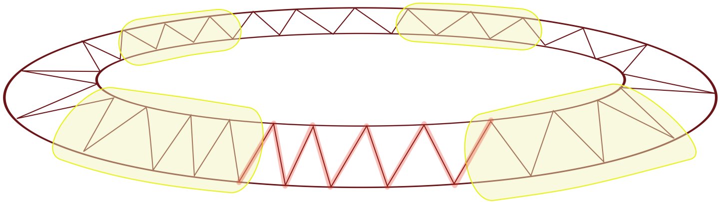

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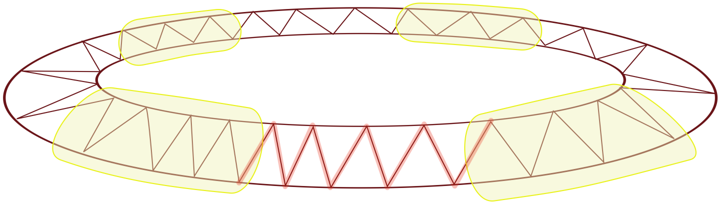

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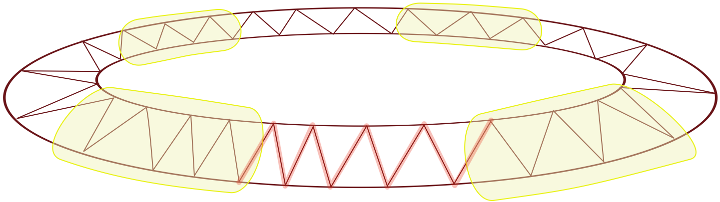
- this is even possible under more restrictions (to be specified shortly).



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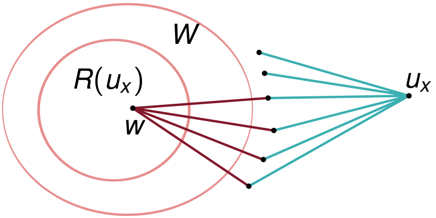

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- hence Step 2 still goes through when $x \in V(F) - V(F^*)$ is image restricted to $R(u_x)$.

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**Theorem** — B, Montgomery, Parczyk, Person

If $\alpha > 0$, $\Delta \neq 4$, $p = \omega(n^{-2/(\Delta+1)})$, $F \in \mathcal{F}(n, \Delta)$,

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  - use Riordan to embed the sparse part,
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- They only get almost spanning.
  - But we use absorption with the help of $G_\alpha$.

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- plus strategy of Ferber, Luh, Nguyen '18:
  - Riordan's Theorem: powerful tool to embed spanning $H$ in $G(n,p)$,
  - but does "not work" if there are dense spots in $H$,
  - so vertex-decompose $H$ into a sparse part and dense spots,
  - use Riordan to embed the sparse part,
  - use Janson and hypergraph Hall to extend to the dense spots.
- They only get almost spanning.
  - But we use absorption with the help of $G_\alpha$.
- They work with a higher probability.
  - But we we don't care which dense spots are used for completion and we can use $G_\alpha$.

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**Theorem** — B, J. Han, Montgomery, Kohayakawa, Parczyk, Person

For each $\alpha > 0$, $\Delta$, there is $C > 0$ s.t. for $p \geq C/n$

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  - $v(G) = n$, $\Delta(G) \leq Dpn$,
  - $\forall U, W \subseteq V(G)$ with $|U|, |W| \geq \varepsilon n$ we have $e(U, W) \geq (p/D)|U||W|$.

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  - randomly embed a small linear sized subtree,
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  - randomly embed a small linear sized subtree,
  - since this is done randomly, we can use it to construct reservoir sets,
  - extend to an almost spanning subtree using a theorem of Haxell,
  - complete using greedy and swapping.

## Concluding remarks

- so, often one saves a log-factor in $p$,
  but not always, and sometimes even more

When? And why?

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Other hypergraphs?

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Thank you!