Upper embedding of symmetric configurations with block size 3

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Joint work with Terry Griggs and Jozef Širáň

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A *blocking set* in a configuration of triples is a subset of V which intersects each block $B \in \mathcal{B}$ in 1 or 2 points.

Associated graph

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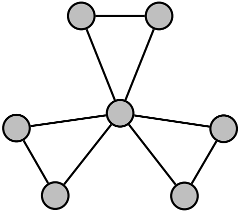

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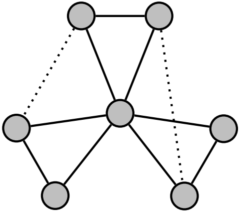

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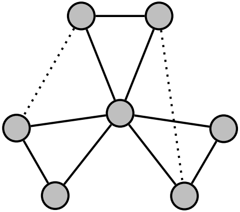

The associated graph of the $STS(7)$ is the complete graph K_7.

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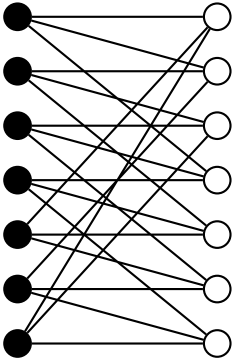
The Levi graph of the $STS(7)$ is the Heawood graph:



Upper embedding

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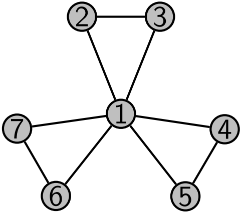
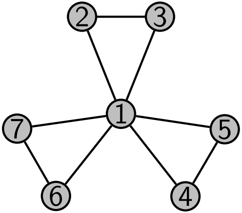
Orientation of the triples is important. In principle, a configuration may be embeddable given one choice of orientations, but not another.





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A configuration $\mathcal{X}$ admits an upper embedding in some orientation if and only if its Levi graph $G(\mathcal{X})$ admits a spanning tree such that each of its co-tree components has an even number of edges.

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Theorem (Griggs, McCourt, Širáň 2019+)

Let $\mathcal{X}$ be a configuration. If its Levi graph $G(\mathcal{X})$ admits a spanning tree such that every point vertex has even valency in the corresponding co-tree, then $\mathcal{X}$ admits an upper embedding in every orientation of triples.

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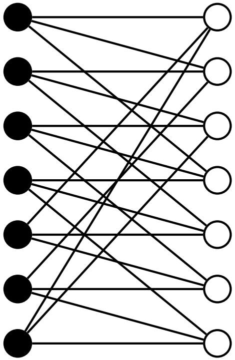
Theorem (E., Griggs, Širáň 2019+)

Let $\mathcal{X} = (V, \mathcal{B})$ be a symmetric configuration v_3 for some odd $v \geq 7$, and let G be its Levi graph. Suppose that there exists a subset S of V of size $(v-1)/2$ with the property that every block $B \in \mathcal{B}$ contains a point of S, and the subgraph of G induced by the points of S and all the blocks in $\mathcal{B}$ is connected. Then $\mathcal{X}$ is upper embeddable in every orientation.



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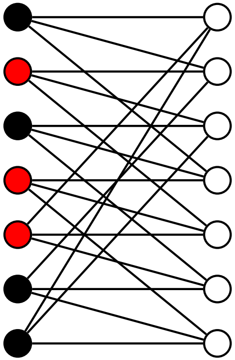

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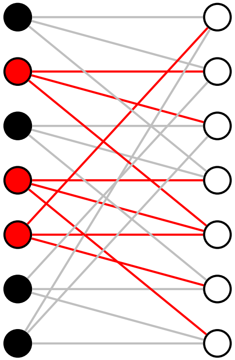

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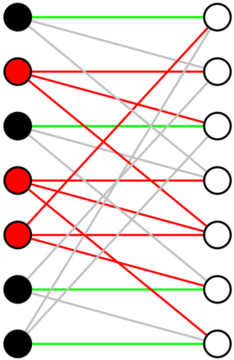

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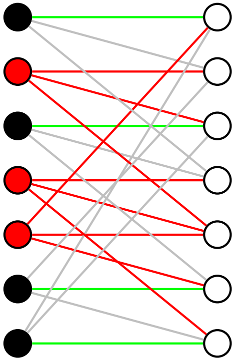


Question: when does a dominating set S with these properties exist?

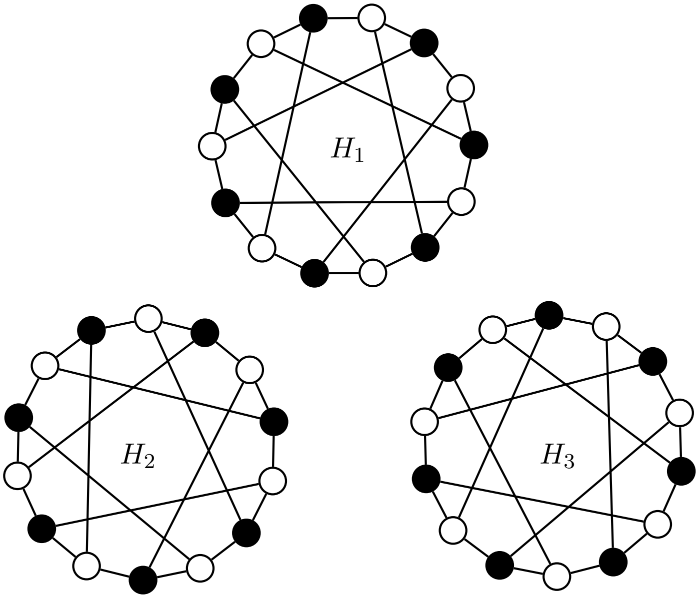
Results

v	# configs	no set
7	1	0
9	3	0
11	31	0
13	2,036	0
15	245,342	0
17	38,904,499	0
19	7,597,040,188	0
21	???	≥ 1

There is at least one configuration not embeddable in any orientation for any odd $v \geq 21$.

There are infinitely many configurations embeddable in every orientation.

A non-embeddable configuration on 21 points



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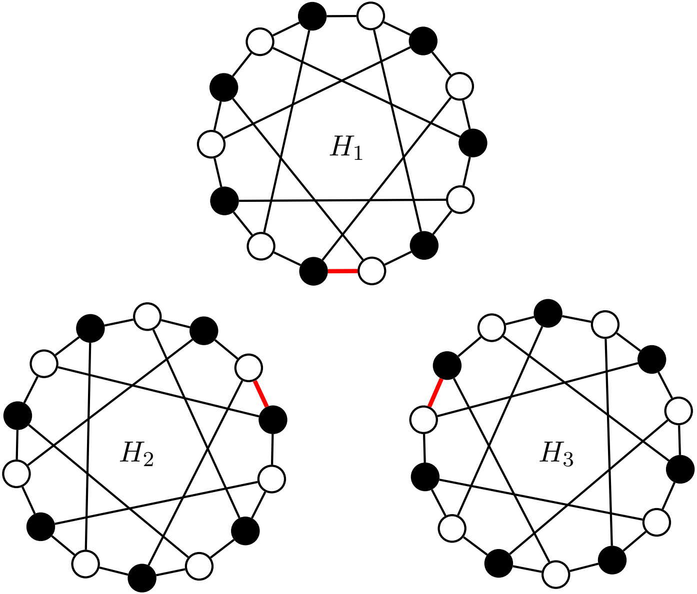

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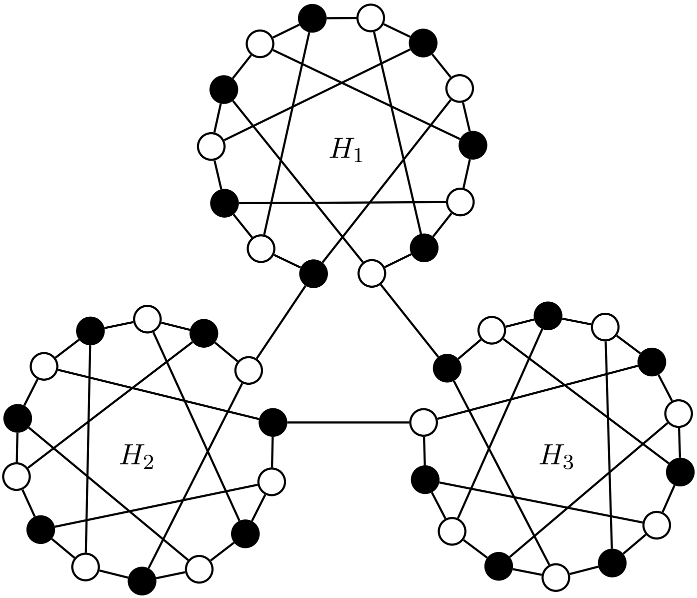
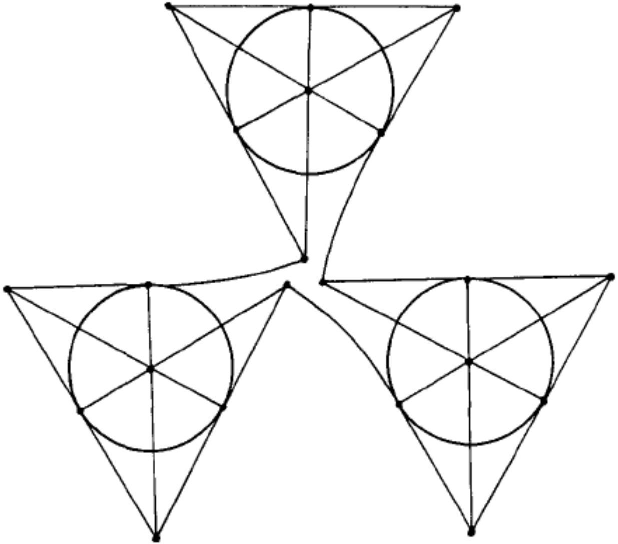


A 3-chromatic 3-regular, 3-uniform hypergraph on 21 vertices (Bollobás/Harris 1985)



Open questions

The Bollobás/Harris construction of 3-chromatic hypergraphs is equivalent to blocking set-free configurations of triples. The existence of blocking set-free configurations on 20, 23, 24 or 26 points is open. What (if anything) is the link to our embeddability question?

Is our "cyclic stitching" construction in some sense the only thing that can generate non-embeddable configurations?

By a result of Archdeacon et al (2004) there is always a set of $(v-1)/2$ points which dominates the blocks. Is there a simple graph-theoretic property which would guarantee our connectedness criterion?

We have no examples of a configuration which is embeddable in some orientation but not all. Is this possible?