Resolution of the Oberwolfach problem

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Decompositions

Definition

An *F-decomposition* of a graph G is a partition of the edge set of G where each part is isomorphic to F.

- If $G = K_n$ and $F = K_3$, this is a Steiner triple system of order n
- Kirkman's schoolgirl problem (1850): Does K_{15} decompose into triangle factors?
- Walecki's theorem (1892): K_n has a decomposition into Hamilton cycles for every odd n

Common generalization: Oberwolfach problem

Formal statement

cycle factor = vertex disjoint cycles spanning all vertices

Oberwolfach problem (Ringel, 1967)

Let F be any cycle factor on n vertices. Does K_n have an F-decomposition?

posed at Oberwolfach conference and can be rephrased as:

Oberwolfach problem (Ringel, 1967)

Given round tables with n seats in total and n people who eat $\frac{n-1}{2}$ meals together, is it possible to find a seating chart such that everyone sits next to everyone else exactly once?

Formal statement

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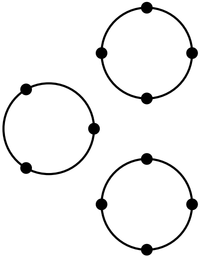
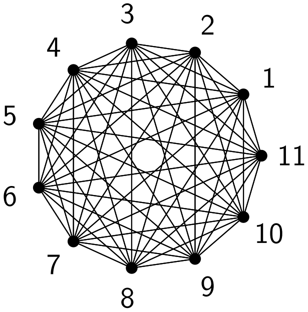
Example: $F = C_3 \,\dot\cup\, C_4 \,\dot\cup\, C_4$





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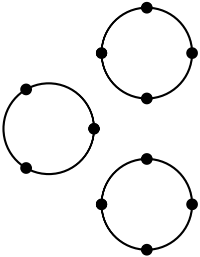

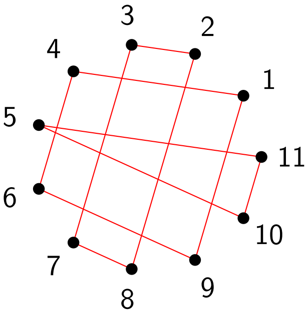

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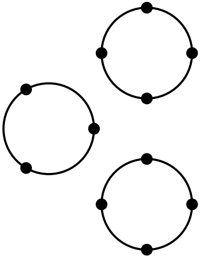

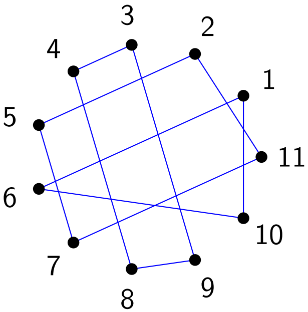

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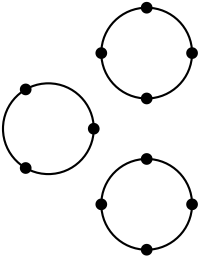

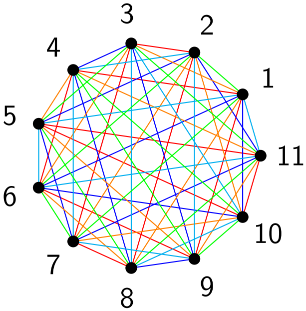

Partial results

- $F =$ Hamilton cycle: Walecki (1892)
- $F =$ triangle factor: Ray-Chaudhuri & Wilson, and Lu (1970s)

$\vdots$

Theorem (Bryant and Scharaschkin, 2009)

$\exists$ infinitely many n such that for any cycle factor F on n vertices, K_n has F-decomposition.

$\vdots$

- Traetta (2013): solution if F consists of two cycles only
- approximate versions by Ferber–Lee–Mousset and Kim–Kühn–Osthus–Tyomkyn (2017)

$\vdots$

≥ 100 research papers covering many partial results

Resolution

The Oberwolfach problem has a solution for all sufficiently large n.

Theorem (Glock, Joos, Kim, Kühn, Osthus, 18^+)

$\exists n_0$ such that for all odd $n \geq n_0$ and any cycle factor F on n vertices, K_n has an F-decomposition.

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- for even n, one can ask for a decomposition of $K_n -$ perfect matching
- **Hamilton-Waterloo problem:** two cycle factors F_1, F_2 given, and prescribed how often each of them is to be used in the decomposition

We also solve these problems (for sufficiently large n).

Most general statement:

Theorem

Suppose $1/n \ll \xi \ll 1/\Delta, \alpha < 1$. Let G be an r-regular n-vertex graph with $r \geq (1-\xi)n$ and let $\mathcal{F}, \mathcal{H}$ be collections of graphs satisfying the following:

- *$\mathcal{F}$ is a collection of at least αn copies of F, where F is a 2-regular n-vertex graph;*
- *each $H \in \mathcal{H}$ is a ξ-separable n-vertex r_H-regular graph for some $r_H \leq \Delta$;*
- *$e(\mathcal{F} \cup \mathcal{H}) = e(G)$.*

Then G decomposes into $\mathcal{F} \cup \mathcal{H}$.

- $\Rightarrow$ can choose first ξn factors greedily
- 'Separable'='small bandwidth'=2-factors, powers of cycles, H-factors...

Proof sketch: simplified setup

A C_ℓ-decomposition of G is resolvable if it can be partitioned into C_ℓ-factors.

So if F is a C_ℓ-factor, then an F-decomposition is precisely a resolvable C_ℓ-decomposition.
(existence of resolvable C_ℓ-decompositions in K_n proved by Alspach, Schellenberg, Stinson, Wagner)

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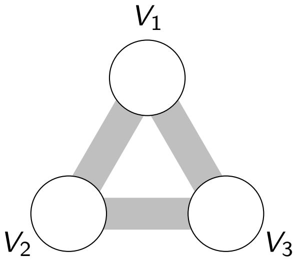
But F might consist of cycles of arbitrary lengths.

Approach: reduce the problem of finding F-decomposition to finding resolvable C_ℓ-decompositions in a quasi-random graphs, for $\ell \in \{3, 4, 5\}$.

Rewiring: simplified setup

Suppose all cycle lengths $\ell_1, \dots, \ell_t$ in F are divisible by 3, and we seek F-decomposition of $K_{n,n,n}$, where $n = \sum \ell_i / 3$.



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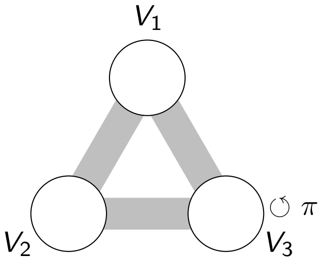

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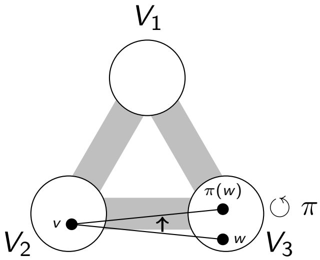

Let π be a permutation on V_3 with cycles of lengths $\ell_1/3, \ldots, \ell_t/3$. Given $H \subseteq K_{n,n,n}$, obtain $\pi(H)$ by replacing vw with $v\pi(w)$ whenever $v \in V_2, w \in V_3$

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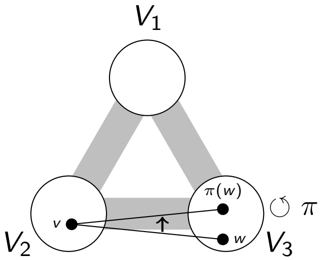

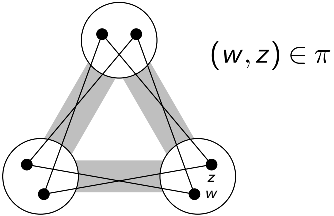

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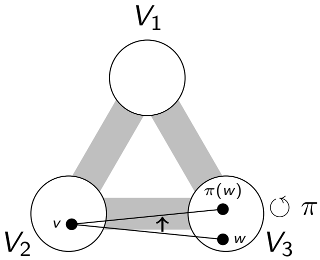

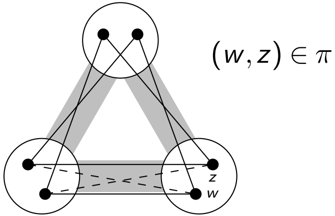

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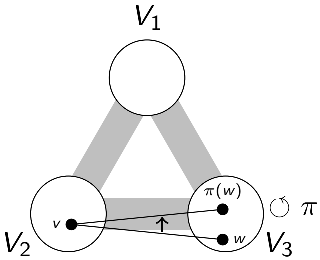

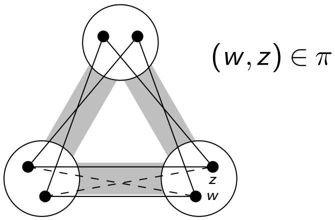

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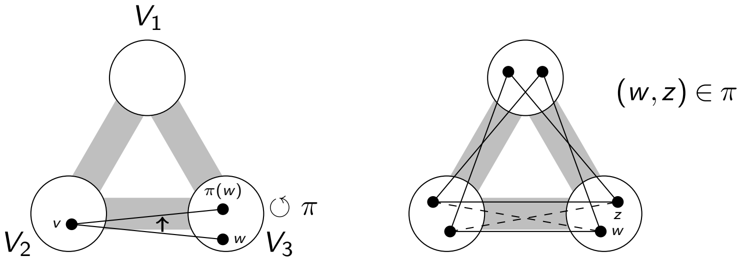


$H = C_3$-factor $\Rightarrow \pi(H) = F$
resolvable C_3-decomposition $\Rightarrow$ F-decomposition
(resolvable C_3-decomposition if $K_{n,n,n}$ exists, eg by Bose, Shrikhande and Parker)

Proof sketch: general setup

Suppose aim to find F-decomposition of K_n.

Absorption approach

1. Take out highly structured absorbing subgraph A.
2. Find approximate decomposition of $K_n - A$ into copies of F to leave a sparse leftover L.
3. Use 'structure' of A to find F-decomposition of $A \cup L$?

First used in context of decompositions for proof of Kelly's conjecture on Hamilton decompositions (Kühn, Osthus'13)

Tool for approximate decomposition

Can apply special case of 'bandwidth theorem for approximate decompositions':

Theorem (Condon, Kim, Kühn, Osthus, 2017^+)

Suppose $H_1, \dots, H_s$ is a collection of n-vertex 2-regular graphs and G is an n-vertex d-regular graph such that

$$d \geq (1 + o(1))9n/10 \text{ and } s \leq (1 - o(1))d/2.$$

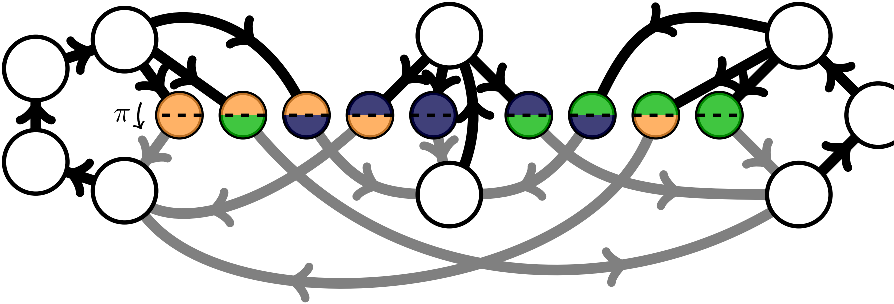
Then $H_1, \dots, H_s$ pack into G.

Proof is based on:

- blow-up lemma for approximate decompositions
- Szemerédi's regularity lemma

actually prove version for general bounded degree graphs of small bandwidth

The absorbing structure A:
grey/black graphs between classes are quasi-random



Can show that A has an F-decomposition via a generalization of switching permutation argument described earlier
(by reducing to resolvable C_3, C_4 and C_5-decompositions)

Tool for finding resolvable decomposition

To show absorber has an F-decomposition (via switching permutation argument), we use:

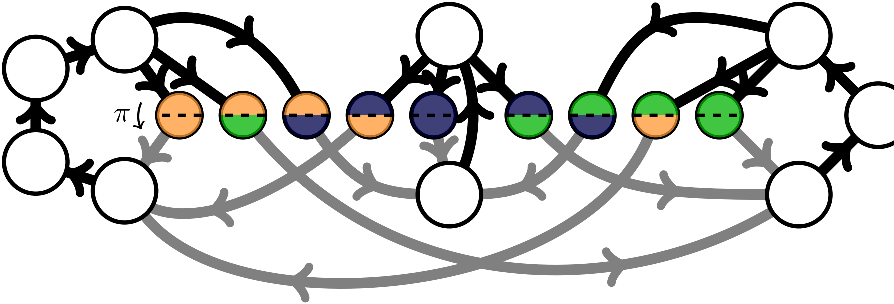
Resolvable C_ℓ-decompositions exist in quasirandom partite graphs

Theorem (Keevash, 2018^+)

Suppose that G is a blow-up of an ℓ-cycle so that each blown-up pair is quasirandom and regular. Then G has a resolvable C_ℓ-decomposition.

Note this follows from:
the existence of ℓ-wheel decompositions in a quasi-random blow-up of an ℓ-wheel

The absorbing structure A:



Let L be the leftover from the approximate decomposition step.

(1) Use suitable edges E of A to cover L with copies of F

(2) Then decompose $A - E$ into copies of F,
(by reducing to resolvable C_3, C_4, and C_5-decomposition).

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Challenges/Problems:

- Need to augment A by adding edges inside clusters in order to cover edges of L between clusters
- Need to do the extension in a 'globally balanced' way, i.e. $E = \cup_i E_i$ is 'balanced' with respect to A. This ensures that $A-E$ is still F-decomposable.
- Above approach only works if there are ηn vertices of F in long cycles

Open problems and related questions

Not only 2-regular graphs?

Conjecture

Suppose $\Delta \ll n$. Let $F_1, \ldots, F_t$ be n-vertex graphs such that F_i is r_i-regular for some $r_i \leq \Delta$ and $\sum_{i \in [t]} r_i = n - 1$. Then there is a decomposition of K_n into $F_1, \ldots, F_t$.

Interesting special case

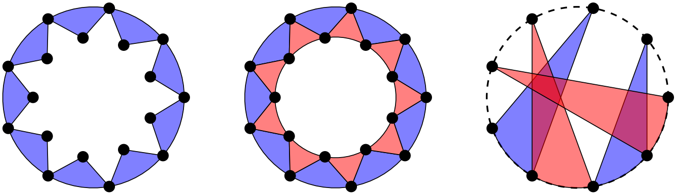
- F_i is a kth power of a Hamilton cycle

Open problems: Hamilton decompositions of hypergraphs

Theorem (Walecki, 1892)

Complete graph K_n has a Hamilton decomposition $\Leftrightarrow$ n odd

Problem: Prove a hypergraph version of Walecki's theorem.



loose Hamilton cycle | tight Hamilton cycle | Berge Hamilton cycle

open | open | solved (K+O)

approximate versions exist for the loose and tight case
(Bal, Frieze, Krivelevich, Loh) for infinitely many n

Open problems: Euler circuits

Conjecture (Chung, Diaconis and Graham, 1989 ($100))

For sufficiently large n, K_n^k has a tight Euler tour iff $k \mid \binom{n-1}{k-1}$.

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Theorem (Glock, Joos, Kühn, Osthus, 18$^+$)

The conjecture is true.

based on existence of F-designs (Glock, Lo, Kühn, Osthus, 17$^+$)

Open problems: Euler circuits

Theorem (Glock, Joos, Kühn, Osthus, 18^+)

For sufficiently large n, K_n^k has a tight Euler tour if and only if $k \mid \binom{n-1}{k-1}$.

Conjecture

Every k-graph G with $\delta_{k-1}(G) \geq (1/2 + o(1))n$ has a tight Euler tour if all vertex degrees are divisible by k.

Bon appetit!