Decidability of the WQO problem for permutations under the consecutive pattern involvement

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Introduction: well ordering for posets

- Well order: a totally ordered set with no infinite descending chains – well ordering principle, ordinals,...
- Partial well order: no infinite descending chains and no infinite antichains.
- Alternative term: well quasi order – WQO for short.
- Cherlin (2011): 'tame' (WQO) vs 'wild' (non-WQO).

Substructure orderings in combinatorics

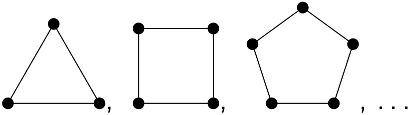
- WQO in combinatorics usually arises in connection with substructure orderings (subgraph, induced subgraph, subpermutation, etc.)
- Automatically no infinite descending antichains (size).
- WQO $=$ no infinite antichains.

Famous example: graph minors

Theorem (Robertson, Seymour)

The set of all finite graphs under the minor ordering is WQO.

However, under subgraph ordering and induced subgraph ordering there are antichains; e.g.: cycles C_n, $n = 3, 4, \ldots$.



WQO problem

Informally: If the entire class is not WQO, can it be algorithmically decided which downward closed subclasses are WQO?

Problem

Given a class $\mathcal{C}$ of combinatorial objects and a partial ordering on $\mathcal{C}$ is the following algorithmic problem decidable?

- INPUT: A finite collection $S_1, \dots, S_m$ of structures from $\mathcal{C}$, which define a downward closed class

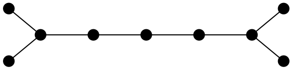
$$\mathcal{D} = \mathrm{Av}(S_1, \dots, S_m) = \{S \in \mathcal{C} \, : \, S_i \not\leq S \text{ for all } i = 1, \dots, m\}.$$

- OUTPUT: YES if $\mathcal{D}$ is WQO, NO if $\mathcal{D}$ is not WQO.

Example: subgraph ordering

Theorem (Ding 1992)

A downward closed set of graphs under the subgraph relation is WQO iff it contains only finitely many cycles and double-ended forks.



Corollary

The WQO problem is decidable for graphs under the subgraph relation.

HOWEVER: the problem is OPEN for the induced subgraph relation (Lozin et al.), digraphs, tournaments (Cherlin & Latka), . . .

Words: subword ordering

A – a finite alphabet; A^* – all words over A.
(Scattered) subword ordering:

$$x_1x_2\dots x_m \leq y_1y_2\dots y_n \Leftrightarrow x_1x_2\dots x_m = y_{i_1}y_{i_2}\dots y_{i_m} \text{ for some } 1 \leq i_1 < i_2 < \dots < y_m \leq n.$$

Example: $aaa \leq ababa$, $bbb \not\leq ababa$.

Theorem (Higman 1952)

A^* is WQO under the subword ordering.

This (and/or Kruskal's Tree Theorem) underpin all non-trivial WQO results.

Words: factor ordering

Factor (or contiguous subword) ordering on A^*:

$$x_1x_2\dots x_m \leq y_1y_2\dots y_n \Leftrightarrow x_1x_2\dots x_m = y_iy_{i+1}\dots y_{i+m-1}$$

for some i.

Example: $aaa \not\leq ababa$, $bab \leq ababa$.

WQO problem for factor ordering (1)

Given: $\mathcal{C} = \mathrm{Av}(w_1, \ldots, w_m)$ – a downward closed set under factor ordering.
Note: $\mathcal{C}$ is a regular language.

Define a directed graph $\Gamma(\mathcal{C})$ as follows.
Let $\ell = \max\{|w_1|, \ldots, |w_m|\}$.

Vertices: $\mathcal{C} \cap A^\ell$.
Edges: $a_1 a_2 \ldots a_\ell \to a_2 \ldots a_\ell a_{\ell+1}$.

Facts

- Every word $w \in \mathcal{C}$ with $|w| \geq \ell$ defines a path in $\Gamma(\mathcal{C})$.
- Every path in $\Gamma(\mathcal{C})$ defines a unique word $w \in \mathcal{C}$ with $|w| \geq \ell$.

WQO problem for factor ordering (2)

Definition

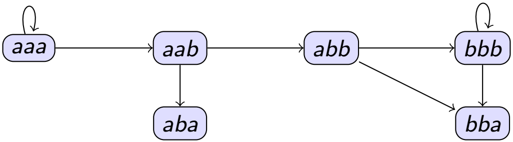
A directed cycle is said to be an in-out cycle if it contains a vertex of indegree > 1 and a vertex of out-degree > 1.

Fact

In-out cycles in $\Gamma(\mathcal{C})$ lead to antichains.

Example

Let: $\mathcal{C} = \text{Av}(baa, bab)$.



An in-out cycle: $bbb \to bbb$.
Antichain: $ab^i a$, $i \geq 3$.

WQO problem for factor ordering (3)

Theorem

$\mathcal{C} = \mathsf{Av}(w_1, \ldots, w_m)$ contains an antichain if and only if $\Gamma(\mathcal{C})$ contains an in-out cycle.

Corollary

WQO problem is decidable for A^* under the factor ordering.

This result is a special case of the following:

Theorem (Atminas, Lozin, Moshkov 2013)

It is decidable in polynomial time whether a regular language over A contains an antichain under the factor ordering.

Permutations

Permutation $=$ a sequence $\sigma = s_1 \dots s_n$ s.t.

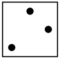
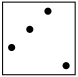
$$\{s_1, \dots, s_n\} = \{1, \dots, n\}.$$

$\mathcal{S}$ $=$ the set of all permutations.

$\mathcal{S}_n$ $=$ all permutations of length n; $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$.

Canonical representatives of non-repeating sequences.

Example

$\mathrm{perm}(2, 7, 5) = 132 =$ , $\mathrm{perm}(1, e, \pi, i^2) = 2341 =$ .

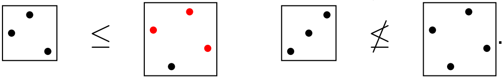
Permutations: involvement ordering

Analogous to subword ordering.

$$s_1 \dots s_m \leq t_1 \dots t_n \Leftrightarrow s_1 \dots s_m = \text{perm}(t_{i_1} \dots t_{i_m}) \text{ for some } 1 \leq i_1 < \dots < i_m \leq n.$$

Example

$231 \quad \leq \quad 3142 \qquad\qquad 123 \quad \not\leq \quad 3142$



Open Problem

Is the WQO problem decidable for permutations under the involvement ordering?

Permutations: consecutive involvement ordering

$$s_1 s_2 \dots s_m \leq t_1 t_2 \dots t_n \Leftrightarrow s_1 s_2 \dots s_m = \text{perm}(t_i t_{i+1} \dots t_{i+m-1})$$
$$\text{for some } i.$$

Example

$231 \not\leq 3142$, $213 \leq 3142$.

Question

Is the WQO problem decidable for permutations under the consecutive involvement ordering?

Graph $\Gamma(\mathcal{C})$

$\mathcal{C} = \mathrm{Av}(\pi_1, \dots, \pi_m)$; $\ell = \max\{|\pi_1|, \dots, |\pi_m|\}$.

Vertices: $\mathcal{C} \cap S_\ell$;
Edges: $a_1 \dots a_\ell \to b_1 \dots b_\ell \Leftrightarrow \mathrm{perm}(a_2 \dots a_\ell) = \mathrm{perm}(b_1 \dots b_{\ell-1})$.

Facts

- Every permutation $\sigma \in \mathcal{C}$ with $|\sigma| \geq \ell$ defines a path in $\Gamma(\mathcal{C})$.
- BUT: a path in $\Gamma(\mathcal{C})$ may correspond to several σ (an ambiguous path).

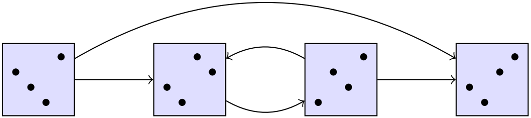
First obstacle to WQO: in-out cycles

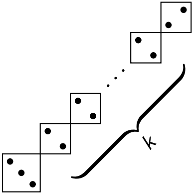
Fact

If $\Gamma(\mathcal{C})$ has an in-out cycle then $\mathcal{C}$ contains an infinite antichain.

Example

$\mathcal{C} = \mathsf{Av}(231, 312, 1234, 1243, 1432, 2431, 3142, 4213, 4321)$.

$\Gamma(\mathcal{C}) =$ 

Antichain: 

for $k = 1, 2, \ldots$.

Bicyclic classes

Bicycle: digraph consisting of two simple cycles connected by a single non-trivial path.

$\mathcal{C}$ is bicyclic if $\Gamma(\mathcal{C})$ is a bicycle (or a degenerate form, where one or both cycles are not present).

Fact

If $\mathcal{C}$ has no in-out cycles then it is a finite union of bicyclic classes.

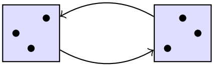
So we may restrict our WQO considerations to bicyclic classes.

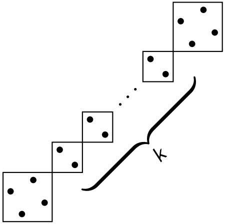
Second obstacle to WQO: ambiguous paths

Fact

If a bicyclic class $\mathcal{C}$ has an ambiguous path which begins and ends on the same cycle then $\mathcal{C}$ contains an infinite antichain.

Example



Antichain: 

for $k = 1, 2, \ldots$.

Going around a cycle

Consider a cycle with no ambiguous paths.

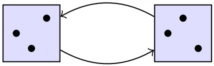
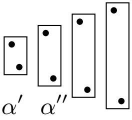
The effect of repeatedly going around the cycle can be viewed as a permutation $\alpha = a_1 \dots a_n$ which is repeatedly juxtaposed with itself according to a fixed rule.

This in turn can be represented as a juxtaposition $\alpha'\alpha'' = a'_1 \dots a'_n a''_1 \dots a''_n$ of two copies of α.

Let a_i, a_j be two entries, consecutive in value.

If $a''_i < a'_i < a'_j < a''_j$ we say that (a_i, a_j) is a nested interval of α.

Example

Cycle:  Juxtaposition: 

$\cdots$

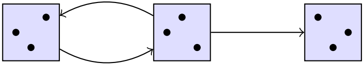
Note: No antichains here.

Third obstacle to WQO: inserting a point into a nested interval

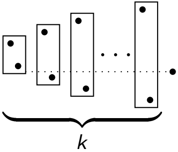
Fact

If a bicyclic class $\mathcal{C}$ has an ambiguous path which begins on the initial cycle and ends on the connecting path which allows insertion into a nested interval of α then $\mathcal{C}$ contains an infinite antichain.

Example



Antichain:



No more obstacles to WQO

Theorem (McDevitt, NR)

A downward closed class $\mathcal{C} = \mathsf{Av}(\pi_1, \ldots, \pi_m)$ *of permutations under the consecutive involvement ordering is WQO iff the following three conditions are satisfied:*

(1) $\Gamma(\mathcal{C})$ *has no in-out cycles;*

(2) *no bicyclic component of* $\Gamma(\mathcal{C})$ *has an ambiguous path starting and ending on the same cycle;*

(3) *no bicyclic component of* $\Gamma(\mathcal{C})$ *permits insertion into a nested interval.*

Corollary (McDevitt, NR)

WQO problem is decidable for permutations under the factor ordering.

Concluding remarks

Similar techniques, involving the graph $\Gamma(\mathcal{C})$, can be used to prove that the atomicity problem is decidable for: (a) permutations under the consecutive factor ordering; and (b) words under factor ordering.

A downward closed set is atomic if it is not a union of two proper downward closed subsets; equivalently: Joint Embedding Property.

Braunfeld (2019) proved that atomicity is undecidable for: (a) graphs under the induced subgraph ordering; and (b) 3-dimensional permutations under the involvement ordering.

Questions

Questions

Are the atomicity and WQO problems decidable for 3-dimensional permutations, where in two dimensions the ordering is consecutive, and in the remaining one it is not? What can be said about higher-dimensional permutations?

Question

To what extent can the WQO and atomicity results be extended to infinitely based classes? E.g. $\mathrm{Av}(B)$ where B is a periodic antichain?

THANK YOU!