Counting Elements of Thompson's Group F

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Thompson's Group F

Thompson's group F was discovered in the 1960s by Richard Thompson. It has a number of interesting properties:

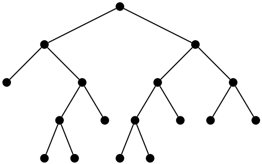
- It is finitely presented
- It has a simple commutator subgroup
- Every proper quotient is abelian
- It is torsion-free
- It can be viewed as equivalence classes of pairs of trees
- It can be viewed as a group of piecewise linear homeomorphisms of the unit interval

Growth of Groups

- If G is a group with a finite generating set S, we can look at the minimal length of a word over S representing each element g of G; called the *word length* of g.
- Define the *growth function* of G to be the function that maps each non-negative integer n to the number of elements of G with word length n.
- The growth function of Thompson's group F is known to be exponential, although the exact function is not known.

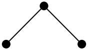
Ordered Rooted Binary Trees

Example



Example

A *caret* is the ordered rooted binary tree with two leaves:

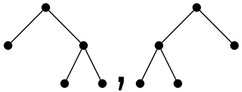


Tree Pairs

Definition

A *tree pair* is an ordered pair (R, S) of finite ordered rooted binary trees, each with the same number of leaves.

Example

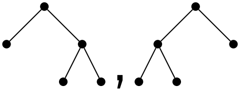


Tree Pairs

Definition

Let (R, S) be a tree pair. If we number the leaves of R (resp. S) from left to right, and fix an index i, then an *elementary expansion* of (R, S) (at i) is the tree pair obtained by attaching a caret to the ith leaves of R and S.

Example



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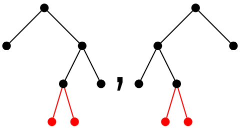


Tree Pairs

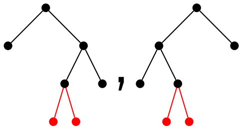
Definition

Let $(R,\ S)$ be a tree pair. Number the leaves of R (resp. S) from left to right, and fix an index i, such that the ith and $(i+1)$th leaves of both R and S lie in carets. An *elementary contraction* of $(R,\ S)$ (at i) is the tree pair obtained by removing the carets in the ith positions of R and S.

Definition

We say a tree pair is *reduced* if it has no elementary contractions

Example



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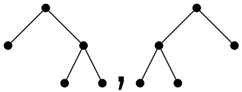


Tree Pairs

Definition

We say two tree pairs are *equivalent* if there is a finite sequence elementary expansions and contractions from one to the other.

Lemma

Equivalence of tree pairs is an equivalence relation.

Lemma

Each equivalence class of tree pairs contains a unique reduced tree pair.

Thompson's Group F

Definition

If $[(R, \ S)]$ and $[(S, \ T)]$ are equivalence classes of tree pairs, we can define a binary operation on the set of all equivalence classes by $[(R, \ S)] \cdot [(S, \ T)] = [(R, \ T)]$.

Lemma

The set of equivalence classes of tree pairs under the above multiplication forms a group, called Thompson's group F.

Exponents

Definition

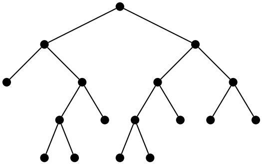
Let R be an ordered rooted binary tree. Let $l_0, \ldots, l_{n-1}$ be the leaves of R, in order, where $n \in \mathbb{N}\backslash\{0\}$. For each $k \in \mathbb{N} \cup \{0\}$ $k \leq n-1$, define the *kth exponent* of R to be the length of the longest path in R, including l_k, which comprises entirely left edges, and never reaches a vertex in the right-side of R.

Definition

The *exponent list* of an ordered rooted binary tree R is the ordered set of exponents of R, under the left to right ordering.

Exponents

Example



The exponent list of this ordered rooted binary tree is

$$(1,\ 2,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0,\ 0).$$

Exponents

Theorem (L.)

Let $n \in \mathbb{N}$, $n \geq 2$, and $(a_0, \ldots, a_{n-1})$ be a sequence of non-negative integers. Then $(a_0, \ldots, a_{n-1})$ is the exponent list if and only if

1. $a_{n-2} = a_{n-1} = 0$;
2. $\sum_{i=k}^{n-3} a_i \leq (n-2) - k$, for all $k \in \{0, \ldots, n-3\}$.

In addition, ordered rooted binary trees have the same exponent list if and only if they are equal.

Exponents

Theorem

A tree pair $(R,\ S)$ with exponent lists $(a_0,\ \dots,\ a_{n-1})$ and $(b_0,\ \dots,\ b_{n-1})$ is reduced if and only if:

1. If k is maximal such that $a_k > 0$, and $a_k < n - 2 - k$ or no such k exists (that is $a_i = 0$ for all i), then there exists l such that $b_l > 0$, and the maximal such l satisfies $b_l = n - 2 - l$;
2. If $a_k,\ b_k > 0$, for some $k \in \{0,\ \dots,\ n-3\}$, then $a_{k+1} > 0$ or $b_{k+1} > 0$.

Counting Trees

Notation

Let $s(n,\ j)$ denote the number of ordered rooted binary trees with n leaves, and whose exponent list sums to j.

Lemma

Let $n \in \mathbb{N}$ be such that $n \geq 3$. Then

1. $s(n,\ 0) = 1$;
2. $s(n,\ 1) = n - 2$;
3. $s(n,\ j) = 0$ *for all* $j > n - 2$;
4. $s(n,\ j) = s(n - 1,\ j) + s(n,\ j - 1)$ *for all* $j \leq n - 2$.

Counting Trees

Proof

We have that

$$\begin{aligned} s(n,\ j) &= \sum_{i=0}^{j} s(n-1,\ i) \\ &= s(n-1,\ j) + \sum_{i=0}^{j-1} s(n,\ i) \\ &= s(n-1,\ j) + s(n,\ j-1). \end{aligned}$$

Further Questions

- What is the generating function for $s(n, j)$?
- What is the generating function for the number of reduced tree pairs with n leaves?

J. W. Cannon, W. J. Floyd, and W. R. Parry.
Introductory notes on Richard Thompson's groups.
Enseign. Math. (2), 42(3-4):215–256, 1996.