Hermitian Laplacians, Cheeger inequalities, and 2-variable linear equations

Luca Zanetti (Cambridge)

Joint work with Huan Li (Fudan) and He Sun (Edinburgh)

The Max-2-Lin(k) problem

We are given a set of m linear equations of the form

$$x_i - x_j \equiv c_{ij} \mod k$$

Our goal is to find an assignment to $\{x_i\}_{i=1}^n$ maximising the number of satisfied equations

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- if the system is satisfiable, it is trivial to find a satisfying assignment
- otherwise, finding an optimal solution is NP-hard
- there exists an SDP-based algorithm that, given a $(1-\epsilon)$-satisfiable instance, finds an assignment satisfying a $1 - O\left(\sqrt{\epsilon \log k}\right)$ fraction of equations
- almost optimal according to the Unique Games Conjecture

Max-2-Lin(k) and Max-Cut

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- the system is satisfiable if and only if G is bipartite (i.e., there exists a cut containing all edges in the graph)
- Maximising the number of satisfied equations is equivalent to finding the largest cut in the graph

  Recall: cut = "edges across a bipartition of the vertices"

Max-Cut and the smallest eigenvalue

Suppose (wlog) G is connected and d-regular

- Let A be the adjacency matrix of G

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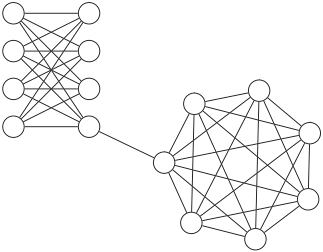
- for the $\Rightarrow$ part we need to be more careful

Max-Cut and the smallest eigenvalue: Trevisan's result

Example:



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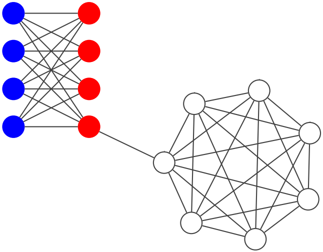

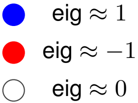

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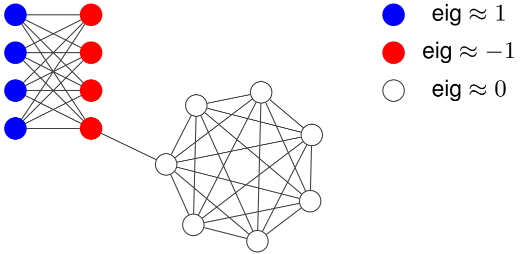

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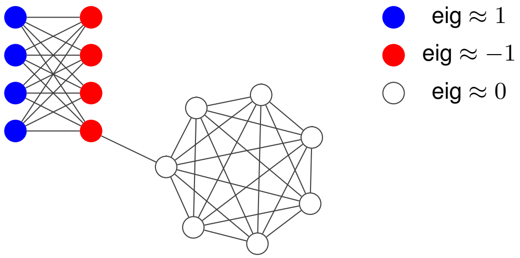

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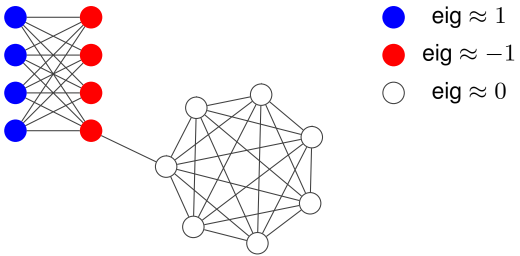

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Q? Can we generalise this framework to arbitrary 2-variable linear systems?

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$$A_{\mathcal{S}}(i,j) = \begin{cases} \omega_k^{c_{ij}} & \text{if } i \xrightarrow{c_{ij}} j \\ \overline{\omega_k}^{c_{ij}} & \text{if } j \xrightarrow{c_{ij}} i \\ 0 & \text{o.w.} \end{cases}$$

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We then define the Laplacian operator $L_{\mathcal{S}} \in \mathbb{C}^{n \times n}$ as

$$L_{\mathcal{S}} = I - D_{\mathcal{S}}^{-1/2} A_{\mathcal{S}} D_{\mathcal{S}}^{-1/2}$$

where $D_{\mathcal{S}}$ is diagonal and $D_{\mathcal{S}}(i,i) = \sum_j |A_{\mathcal{S}}(i,j)|$.

Eigenvalues of the Hermitian Laplacian

Let λ be the smallest eigenvalue of $L_{\mathcal{S}}$. By Courant-Fischer,

$$\lambda = \min_{y \in \mathbb{C}^n \setminus \{0\}} \frac{\sum_{i \to j} \left| y(i) - \omega_k^{c_{ij}} y(j) \right|^2}{\sum_{i \to j} |y(i)|^2}$$

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Can we prove a robust version of this fact?

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Let $\mathcal{S}$ be a system of 2-variable equations modulo k and $\phi : [n] \to [k] \cup \{\perp\}$ a partial assignment.

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Theorem

Let $\mathcal{S}$ be a system of 2-variable equations modulo k, and $\lambda(L_{\mathcal{S}})$ the smallest eigenvalue of its Laplacian. Then,

$$\lambda(L_{\mathcal{S}}) \lesssim \min_{\phi:[n]\to[k]\cup\{\perp\}} u(\phi) \lesssim k\sqrt{\lambda(L_{\mathcal{S}})}$$

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- if $|y(i)| \approx 0$, assign $\phi(i) = \perp$
- Otherwise, (randomly) divide the complex plane in k regions corresponding to the k possible assignments

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Let $y \in \mathbb{C}^n$ be the bottom eigenvector of $L_{\mathcal{S}}$. Assume (w.l.o.g.) $\max_i |y(i)| = 1$.

Rounding algorithm:

- Draw $t \in [0,1]$ such that t^2 is distributed uniformly over $[0,1]$
- Set $\phi(i) = \perp$ for any i s.t. $|y(i)| < t$
- Draw $\eta \in [0, 2\pi/k]$ u.a.r.
- Set $\phi(i) = j \iff |y(i)| \geq t \wedge \Theta(y(i), \mathrm{e}^{i\eta}) \in [2\pi j/k, 2\pi(j+1)/k)$

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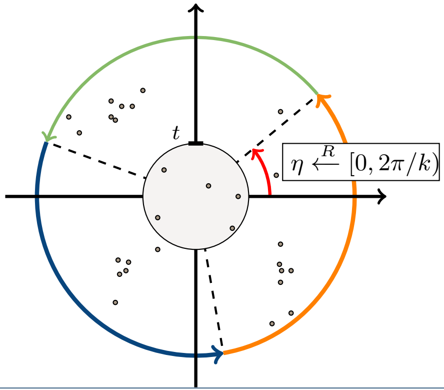
- Draw $\eta \in [0, 2\pi/k]$ u.a.r.
- Set $\phi(i) = j \iff |y(i)| \geq t \wedge \Theta(y(i), \mathrm{e}^{i\eta}) \in [2\pi j/k, 2\pi(j+1)/k)$



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Algorithm

INPUT: a system of equations $\mathcal{S}$

1. Compute the eigenvector y corresponding to $\lambda(L_{\mathcal{S}})$
2. Apply the rounding procedure to find a partial assignment ϕ
3. Let $\mathrm{vol}(\phi) = \sum_{i \to j} \mathbf{1}\{\phi(i) \neq \perp\}$
4. if $u(\phi) \geq (1 - 1/k)\,\mathrm{vol}(\phi)$ then Return a full random assignment
5. else if ϕ is a full assignment then Return ϕ
6. else Recurse on a set of equations defined on variables $\{x_i \colon \phi(i) = \perp\}$.

Thank you