Resilience of random graphs with respect to Hamiltonicity.

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joint work with
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Definition (Local resilience)

The local resilience of a graph G with respect to some property $\mathcal{P}$ is the maximum number r such that for any subgraph $H \subseteq G$ with $\Delta(H) < r$, the graph $G \setminus H$ satisfies $\mathcal{P}$.

This talk: G will be random and $\mathcal{P}$ will be Hamiltonicity.

Dirac's theorem: resilience version

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Equivalently, we can state Dirac's theorem in the language of resilience.

Theorem (Dirac)

The complete graph K_n is $\lfloor n/2 \rfloor$-resilient with respect to Hamiltonicity.

Dirac's theorem for random graphs

Hamiltonicity in the binomial random graph $G_{n,p}$ is well studied.

Theorem (Pósa, 1976; Koršunov, 1976)

For $p \gg \log n/n$ we have that $G_{n,p}$ contains a Hamiltonian cycle asymptotically almost surely.

Note: $p \ll \log n/n \implies G_{n,p}$ will contain isolated vertices a.a.s.

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Theorem (Lee and Sudakov, 2012)

For $p \gg \log n/n$, the random graph $G_{n,p}$ is a.a.s. $(1/2 - o(1))np$-resilient with respect to Hamiltonicity.

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For $p \gg \log n/n$, the random graph $G_{n,p}$ is a.a.s. $(1/2 - o(1))np$-resilient with respect to Hamiltonicity.

Note that the above threshold is tight: if we could delete anymore edges we could disconnect the graph.

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Theorem ($G_{n,d}$ is Hamiltonian)

For all $3 \leq d \leq n-1$ we have that $G_{n,d}$ is Hamiltonian a.a.s.

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For every $\varepsilon > 0$ and d sufficiently large, a.a.s. $G_{n,d}$ is $(1-\varepsilon)d/6$-resilient with respect to Hamiltonicity.

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They conjectured that the true value should be closer to $d/2$.

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Theorem (Condon, Espuny-Díaz, Girão, Kühn and Osthus, 2019$^+$)

For every $\varepsilon > 0$ there exists D such that, for every $d > D$, the random graph $G_{n,d}$ is a.a.s. $(1/2 - \varepsilon)d$-resilient with respect to Hamiltonicity.

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Theorem (Condon, Espuny-Díaz, Girão, Kühn and Osthus, 2019^+)

For any odd $d > 2$, the random graph $G_{n,d}$ is not a.a.s. $(d-1)/2$-resilient with respect to Hamiltonicity.

Theorem (Pósa, 1962)

Let G have degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ such that $d_i \geq i+1$ for all $i < n/2$. Then, G is Hamiltonian.

Beyond Dirac

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Theorem (Chvátal, 1972)

Let G have degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ such that, for all $i < n/2$, we have that $d_i \geq i+1$ or $d_{n-i} \geq n-i$. Then, G is Hamiltonian.

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Question: Do Pósa's and Chvátal's results have corresponding analogues in $G_{n,p}$, like Dirac's result?
Answer: YES for Pósa, NO for Chvátal.

Beyond Dirac: $G_{n,p}$

Pósa's theorem for random graphs.

Theorem (Condon, Espuny Díaz, Kim, Kühn, Osthus, '18$^+$)

For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C \log n/n$, a.a.s. every subgraph G of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ with $d_i \geq (i + \varepsilon n)p$ for all $i < n/2$ is Hamiltonian.

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There exist counterexamples to Chvátal for random graphs.

Theorem (Condon, Espuny Díaz, Kim, Kühn, Osthus, '18$^+$)

For $p \gg \log n / n$, a.a.s. there exist subgraphs G of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ satisfying $d_i \geq (i + \varepsilon n)p$ or $d_{n-i} \geq (n - i + \varepsilon n)p$ for all $i < n/2$ which are not Hamiltonian.

In fact, there exist subgraphs not containing a perfect matching.

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Definition (3-expander)

An n-vertex graph G is called a 3-expander if it is connected and, for every $S \subseteq [n]$ with $|S| \leq n/400$, we have $|N_G(S)| \geq 3|S|$.

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An n-vertex graph G is called a 3-expander if it is connected and, for every $S \subseteq [n]$ with $|S| \leq n/400$, we have $|N_G(S)| \geq 3|S|$.

- We show there exists a 'sparse' subgraph $R \subseteq G'$ which is a 3-expander.

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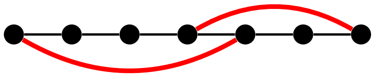
$\Longrightarrow$ there is a 'large' set edges whose inclusion would make R Hamiltonian, or increase the length of a longest path.

In fact, we consider 'booster' pairs of edges, which have the same effect.



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- By passing from R to G' we argue that some of these booster pairs must exist.
- We add these edges to R to make it Hamiltonian or else to increase the length of a longest path.
- We iterate this process at most n times.

Shifted Chvátal resilience.

Conjecture (Condon, Espuny Díaz, Kim, Kühn, Osthus, '18$^+$)

For $p \gg \log n/n$, a.a.s. every subgraph G of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ satisfying $d_i \geq (i + \varepsilon n)p$ or $d_{n-i-\varepsilon n} \geq (n - i + \varepsilon n)p$ for all $i < n/2$ is Hamiltonian.

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The conjecture holds for perfect matchings.

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Open problems: $G_{n,d}$

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Question

Given any fixed even d, determine whether the graph $G_{n,d}$ is a.a.s. $(d/2-1)$-resilient with respect to Hamiltonicity.

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Question

What is the likely resilience of $G_{n,4}$ with respect to Hamiltonicity? Is a graph obtained from $G_{n,4}$ by removing any matching a.a.s. Hamiltonian?