Categorical Perspectives

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Abstract. Quantales provide an abstract algebra of actions equipped with a binary operation of sequential composition and an infinitary operation (sup) of non-deterministic amalgamation. Formally, quantales are monoids in the category of complete sup-lattices. Quantales have provided a setting for studying ontic actions and various process equivalences. More recently, they have been used as a semantic setting for discussion of epistemic actions and quantum logics.

The archetypical example is given by the monoid of binary relations on a set S. We think of these as non-deterministic actions, acting on states that are elements of S. It is known that every quantale may be represented as a quantale of relations - indeed, Q has several representations derived from the Cayley representation of the underlying monoid, as a set of relations on Q. However, these representations uses a subset of relations that is not, in general, closed under suprema, so non-determinism is not faithfully represented.

We seek to interpret Q as a quantale of relations over a non-classical set. Given a quantale, Q, we construct the classifying topos for a set equipped with relations reflecting the structure of Q. We represent Q as a quantale of global sections of relations on the generic object in this classifying topos. This provides a universal, or generic, relational representation of Q, in the normal sense of classifying topoi.

The site supporting this classifying topos has as objects finitely presented transition systems that represent lax quotients of Q. We interpret these as perspectives, representing a local focus on some aspects of the world - a finitely-observable set of observations of the effects of some actions, compatible with the structure of Q. This category is equipped with a Grothendieck topology that forces the representation to be strict,

We conclude with a discussion of conditions neccessary for the generic representation to be faithful.

Quantale
Set Topos Geometric Logic
Global sections

1 Introduction

We begin with a brief discussion of quantales, and in particular quantales of relations, in the category of sets, **Set**.

Quantales of relations may be defined in elementary topos, and we observe that our **Set**-based discussion is constructive, in the sense that it can be interpreted within an elementary topos.

The global sections of a quantale form a quantale, and some properties of quantales are preserved by the global sections functor.

We present some examples, then describe the construction of a generic representation of a quantale in the quantale of global sections of relations on some sheaf.

Quantales in topoi.

Representations lax quotients, equivalence relations, generic representation, generators

2 Quantales

Quantales are abstract algebras. We view them as algebras of non-deterministic actions, with a specialisation ordering ($\alpha \leq \beta$ if β may do everything that α may do), and operations of sequential and non-deterministic composition. We call the elements *actions*. Non-deterministic composition gives joins (least upper bounds) for the specialisation order. For sequential composition, we use multiplicative notation, writing 1 for the identity,[1] and $x\,;y$, or often simply xy, for the composite action, *x then y*.

Definition 1 ([Mul86]). *A* quantale*, $\mathfrak{Q}$ is a monoid in the category of $\bigvee$-lattices: that is, a complete sup-lattice equipped with an associative, binary operation (;) with* identity*, 1, such that $1\,;x = x = x\,;1$, that preserves joins ($\bigvee$) in each argument:*[2]

$$(x\,;y)\,;z = x\,;(y\,;z) \qquad \Big(\bigvee_i x_i\Big)\,;y = \bigvee_i (x_i\,;y) \qquad x\,;\bigvee_i y_i = \bigvee_i (x\,;y_i) \quad (1)$$

[1] A note on notation: Mulvey (*op. cit.*), and others, use e for the identity, and 1 for the top element of the lattice—which we denote by 1, and $\top$, respectively.

[2] All our quantales are *unital* in the sense of Mulvey (*op. cit.*). However, our key construction can also be applied to a general quantale. We abuse notation, and write $\mathfrak{Q}$ also for the underlying set, poset, sup-lattice, or monoid of $\mathfrak{Q}$ when we need to refer to these.

Note that composition (;) is monotone in each argument, since it preserves $\vee$. We write $\top = \bigvee Q$ for the top element of the $\bigvee$-lattice, Q, and 0 for the bottom element, $0 = \bot = \bigvee \emptyset$, since it satisfies $0\,;x = 0 = x\,;0$.

An involution *on* $\mathfrak{Q}$ *is a map* $\mathfrak{Q} \xrightarrow{(-)^*} \mathfrak{Q}$ *such that:*

$$x^{**} = x \qquad (xy)^* = y^* x^* \qquad (\bigvee x_i)^* = \bigvee x_i^* \tag{2}$$

A quantale equipped with an involution is said to be involutive. *A quantale whose underlying sup-lattice is a frame—which means simply that it is completely distributive:* $x \wedge \bigvee y_i = \bigvee (x \wedge y_i)$*—is called a* quantal frame.

Example 1. The binary relations, $R \subseteq \mathcal{X} \times \mathcal{X}$, on a set $\mathcal{X}$, ordered by set inclusion and composed by relational composition, form an involutive quantal frame, $\mathfrak{R}(\mathcal{X})$, where $\mathbf{r}^*$ is the *reciprocal* of $\mathbf{r}$, defined by $x\,\mathbf{r}^*\,y \iff y\,\mathbf{r}\,x$.

Example 2. The (sup-preserving) automorphisms of any sup-lattice $\mathit{\Lambda}$, equipped with the pointwise ordering and function composition, form a quantale, $\mathfrak{A}(\mathit{\Lambda})$.

There is an obvious isomorphism, $\mathfrak{R}(\mathcal{X}) \equiv \mathfrak{A}(\wp(\mathcal{X}))$, between the automorphisms of the powerset and the quantale of relations—take the image of a set under a relation, or apply an automorphism to a singleton set.

Definition 2. *A* (strict) morphism *of quantales* $\mathfrak{Q} \xrightarrow{f} \mathfrak{R}$ *is a map that is both a morphism of sup-lattices, and a monoid homomorphism: it preserves* $\bigvee, ;, 0, 1$. *A* lax morphism *of quantales is a morphism* f *of sup-lattices such that* $1_{\mathfrak{R}} \leq f(1_{\mathfrak{Q}})$ *(*f *is* 1-lax*) and* $f(\alpha) ; f(\beta) \leq f(\alpha ; \beta)$ *(*f *is* ;-lax*). An* op-lax morphism *of quantales is a morphism* f *of sup-lattices, such that* $f(1_{\mathfrak{Q}}) \leq 1_{\mathfrak{R}}$ *(*f *is* 1-strict*) and* $f(\alpha ; \beta) \leq f(\alpha) ; f(\beta)$ *(*f *is* ;-strict*).*

Maps from a set X to (the underlying set of) a quantale, $\mathfrak{Q}$ form a sup-lattice under the pointwise partial order, so concrete categories of quantales are naturally enriched (hom-sets form posets, and often sup-lattices). Each of these three classes of morphism (lax, op-lax, strict) is closed under sup. So in each case the hom-sets are naturally sup-lattices. Clearly, lax morphisms are also closed under meets $\bigwedge$, so we can make the following definition.

Definition 3. *Given quantales,* $\mathfrak{Q}$*,* $\mathfrak{R}$*, a set* $\mathit{\Lambda} \xrightarrow{i} \mathfrak{Q}$ *of actions in* $\mathfrak{Q}$*, and a map* $\mathit{\Lambda} \xrightarrow{\lambda} \mathfrak{R}$*, we define* $\mathfrak{Q} \xrightarrow{\ulcorner\lambda\urcorner} \mathfrak{R}$*, the* extension of λ along i*, to be the minimal lax morphism such that* $\lambda \leq i \circ \ulcorner\lambda\urcorner$.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, "\lambda"'] & \mathfrak{Q} \arrow[d, "{\ulcorner\lambda\urcorner = \bigwedge\{\rho \mid \mathfrak{Q} \xrightarrow[lax]{\rho} \mathfrak{R} \text{ and } \lambda \leq \rho\}}"] \\
& \mathfrak{R}
\end{tikzcd}$$

We will use this construction later.

Example 3. Given a function $X \xrightarrow{f} Y$, we have three order-preserving maps

$$\wp(X) \begin{array}{c} \xrightarrow{\forall_f} \\ \xleftarrow{f^*} \\ \xrightarrow{\exists_f} \end{array} \wp(Y)$$

These form a stack of adjoints, $\exists_f \dashv f^* \dashv \forall_f$. As left adjoints preserve limits (in this setting, suprema), the two left adjoints form an adjoint pair of sup-lattice morphisms.

Given $\mathcal{X} \xrightarrow{f} \mathcal{Y}$, the image, $\wp(\mathcal{X} \times \mathcal{X}) \xrightarrow{\exists_{f \times f}} \wp(\mathcal{Y} \times \mathcal{Y})$, acts as an op-lax quantale morphism $\mathfrak{R}(\mathcal{X}) \xrightarrow{f} \mathfrak{R}(\mathcal{Y})$; the inverse image $\wp(\mathcal{Y} \times \mathcal{Y}) \xrightarrow{(f \times f)^*} \wp(\mathcal{X} \times \mathcal{X})$ is a lax quantale morphism.

Example 4. Given an adjoint pair of sup-lattice morphisms $\Gamma \underset{f}{\overset{g}{\leftrightarrows}} \Lambda$ ($\top$), we have a lax morphism $\mathfrak{A}(\Gamma) \xrightarrow{[f,g]} \mathfrak{A}(\Lambda)$, and an op-lax morphism $\mathfrak{A}(\Lambda) \xrightarrow{[g,f]} \mathfrak{A}(\Gamma)$. For example, the adjoint pair $\wp(\mathcal{X}) \underset{\exists_f}{\overset{f^*}{\leftrightarrows}} \wp(\mathcal{Y})$ ($\top$), arising from a map $\mathcal{X} \xrightarrow{f} \mathcal{Y}$, gives rise to the $\begin{array}{c}\text{op-lax}\\ \text{lax}\end{array}$morphisms $\mathfrak{A}(\wp(\mathcal{X})) \underset{[\exists_f, f^*]}{\overset{[f^*,\exists_f]}{\rightleftarrows}} \mathfrak{A}(\wp(\mathcal{Y}))$.

Unsurprisingly, the constructions of $\mathfrak{R}(\mathcal{X})$ and $\mathfrak{A}(\wp(\mathcal{X}))$ are functorial. The isomorphism, $\mathfrak{R}(\mathcal{X}) \equiv \mathfrak{A}(\wp(\mathcal{X}))$, between the automorphisms of the powerset and the quantale of relations is a natural isomorphism for the functorial constructions described the in the previous example.

Definition 4 (Labeled Transition System). *A labeled transition system* (LTS)*, $\mathfrak{L}$, is a set $\mathcal{X}$ together with a labeled set of actions, given by a map $\Lambda \longrightarrow \mathfrak{R}(\mathcal{X})$ from a set, Λ, of labels to the set of binary relations on $\mathcal{X}$. We say $\mathfrak{L}$ is* finite *if both $\mathcal{X}$ and Λ are finite sets.*

2.1 Constructive Properties

A quantale of relations, $\mathfrak{R}(\mathcal{X})$, has some particular properties. Algebraically, it has an involution, given by taking the transpose of each relation: $x\,\mathbf{r}\,y \vdash y\,\mathbf{r}^*\,x$. $\wp(\mathcal{X} \times \mathcal{X})$, like any powerset, is an atomic distributive sup-lattice, or atomic frame. Algebraically, the atoms, which are the singletons $\{\langle x, y\rangle\}$, are sub-units, so $\mathfrak{R}(\mathcal{X})$ is an inverse quantale frame, in the sense of Resende [Res07].

3 Quantales in a Topos

The construction and properties of quantales in **Set** discussed above can be formalised entirely constructively—in the sense that it can be interpreted in any topos. If $\mathbb{E} \xrightarrow{\Gamma} \mathbb{B}$ is a topos over a "base" topos $\mathbb{B}$, and $\mathfrak{Q}$ is a quantale in $\mathbb{E}$

then $\Gamma\mathfrak{Q}$ is a quantale in $\mathbb{B}$. For any object $\mathcal{X}$ of $\mathbb{E}$, the object $\wp(\mathcal{X} \times \mathcal{X})$ of relations on $\mathcal{X}$ is a quantale, $\mathfrak{R}(\mathcal{X})$ in $\mathbb{E}$. So we can construct quantales in $\mathbb{B}$ by taking global sections of quantales of relations in $\mathbb{E}$, or represent quantales in $\mathbb{B}$ as global sections of a quantale of relations in $\mathbb{E}$.

In particular, an object of a Grothendieck topos is a sheaf. The binary relations on a sheaf on a site $\mathbf{C}$ form a quantale in the topos $\mathrm{Sh}(\mathbf{C})$, whose global sections form a quantale in **Set**.

3.1 Geometric and Cartesian Properties

A *geometric* formula is built from atoms using the connectives $\wedge, \bigvee, \exists$. A *cartesian* formula is built using $\wedge, \exists!$, where $\exists!$ is unique existence. Geometric formulae are preserved by inverse images, cartesian formulae by direct images, or global sections.

Quantales are cartesian, in the sense that they can be axiomatised by entailments between cartesian formulae, so the global sections of a quantale form a quantale in **Set**. Similarly, every quantale of sections of a relational quantale is a regular quantal frame.

Quantales of relations are, internally, inverse quantal frames. However, their global sections can be more general.

3.2 Grothendieck topoi

We consider Grothendieck topoi $Sh(C) \longrightarrow \mathbf{Set}$. An object $\mathcal{X}$ of this topos is a sheaf on the site C. The object $\wp(\mathcal{X} \times \mathcal{X})$ with its powerset ordering and relational composition (given by internal existential quantification) is an internal quantale. Just as in sets, it is internally isomorphic to the quantale of sup-preserving functions $\wp(\mathcal{X}) \longrightarrow \wp(\mathcal{X})$. Its global sections correspond to subobjects of $\mathcal{X} \times \mathcal{X}$. These form a quantale in **Set**.

We look first at the examples of presheaves over a category and sheaves over a topological space or locale.

Presheaves A pre-sheaf on $\mathbf{C}$ is a functor $A \in \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}$. For $f : Y \longrightarrow X$ in $\mathbf{C}$ we have sets $A(X), A(Y)$ and restriction maps $\upharpoonright_f\colon A(X) \longrightarrow A(Y)$ that are functorial, $\upharpoonright_{(f\circ g)}=\upharpoonright_f \mathbin{;} \upharpoonright_g$.[3]

If A, B are presheaves on $\mathbf{C}$, we describe the function-space presheaf B^A of functions $A \longrightarrow B$. An element F of the set $B^A(X)$ is a family of functions

$$F_f : A(Y) \longrightarrow B(Y), \text{ indexed by arrows } Y \xrightarrow{f} X \text{ in } \mathbf{C}$$

satisfying the *naturality condition*, that $(F_f y) \upharpoonright_g = F_{f\circ g}(y \upharpoonright_g)$. Restrictions are given by $(F \upharpoonright_f)_g = F_{f\circ g}$.

Consider the powerset pre-sheaf, $\wp(A) = \Omega^A$, of a presheaf A. An element U of $\wp(A)(X)$ is given by a family of subsets $U_f \subseteq \wp(A(Y))$, indexed by morphisms

[3] We write restriction with right-application.

$f : Y \longrightarrow X$ in $\mathbf{C}$, such that $x \in U_f \Rightarrow x\restriction_g \in U_{f\circ g}$. The restriction maps are given by $(U\restriction_f)_g = U_{f\circ g}$, and are thus functorial by construction.

We consider the Kripke-Joyal-Beth semantics for a many-sorted language, $\mathcal{L}$, whose sorts are interpreted by sheaves, with each constant of sort A interpreted by global section of the sheaf A. We define an extension $\mathcal{L}(X)$ of $\mathcal{L}$ for each object X of $\mathbf{C}$, given by adding a constant of sort A for each element of $A(X)$. For $f : Y \longrightarrow X$, we have a restriction map $\restriction_f$: $\mathcal{L}(X) \longrightarrow \mathcal{L}(Y)$, taking a formula φ of $\mathcal{L}(X)$ to the formula $\varphi\restriction_f$ of $\mathcal{L}(X)$ obtained by making the substitution $[(a\restriction_f)/a]$ for each constant a in φ. Each global section of A can be considered as a constant invariant under restrictions, and so common to every $\mathcal{L}(X)$. Each morphism $a : A \longrightarrow B$ in $\mathrm{Sh}(\mathbf{C})$ corresponds to a global section of B^A, and hence to an invariant function symbol, of sort B^A. Each subobject $A' \rightarrowtail A$ corresponds to a global section of Ω^A, and hence to an invariant constant of sort $\wp(A)$.

The Kripke-Joyal-Beth semantics uses sheaves over a site to interpret higher-order logic (HOL), a many-sorted logic with sorts, $A, B, \ldots$, function sorts, B^A, and power sorts, $\wp(A)$. It is the instantiation in Grothendieck topoi of the general interpretation of higher-order logic in elementary topoi discovered by Lawvere and Tierney. The interpretation in a presheaf topos generalises Kripke's semantics of possible worlds. The extension to *sites*, categories with a notion of covering given by a Grothendieck topology, generalises models introduced by Beth to interpret intuitionistic analysis.

We give first the definition for Kripke-Joyal semantics using presheaves. Forcing is a relation $f \Vdash \varphi$, where $f : Y \longrightarrow X$ in $\mathbf{C}$ and φ is a sentence (a formula with no free variables) in $\mathcal{L}(X)$. It has the fundamental *stability* property that $f\circ g \Vdash \varphi \Leftrightarrow g \Vdash \varphi\restriction_f$, for every $g : Z \longrightarrow Y$ in $\mathbf{C}$. The forcing relation is defined to satisfy this property for atomic formulae. For example, for $f : Y \longrightarrow X$, $a \in A(X)$, and $P \in \wp(A)(X)$, we specify $f \Vdash a \in P$ **iff** $a\restriction_f \in P_f$. Stability follows immediately from the definition of the powerset presheaf.

The forcing relation is extended to a compound formula, φ, by an inductive definition on the structure of φ, to satisfy the following rules, where $Z \xrightarrow{f\circ g} X$ is the composite $Z \xrightarrow{g} Y \xrightarrow{f} X$:

$$f \Vdash \top \qquad f \Vdash a = b \iff a = b \qquad f \Vdash a \in U \iff a \in U_f \tag{3}$$

$$f \Vdash \varphi \wedge \psi \iff f \Vdash \varphi \text{ and } f \Vdash \psi \qquad f \Vdash \varphi \vee \psi \iff f \Vdash \varphi \text{ or } f \Vdash \psi \tag{4}$$

$$f \Vdash \varphi \rightarrow \psi \iff \forall g. f\circ g \Vdash \varphi\restriction_g \Rightarrow f\circ g \Vdash \psi\restriction_g \tag{5}$$

$$f \Vdash \exists x.\varphi \iff \exists a \in A(Y).\, f \Vdash \varphi[a/x] \tag{6}$$

$$f \Vdash \forall x.\varphi \iff \forall a \in A(Z).\, f\circ g \Vdash \varphi[a/x] \tag{7}$$

Terms are interpreted We will sometimes write $f \Vdash \varphi$, where φ is a formula with free variables, to mean that for every $g : Z \longrightarrow Y$ and for every closed substitution instance $\varphi[\bar{a}/\bar{x}]$ of φ in $\mathcal{L}(Z)$, we have $f\circ g \Vdash \varphi[\bar{a}/\bar{x}]$, or equivalently, that $f \Vdash \forall\bar{x}.\varphi$, where $\bar{x}$ includes all free variables of φ.

Example 5. Let M be a monoid, a category with one object, M; as arrows, we take elements $a, b, \ldots,$ of M composing thus: $M \xleftarrow{a} M \xleftarrow{b} M = M \xleftarrow{ab} M$. The presheaves form the category of M-sets: an object is a set, $\mathcal{X} = X(M)$, equipped with a right M-action. The global sections of an object are its M-invariant elements. Products are **Set**-products, with point-wise action: $\langle x, y\rangle g = \langle xg, yg\rangle$. Global relations between M-sets are given by relations on the underlying sets that are invariant under the action. The powerset object is given by families $\mathcal{X}'_f \rightarrowtail \mathcal{X}$ such that

So, for an M-set $\mathcal{X}$, we have the internal quantale $\mathfrak{R}(\mathcal{X}) = \wp(\mathcal{X} \times \mathcal{X})$. The global sections of $\mathfrak{R}(\mathcal{X})$ correspond to subobjects of $\mathcal{X} \times \mathcal{X}$; that is, to subsets of $\mathcal{X} \times \mathcal{X}$ in **Set** invariant under the action of M.

Consider the topos Z_2-sets, of sets with an automorphism of order 2. The generic Z_2-set is a doubleton $\{a, b\}$ with the non-trivial Z_2-action. This has precisely two, indistinguishable elements: it satisfies $\exists x, y.\, x \neq y \wedge \forall z.\, z = x \vee z = y$.

This quantale of global sections of the quantale of relations on the generic Z_2-set are the subsets is precisely the quantale used by Brown and Gurr [BG91] as an example of a non-relational quantale. The quantale has two atoms, s is the relation $x \neq y$, or $\{\langle a, b\rangle, \langle b, a\rangle\}$ which swaps the two elements; $1_{a,b}$ is the relation $x = y$, or $\{\langle a, a\rangle, \langle b, b\rangle\}$. The only other invariant relations, and so the only other constructively definable relations on the two-element set Z_2 are the trivial relations, $\top, \bot$. Observe that {s,1} forms a submonoid of the quantale of global sections, isomorphic to Z_2. The quantale is regular $s = s^*$ and $s^2 = 1$, but s is not a partial unit, nor a join of such.

$$\begin{tikzcd}
 & \top & \\
s \arrow[ur] & & 1 \arrow[ul] \\
 & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

If we adjoin an element, c, (with trivial Z_2 action) to the generic Z_2-set we have two atoms in the lattice of invariant subsets: one $\{a, b\}$ with a non-trivial automorphism; the other $\{c\}$.

Now consider an invariant relation, $\mathbf{r}$. It must satisfy the following constraints:

$$a\,\mathbf{r}\,b \iff b\,\mathbf{r}\,a \quad c\,\mathbf{r}\,a \iff c\,\mathbf{r}\,b \quad a\,\mathbf{r}\,c \iff b\,\mathbf{r}\,c$$

So the quantale of global relations has the two atoms as before, augmented with three new atomic elements $1_c = \{\langle c, c\rangle\}, p = \{\langle a, c\rangle, \langle b, c\rangle\}, p^* = \{\langle c, a\rangle, \langle c, b\rangle\}$.

The identity is the join of the two partial units $1 = 1_{a,b} \wedge 1_c$. Again, p^* is an inverse for p, but neither is a join of partial units.

4 Partial Equivalence Relations

Definition 5. *A* partial equivalence relation (PER) *in an involutive quantale is an action, e, which is both symmetric ($e^* \leq e$) and transitive ($e^2 \leq e$). An* equivalence *is a* PER*, e, which is reflexive ($1 \leq e$). We say a* respects *a* PER*, e, iff $ea \leq ae$.*

The set of actions $a \in \mathfrak{Q}$ that respect a PERe form a sub-quantale of $\mathfrak{Q}$. The map

5 Representations

Definition 6. *A* (lax, op-lax, strict) representation*, ρ, of $\mathfrak{Q}$ is a (lax, op-lax, strict) quantale morphism to a relational quantale $\mathfrak{Q} \xrightarrow{\rho} \mathfrak{R}(\mathcal{X})$. We say ρ, is* faithful *if its underlying map is $1-1$, and* full *if the map is onto.*

Let $\mathfrak{Q}$ be a quantale, and $\mathcal{L}(\mathfrak{Q})$ the geometric language with a binary relation symbol $\mathbf{r}$ for each element r of $\mathfrak{Q}$.

This means that $\mathcal{L}(\mathfrak{Q})$ is a purely relational predicate logic with: binary relations, $\mathbf{r}$(for each $r \in \mathfrak{Q}$), and $=$; finitary conjunction, $\top, \wedge$; infinitary disjunction, $\bigvee$; and existential quantification, $\exists$.

A model of $\mathcal{L}(\mathfrak{Q})$ in $\mathbb{S}$, the topos of sets, is given by a set $\mathcal{X}$ and a map $[\![-]\!] : \mathfrak{Q} \longrightarrow \wp(\mathcal{X} \times \mathcal{X})$. A model in a Grothendieck topos $\mathbb{G} \xrightarrow{\pi} \mathbb{S}$ is given similarly, by an object $\mathcal{X}$ and a morphism $[\![-]\!] : \pi^*\mathfrak{Q} \longrightarrow \wp(\mathcal{X} \times \mathcal{X})$ in $\mathbb{G}$, or equivalently, a map $\mathfrak{Q} \longrightarrow \mathbb{G}[1, \wp(\mathcal{X} \times \mathcal{X})]$, or a $\mathfrak{Q}$-indexed family of subobjects, $[\![\mathbf{r}]\!] \rightarrowtail \mathcal{X} \times \mathcal{X}$, where $r \in \mathfrak{Q}$.

We recall that $[\![-]\!]$ can be extended, by induction on the structure of φ, to define an *interpretation*, as a subobject $[\![\varphi]\!]_{\bar{x}} \rightarrowtail \mathcal{X}^{\bar{x}}$, for each well-formed formula (wff) φ, and sequence $\bar{x}$, of distinct variables of $\mathcal{L}$, containing the free variables of φ.[4]

$$[\![x = y]\!]_{\langle x,y\rangle} = \mathcal{X} \overset{\Delta}{\rightarrowtail} \mathcal{X} \times \mathcal{X} \tag{8}$$

$$[\![x\, \mathbf{r}\, y]\!]_{\langle x,y\rangle} = [\![r]\!] \rightarrowtail \mathcal{X} \times \mathcal{X} \tag{9}$$

$$[\![\varphi \wedge \psi]\!]_{\bar{x}} = [\![\varphi]\!]_{\bar{x}} \wedge [\![\varphi]\!]_{\bar{x}} \tag{10}$$

$$[\![\bigvee \varphi_i]\!]_{\bar{x}} = \bigvee [\![\varphi_i]\!]_{\bar{x}} \tag{11}$$

$$[\![\exists x.\, \varphi]\!]_{\bar{y}} = \exists^x_{\bar{y}} [\![\varphi]\!]_{x,\bar{y}} \tag{12}$$

$$[\![\varphi]\!]_{\bar{y}} = \pi^*_\sigma [\![\varphi]\!]_{\bar{x}} \quad \text{where } \bar{x} \xrightarrow{\sigma} \bar{y} \tag{13}$$

[4] We write $\mathcal{X}^{\bar{y}} \xrightarrow{\pi_\sigma} \mathcal{X}^{\bar{x}}$ for the projection corresponding to a "substitution" map $\bar{x} \xrightarrow{\sigma} \bar{y}$, and $\exists^x_{\bar{y}}$ for $\exists_{\pi_\sigma}$ where σ is the inclusion $\bar{y} \overset{\sigma}{\rightarrowtail} x, \bar{y}$.

We fix an interpretation, and consider *sequents* of the form $\varphi \vdash \psi$. We say the sequent $\varphi \vdash \psi$ is *valid* (written, $\bar{x}; \varphi \vDash \psi$) iff $[\![\varphi]\!]_{\bar{x}} \subseteq [\![\psi]\!]_{\bar{x}}$, whenever $\bar{x}$ contains the free variables of φ and ψ.

Lemma 1. *The interpretation* $[\![-]\!]$ *is*

$$
\begin{array}{rlr}
\text{order-preserving } \textit{iff} & x\,\mathbf{r}\,y \vDash x\,\mathbf{s}\,y \qquad \textit{whenever } r \leq s & (14)\\
\text{involutive } \textit{iff} & x\,\mathbf{r}\,y \vDash y\,\mathbf{r}^*\,x & (15)\\
\mathbf{;}\text{ -lax } \textit{iff} & x\,\mathbf{r}\,y \wedge y\,\mathbf{s}\,z \vDash x\,(\mathbf{rs})\,z & (16)\\
\text{1-lax } \textit{iff} & \vDash x\,\mathbf{1}\,x & (17)\\
\text{sup-preserving } \textit{iff} & x\,(\bigvee_{\mathbf{i}} \mathbf{s_i})\,y \vDash \bigvee_i (x\,\mathbf{s_i}\,y) & (18)\\
\mathbf{;}\text{ -strict } \textit{iff} & x\,(\mathbf{rs})\,z \vDash \exists y.\,(x\,\mathbf{r}\,y \wedge y\,\mathbf{s}\,z) & (19)\\
\text{1-strict } \textit{iff} & x\,\mathbf{1}\,y \vDash x = y & (20)
\end{array}
$$

A finitely presented model is a finite set $\mathcal{F}$ together with a monotone map $\mathfrak{Q} \xrightarrow{\rho} \mathfrak{R}(\mathcal{F})$ such that $\rho r\,\mathbf{;}\,\rho s \leq \rho rs$, $\rho r^* = (\rho r)*$, and $\rho 1 = 1$.

5.1 Grothendieck Topology

Let P be a poset, and ν a $\wedge$-preserving map from P to a frame. For $y \in P$, we say a family $x_i \leq y$ ν*-covers* y iff $\nu(y) \leq \bigvee_i \nu(x_i)$. The collection of ν-covers is a *Grothendieck topology* on the poset. Every Grothendieck topology arises in this way—we can construct a suitable frame, and the map, ν, from the topology. An intersection of Grothendieck topologies is a Grothendieck topology—since a product of frames is a frame. The coarse topology, in which every family is covering, is a Grothendieck topology—since the singleton lattice is a frame. So any collection of covering families generates a Grothendieck topology.

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