On the location of the zeros of the independence polynomial of bounded degree graphs

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Partly based on joint works with Viresh Patel, and Han Peters, UvA

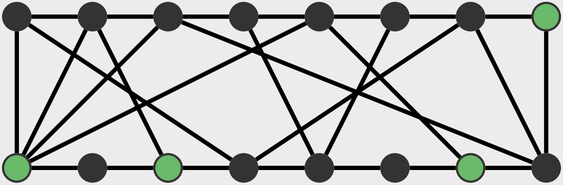
My collaborators

From left to right: Viresh Patel, and Han Peters:

Structure of this presentation

- Definition of the independence polynomial and why do we care about its zeros. With motivation from
  - Statistical physics
  - Computer Science
- Survey of results
- Ingredients of proofs: connection to complex dynamical systems

The independence polynomial



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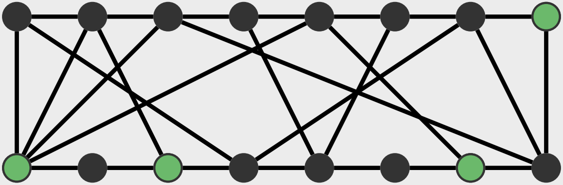

For a graph $G = (V, E)$, the independence polynomial is defined as

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V \\ I \text{ independent}}} \lambda^{|I|} = \sum_{k=0}^{\alpha(G)} i_k \lambda^k.$$

Interpretation from statistical physics

A model for a gas:

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Question

Can one (approximately) compute $Z_G(\lambda)$ efficiently? Can one (approximately) sample from this distribution efficiently?

Independence polynomial

Recall

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Answer

$$\lambda \frac{\frac{d}{d\lambda}\log(Z_G(\lambda))}{|V(G)|}.$$

Studying $\log(Z_G(\lambda))$

In statistical physics one typically considers a sequence of larger and larger subgraphs (G_n) of a fixed infinite graph G and asks

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Theorem (Lee-Yang 1952)

If the complex roots of the polynomials Z_{G_n} do not accumulate on λ, then f is analytic at λ.

Location of zeros of Z_G and computing Z_G

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Answer (Patel and Regts 2019, Barvinok 2016)

Fix an (inifinite) collection of graphs $\mathcal{G}$. Let $\lambda^* > 0$. If there exists an open region $U \subset \mathbb{C}$ containing $[0, \lambda^*]$ such that for all $\lambda \in U$ and $G \in \mathcal{G}$, $Z_G(\lambda) \neq 0$, then there is an algorithm that (approximately) computes $Z_G(\lambda^*)$ efficiently for all $G \in \mathcal{G}$.

Location/absence of zeros of Z_G

For $\Delta \in \mathbb{N}_{\geq 3}$ let $\mathcal{G}_\Delta$ be the collection of graphs of maximum degree at most Δ and let

$$\lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}.$$

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Theorem (Peters and Regts, 2019; Conjectured by Sokal in 2001)

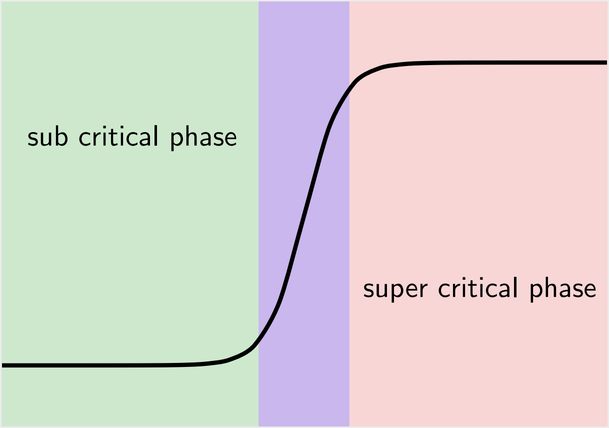
- *There exists an open region D_Δ in $\mathbb{C}$ containing $[0, \lambda_c(\Delta))$ such that for any graph $G \in \mathcal{G}_\Delta$ and $\lambda \in D_\Delta$,*

$$Z_G(\lambda) \neq 0.$$

- *There exists a sequence of graphs $(G_n) \subset \mathcal{G}_\Delta$ and a sequence (λ_n) such that*

$$Z_{G_n}(\lambda_n) = 0 \text{ and } \lambda_n \to \lambda_c.$$

Computational threshold at phase transition



$0 \leq \lambda < \lambda_c$ λ_c $\lambda_c < \lambda$

Weitz '06 Sly and Sun '12

More zero-free regions/ regions with zeros for Z_G

Contributions from:

- Scott and Sokal, 2005
- Peters Regts, 2019
- Bezáková, Galanis, Goldberg and Štefankovič, 2018
- Bencs and Csikvári, 2018
- Buys, 2019
- Vondrák and Srivastava, 2019+

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Fundamental recurrence for Z_G: for a fixed vertex v

$$Z_G(\lambda) = \lambda Z_{G\setminus N[v]}(\lambda) + Z_{G-v}(\lambda).$$

Definition

Let us define, assuming $Z_{G-v}(\lambda) \neq 0$,

$$R_{G,v} := \frac{\lambda Z_{G\setminus N[v]}(\lambda)}{Z_{G-v}(\lambda)}.$$

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A useful observation:

$$R_{G,v} \neq -1 \text{ if and only if } Z_G(\lambda) \neq 0.$$

A recurrence relation

Definition

Let G be a graph with fixed vertex v_0. Let $v_1, \ldots, v_d$ be the neighbors of v_0 in G (in any order). Set $G_0 = G - v_0$ and define for $i = 1, \ldots, d$, $G_i := G_{i-1} - v_i$. Then $G_d = G \setminus N[v_0]$.

Lemma

Suppose $Z_{G_i}(\lambda) \neq 0$ for all $i = 0, \ldots, d$. Then

$$R_{G,v_0} = \frac{\lambda}{\prod_{i=1}^{d}(1 + R_{G_{i-1},v_i})}.$$

Proof sketch of Shearer's bound

Theorem (Shearer 1985, Scott Sokal 2005)

Let $\Delta \geq 3$. *Let* $H = (V, E) \in \mathcal{G}_\Delta$ *and let* λ *be such that* $|\lambda| \leq \lambda^*(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$. *Then* $Z_H(\lambda) \neq 0$.

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Proof.

Idea: assume H connected. Use

$$R_{G,v_0} = \frac{\lambda}{\prod_{i=1}^{d}(1 + R_{G_{i-1},v_i})},$$

to prove inductively that the following holds for all $U \subseteq V \setminus \{u_0\}$ (for some fixed u_0):

(i) $Z_{H[U]}(\lambda) \neq 0$,

(ii) if $u \in U$ has a neighbour in $V \setminus U$, then $|R_{H[U],u}| < 1/\Delta$.

□

Proof of Sokal's conjecture: ideas

Theorem (Peters and Regts 2019)

There exists an open region D_Δ *in* $\mathbb{C}$ *containing* $[0, \lambda_c(\Delta))$ *such that for any graph* G *of max. degree at most* Δ *and* $\lambda \in D_\Delta$, $Z_G(\lambda) \neq 0$.

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4. Show that $\mathcal{U}$ also works for F.

Proof of existence of zeros: ideas

- The function f_λ corresponds to the recurrence for Cayley trees.
- Use chaotic behaviour of complex dynamical system $\{f_\lambda^{\circ n}(\lambda)\}$, where

$$f_\lambda(x) = \frac{\lambda}{(1+x)^d}$$

Future work

- Find a full description of the joint zero-free region of Z_G for $G \in \mathcal{G}_\Delta$.
- Find out for which λ approximating $Z_G(\lambda)$ is hard.
- Extend ideas to other models in statistical physics: Ising model, Potts model.

Thank you for your attention!