SHEAF MODELS FOR ANALYSIS

M.P. Fourman *

Department of Mathematics, Columbia University,
New York, NY 10027, USA

and

J.M.E. Hyland

King's College, Cambridge, England

1. Topological models for intuitionistic analysis were first developed by Scott [17,18]. Analogous models have been studied by Moschovakis [15] and van Dalen [2]. Our interest dates from seminars on topoi and logic organized by Scott 1973-4. The models we shall consider are over complete Heyting algebras (cHa). Mostly they are spatial, i.e. the complete Heyting algebra is $\mathcal{O}(T)$, the open sets of some topological space T. However, in section 4, we make essential use of non-spatial cHa. Topoi provide a general framework for higher order models, and lead to further generalizations which we do not exploit.

The general theory of sheaf models is described in Fourman and Scott [7]. These models provide interpretations for intuitionistic type theory with products, (extensional) power sets and full comprehension as introduced in Fourman [5]. Of course this gives function spaces as subtypes. Choice principles of the form

$$\forall x \exists y \quad \phi(x,y) \rightarrow \exists f \forall x \quad \phi(x,f(x))$$

are not part of this logic. We say that a result is true constructively (or is <u>constructive</u>) when it is provable in this logic. Since sheaves can be used to model intuitionistic set theory (see Grayson [8]) the reader can also consider the models in that setting. Most of our results are themselves constructive and we have phrased our definitions accordingly. The few exceptions are pointed out as they occur.

Much of this paper concerns <u>elementary analysis</u>: that part of analysis expressed in terms of function spaces without use of the power set. We observe that there are two notions of Cauchy sequence – with or without a given modulus of convergence – which lead (respectively) to weak and strong notions of Cauchy completeness. We take the weak notion as basic.

---

* Research supported by NSF Grant MCS 77-02759

We have various independence results. The independence of a proposition may be shown strongly by exhibiting a model where $\neg\phi$ holds, $\models \neg\phi$, or weakly by exhibiting a model where ϕ fails to hold, $\not\models \phi$.

We now outline the paper. In section 2 we give a representation for the models of three basic spaces: R (Dedekind reals), $\mathbb{N}^{\mathbb{N}}$ (Baire space) and $2^{\mathbb{N}}$ (Cantor space). We introduce three principles to give these representations, and discuss them. In the absence of choice from numbers to numbers, the Dedekind reals R may differ from the Cauchy reals R^c . Of course $R^c \subseteq R$. We look at some subfields of R and C (the complex field constructed from R). There may be various Cauchy complete proper subfields of R which are models for elementary analysis. Much of the first half of section 3 ("models over spaces") will have been remarked by anyone looking seriously at this subject, but it seems worthwhile to record it. The remainder is devoted to a discussion of Bar Induction, models being used to compare its various forms. Freyd first showed that non-spatial cHa give rise to higher order properties which can not be obtained with standard spatial models. In the final section we use models over non-spatial cHa to show the independence of the compactness of $2^{\mathbb{N}}$ and the local compactness of R . We believe such models will prove useful in other ways.

Most of our results were obtained in 1973-5 when we were both in Oxford. The last section is more recent. Together we have benefited from discussions with almost every worker in the field. Our main debt is to Dana Scott; his influence pervades the paper. Robin Grayson (our constructive conscience) provided many helpful comments on an earlier draft. A postscript has been added in proof.

2. We now look at some basic examples of higher-order constructs and their sheaf models. Firstly we recall a few fundamental facts. Peano's axioms are categorical by the usual proof. Any sheaf model for them is (isomorphic to) the simple sheaf $\underset{\sim}{\mathbb{N}}$. Internalizing the usual construction of the rationals with their usual structure $(0,1,+,\times,<)$ gives the simple sheaf $\underset{\sim}{Q}$ (henceforth we omit mention of impertinent isomorphisms). In logical terms, these constructions are "absolute".

Let $\underset{\sim}{A}$ be a simple sheaf. For any sheaf B , a section of $B^{\underset{\sim}{A}}$ is just an A-indexed family of sections of B . The internally constructed sheaf $\underset{\sim}{A}^{<\underset{\sim}{\mathbb{N}}}$ of finite sequences of elements of $\underset{\sim}{A}$ is the simple sheaf $(A^{<\mathbb{N}})_{\sim}$. A global section X of $P(\underset{\sim}{A})$ is just a family of truth values $\{ [\![a\in X]\!] \mid a\in A \} \subseteq \Omega$. Finally

$$[\![\forall a\in\underset{\sim}{A} \ \phi]\!] = \bigwedge \{ [\![Ea \to \phi]\!] \mid a\in A \}$$

and (2.1)

$$[\![\exists a\in\underset{\sim}{A} \ \phi]\!] = \bigvee \{ [\![Ea \wedge \phi]\!] \mid a\in A \} .$$

These facts enable us to develop an adequate picture of the sheaf models for the Dedekind reals, Baire space and Cantor space. We make one more remark for later use.

2.2 DEFINITION Let Ω be a cHa. We say $V \in \Omega$ is <u>connected</u> iff

$$\forall W,W' \ ([W \wedge W' = \bot \text{ and } W \vee W' \supseteq V] \rightarrow W \geq V \text{ or } W' \geq V) \ .$$

Furthermore, Ω is <u>locally connected</u> iff for every $U \in \Omega$

$$U = \bigvee \{ V \leq U \mid V \text{ is connected} \} \ .$$

2.3 LEMMA If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are simple sheaves over a locally connected cHa Ω then $\underset{\sim}{B}^{\underset{\sim}{A}}$ is the simple sheaf $(B^A)_{\sim}$.

Proof. Any section of $\underset{\sim}{B}$ over a connected open is constant as the equality on $\underset{\sim}{B}$ is decidable. Thus a section of $\underset{\sim}{B}^{\underset{\sim}{A}}$ is locally a function from A to B .

<u>Dedekind reals</u> $\mathcal{R}$. Starting from $\underset{\sim}{Q}$ we construct the subobject $\mathcal{R} \subseteq \mathcal{P}(\underset{\sim}{Q}) \times \mathcal{P}(\underset{\sim}{Q})$ consisting of those pairs $\langle U,L \rangle$ such that

1)	$\exists\, q,r \in Q \ (q \in U \wedge r \in L)$	inhabited	
2)	$\forall p \ \neg(p \in U \wedge p \in L)$	disjoint	
3)	$\forall p \ (p \in L \leftrightarrow \exists\, q \in L \ \ q > p)$	open lower cut	(2.4)
4)	$\forall p \ (p \in U \leftrightarrow \exists\, q \in U \ \ q < p)$	open upper cut	
5)	$\forall p,q \ (p > q \rightarrow p \in U \vee q \in L)$	near together	

This definition was first introduced in the context of topoi by Tierney who showed that in a spatial topos $Sh(X)$ we have a representation of $\mathcal{R}$ as the sheaf of (germs of) continuous real-valued functions on X . In the presence of choice from numbers to numbers, it is easily seen to be equivalent to the definition in terms of Cauchy sequences (see Bishop [1]). We shall see that this equivalence does not hold in general.

The conditions (2.4) may be translated using (2.1) into conditions on the families of truth values $\{ [\![q \in U]\!], [\![q \in L]\!] \mid q \in Q \}$. This translation gives

1)	$\bigvee \{ [\![q \in U]\!] \wedge [\![r \in L]\!] \mid q,r \in Q \} = \top$		
2)	$[\![p \in U]\!] \wedge [\![p \in L]\!] = \bot$	for $p \in Q$	
3)	$[\![p \in L]\!] = \bigvee \{ [\![q \in L]\!] \mid q > p \}$	for $p \in Q$	(2.5)
4)	$[\![p \in U]\!] = \bigvee \{ [\![q \in U]\!] \mid q < p \}$	for $p \in Q$	
5)	$[\![p \in U]\!] \vee [\![q \in L]\!] = \top$	for $q < p \in Q$	

Tierney's representation may be constructivised by showing that given $r = \langle U,L\rangle$ a global section of $\mathcal{R}$ in $Sh(X)$ and $t \in X$ the pair $r_t = \langle U_t, L_t\rangle$ is a Dedekind real, where

$$U_t = \{ q \in Q \mid t \in [\![q \in U]\!] \} \quad , \quad L_t = \{ q \in Q \mid t \in [\![q \in L]\!] \}$$

and that the mapping $t \to r_t$ is continuous [7] .

We dualise this construction and generalise to sheaves over an arbitrary cHa. In $Sh(\Omega)$ a section of $\mathcal{R}$ over $p \in \Omega$ will turn out to be an $\wedge\bigvee$ map $r^* : \mathcal{O}(R) \longrightarrow \Omega \restriction p$. To show this we shall have to appeal to the local compactness of R . (Our definition is not equivalent to the classical one!)

2.6 DEFINITION A cHa Ω is <u>locally compact</u> iff for every $U \in \Omega$

$$U = \bigvee \{ V \leq U \mid \text{for every cover } \mathcal{U} \text{ of } U \text{ there is a finite subset of } \mathcal{U} \text{ covering } V \} \quad .$$

Classically, $\mathcal{O}(R)$ is locally compact.

We now consider a global section $r = \langle U,L\rangle$ of $\mathcal{R}$. For (p,q) a rational open let

$$[\![r \in (p,q)]\!] = [\![p \in L \wedge q \in U]\!] \qquad ;$$

by (3) and (4) of (2.5) we have

$$(p,q) \subseteq (p',q') \implies [\![r \in (q,p)]\!] \leq [\![r \in (p',q')]\!] \quad .$$

Defining $r^* : \mathcal{O}(R) \longrightarrow \Omega$ by

$$r^*(V) = \bigvee \{ [\![r \in (p,q)]\!] \mid p,q \in Q \text{ and } (p,q) \subseteq V \} \qquad (2.7)$$

thus gives a monotone map such that $r^*(p,q) = [\![r \in (p,q)]\!]$. Furthermore $[\![p \in L]\!] = r^*(p,\infty)$ and $[\![q \in U]\!] = r^*(-\infty,q)$ (by (1) of (2.5)) so r is determined by r^* .

2.8 LEMMA $r^* : \mathcal{O}(R) \longrightarrow \Omega$ is an $\wedge\bigvee$ map and every such map arises uniquely in this way.

Proof. To show r^* is an $\wedge\bigvee$ map it suffices to show that it preserves finite $\wedge$ and existing $\bigvee$ among rational open intervals (and then appeal to 2.7). Finite meets follow easily from 2.5 . The empty meet $\top$ is preserved by (1) . The intersection of (a,b) and (c,d) is either empty in which case (2) applies, or is one of (c,b), (c,d), (a,b) or (a,d) which are treated severally using (3) and (4) .

Joins require more argument. Starting with the simplest non-trivial join of all we show that if $(a,d) = (a,b) \cup (c,d)$ then

$$[\![a \in L \wedge d \in U]\!] = [\![a \in L \wedge b \in U]\!] \vee [\![c \in L \wedge d \in U]\!] \quad .$$

Monotonicity gives us an inclusion $\geq$. Furthermore since $b > c$ we have from (5) (of 2.4) that

$$\models a\in L \wedge d\in U \to (a\in L \wedge b\in U) \vee (c\in L \wedge d\in U)$$

by a short and straightforward logical deduction. Translating gives the inclusion $\leq$. A simple induction now shows that finite joins are preserved. We leave the reader to do this (using the decidability of $=$ on $\underset{\sim}{Q}$).

So far our argument has been wholly constructive. We now use compactness to reduce arbitrary joins to finite ones: Suppose that $\bigvee\{(p_i,q_i) \mid i\in I\} \supseteq (p,q)$ we must show that

$$\bigvee\{[\![r\in(p_i,q_i)]\!] \mid i\in I\} \geq [\![r\in(p,q)]\!] \quad .$$

Happily, $[\![r\in(p,q)]\!] = \bigvee\{[\![r\in(a,b)]\!] \mid p<a \text{ and } b<q\}$ (by (3) and (4) of 2.4). Thus it suffices to show that for $a>p$ and $b<q$ that $[\![r\in(a,b)]\!] \leq \bigvee\{[\![r\in(p_i,q_i)]\!] \mid i\in I\}$. But by compactness we have a finite $F\subseteq I$ such that $(a,b) \subseteq \bigcup\{(p_i,q_i) \mid i\in F\}$ which reduces us to the case of finite joins.

Given $\phi : \mathcal{O}(R) \longrightarrow \Omega$ an $\wedge\bigvee$ map define $r_\phi = \langle U_\phi, L_\phi\rangle$ by

$$[\![q\in U_\phi]\!] = \phi(-\infty,q) \quad \text{and} \quad [\![q\in L_\phi]\!] = \phi(q,\infty) \quad .$$

We assert that r_ϕ is a Dedekind real since the conditions (2.5) follow immediately from corresponding relations between rational intervals which are preserved by the $\wedge\bigvee$ map ϕ . Since

$$[\![r_\phi\in(p,q)]\!] = \phi(p,q)$$

we are done.

2.9 THEOREM Sections of R over $W\in\Omega$ correspond exactly to $\wedge\bigvee$ maps $\mathcal{O}(R) \longrightarrow \Omega\restriction W$. Restrictions (for $V\leq W$) are given by composing with the intersection map $\Omega\restriction W \longrightarrow \Omega\restriction V$.

Proof. The construction of R is local (i.e. commutes with restrictions) as it is carried out in the internal logic. A section of R over W is just a global section of $R\restriction W$ and hence by the lemma an $\wedge\bigvee$ map $\mathcal{O}(R) \longrightarrow \Omega\restriction W$. Finally, since

$$[\![q\in L\restriction W]\!] \wedge W = [\![q\in L]\!] \wedge W \quad \text{(and similarly for } U \text{)}$$

we see that

$$[\![r\in(p,q)]\!] \wedge W = [\![r\restriction W\in(p,q)]\!] \wedge W \quad .$$

We call R the sheaf of $\wedge\bigvee$ maps $\mathcal{O}(R) \longrightarrow \Omega$.

Baire space $\mathbb{N}^{\mathbb{N}}$. The internal Baire space is constructed as the function space $\mathbb{N}^{\mathbb{N}}$. Our representation here depends on the principle of Bar Induction. Consider the tree $\mathbb{N}^{<\mathbb{N}}$ of finite sequences. Each element of $\mathbb{N}^{\mathbb{N}}$ determines a branch of this tree. Horticulturally, Bar Induction tells us that if we prune each branch to a finite node and continually prune any node from which no twigs appear, then we destroy our tree. More formally:

2.10 Bar Induction with monotonicity condition (BI) states: "If B is a predicate on $\mathbb{N}^{<\mathbb{N}}$ satisfying the *hypotheses*

i)	$\forall \alpha \in \mathbb{N}^{\mathbb{N}} \ \exists n \in \mathbb{N} \quad B(\overline{\alpha(n)})$	B is a *bar*
ii)	$\forall u \in \mathbb{N}^{<\mathbb{N}} \ \forall k \in \mathbb{N} \quad B(u) \to B(u{*}k)$	B is *monotonic*
iii)	$\forall u \in \mathbb{N}^{<\mathbb{N}} \ (\ \forall k \in \mathbb{N} \quad B(u{*}k) \to B(u))$	B is *inductive*

then $B(\langle\rangle)$, the empty sequence $\langle\rangle$ satisfies B ".

Here B describes the nodes pruned, $\overline{\alpha(n)} = \langle\alpha(0),\ldots,\alpha(n-1)\rangle$ and $u{*}k = \langle u(0),\ldots,u(m-1),k\rangle$ for u of length m .

Classically (BI) follows easily from dependent choices (show the contrapositive), and the monotonicity condition plays no role. We discuss other forms of Bar Induction at the end of this section, and models in which they hold and fail in section 3 .

Now let $\underline{\alpha}$ be a global section of $\mathbb{N}^{\mathbb{N}}$ in $Sh(\Omega)$. Let $\mathbb{N}^{\mathbb{N}}$ be topologized as usual by the basic opens

$$V(u) = \{ \beta \in \mathbb{N}^{\mathbb{N}} \mid \beta \text{ extends } u \} \quad \text{for } u \in \mathbb{N}^{<\mathbb{N}} .$$

Define $\underline{\alpha}[V(u)] = [\![\underline{\alpha}(0) = u(0) \wedge \ldots. \wedge \underline{\alpha}(n-1) = u(n-1)]\!] = [\![\underline{\alpha} \in V(u)]\!]$, where $u = \langle u(0), \ldots ,u(n-1)\rangle \in \mathbb{N}^{<\mathbb{N}}$.

2.11 LEMMA $\underline{\alpha}$ extends to a $\wedge\bigvee$ map

$$\underline{\alpha} : \mathcal{O}(\mathbb{N}^{\mathbb{N}}) \longrightarrow \Omega ,$$

and every such map arises uniquely in this way.

Proof. To show that $\underline{\alpha}$ extends uniquely to an $\wedge\bigvee$ map it suffices to show that finite intersections and existing unions of basic opens are preserved by $\underline{\alpha}$. Finite intersections are easy. The joins require some work:

Suppose that for a , $b_i \in \mathbb{N}^{<\mathbb{N}}$ we have

(i) $V(a) \subseteq \bigcup \{ V(b_i) \mid i \in I \}$.

We must show that

(ii) $[\![\underline{\alpha} \in V(a)]\!] \leq \bigvee [\![\underline{\alpha} \in V(b_i)]\!]$.

We define a predicate B by

$$B(u) \quad \text{iff} \quad [\![\underline{\alpha} \in V(u)]\!] \leq \bigvee [\![\underline{\alpha} \in V(b_i)]\!] \quad .$$

B is monotone and by (i) B bars a . By distributivity

$$[\![\underline{\alpha} \in V(u)]\!] = \bigvee \{ [\![\underline{\alpha} \in V(u*k)]\!] \mid k \in N \} \quad ,$$

so that B is inductive. Hence applying (BI) below a , we conclude $B(a)$, which is (ii) .

Given an $\wedge\bigvee$ map $\phi : \mathcal{O}(\mathbb{N}^{\mathbb{N}}) \longrightarrow \Omega$, the corresponding global section a_ϕ of $\mathbb{N}^{\mathbb{N}}$ is determined by setting

$$[\![a_\phi(n) = m]\!] = \phi(\{ \alpha \mid \alpha(n) = m \}) \ .$$

This determines a functional relation on $\mathbb{N} \times \mathbb{N}$ and hence a global section of $\mathbb{N}^{\mathbb{N}}$. It is routine to check that for $u \in \mathbb{N}^{\mathbb{N}}$,

$$[\![a_\phi \in V(u)]\!] = \phi(V(u)) \ ,$$

so our circle is complete.

Given the lemma which represents global sections of $\mathbb{N}^{\mathbb{N}}$ we localize just as we did for $\mathcal{R}$ (Theorem 2.9) to get the general representation:

2.12 THEOREM $\mathbb{N}^{\mathbb{N}}$ is represented by the sheaf of $\wedge\bigvee$ maps
$\mathcal{O}(\mathbb{N}^{\mathbb{N}}) \longrightarrow \Omega$.

Cantor space $2^{\mathbb{N}}$. Here the representation works just as for $\mathbb{N}^{\mathbb{N}}$. However, in place of Bar Induction we can use the (weaker) compactness of $2^{\mathbb{N}}$ or Fan Theorem (FT) . This states that for a predicate B on $2^{<\mathbb{N}}$ we have

$$(\forall \alpha \in 2^{\mathbb{N}})(\exists n)\, B(\overline{\alpha(n)}) \rightarrow (\exists k)(\forall \alpha \in 2^{\mathbb{N}})(\exists n \leq k)\, B(\overline{\alpha(n)}) \ . \quad (2.13)$$

We leave the reader to apply FT and obtain

2.14 THEOREM $2^{\mathbb{N}}$ is represented by the sheaf of $\wedge\bigvee$ maps
$\mathcal{O}(2^{\mathbb{N}}) \longrightarrow \Omega$.

REMARKS 1) In the case where Ω is spatial (i.e. $\Omega = \mathcal{O}(T)$ for some T), the representations above take on a familiar form. The space X (= Dedekind reals, Baire space or Cantor space) is represented by the sheaf $X_T = C(X,T)$ of continuous X-valued functions on T . Equivalently X_T is represented as the sheaf of continuous sections of the projection map $X \times T \longrightarrow T$ so we have special cases of the general representation of a topological space in $Sh(T)$ described in Fourman and Scott [7] . The internal topology corresponding to this representation is in each case just the usual topology (based on finite positive information). The general theory tells us that internal continuous maps of sober spaces $X_T \longrightarrow Y_T$ are represented by commuting triangles

$$\begin{tikzcd} X \times T \arrow[rr] \arrow[dr, "\pi"'] & & Y \times T \arrow[dl, "\pi"] \\ & T & \end{tikzcd}$$

Such a triangle is equivalent to a continuous map $X \times T \longrightarrow Y$. This representation of spaces over T and continuous maps between them extends to the case when X and Y are objects in a (cartesian closed) category of filter spaces (Hyland [11]).

2) In order to give a good representation of our spaces, we have introduced three important (and classically valid) principles: the local compactness of R , bar induction and the fan theorem. In general, it is possible to give similar representations using cHa of "formal spaces" $F(R)$, $F(\mathbb{N}^{\mathbb{N}})$, $F(2^{\mathbb{N}})$ in place of actual opens $\mathcal{O}(R)$, etc. In fact each of our principles is equivalent to the isomorphism of the corresponding formal and actual opens (Fourman [6]).

Classically Bar Induction holds in the form

$(BI)_C$ $\quad (\forall \alpha)(\exists n)\ C(\overline{\alpha(n)}) \wedge (\forall u)\ [(\forall k)\ C(u*k) \rightarrow C(u)] \rightarrow C(< >)$

(i.e. without the monotonicity condition). In intuitionism, bar induction is considered in forms $(BI)_M$ and $(BI)_D$, involving two predicates P and Q .

$(BI)_M$ $\quad [(\forall \alpha)(\exists n)\ P(\overline{\alpha(n)}) \wedge (\forall u)(\forall k)\ (P(u) \rightarrow P(u*k))$

$\quad \wedge (\forall u)(P(u) \rightarrow Q(u)) \wedge (\forall u)((\forall k)\ Q(u*k) \rightarrow Q(u))] \rightarrow Q(< >)$,

is equivalent to Bar Induction with Monotonicity (BI) , as introduced above (2.10). (Let $B(u)$ hold iff $\forall v \supseteq u\ Q(v)$.) In the related Decidable Bar Induction

$(BI)_D$ $\quad [(\forall \alpha)(\exists n)\ P(\overline{\alpha(n)}) \wedge (\forall u)(P(u) \vee P(u)) \wedge (\forall u)(P(u) \rightarrow Q(u))$

$\quad \wedge (\forall u)((\forall k)\ Q(u*k) \rightarrow Q(u))] \rightarrow Q(< >)$,

the use of two predicates <u>is</u> essential. Intuitionists also consider extensions of these principles (BI(A) etc.) in which $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}$ are replaced by $A^{\mathbb{N}}$, $A^{<\mathbb{N}}$ for some inhabited $A \subseteq \mathbb{N}$.

Constructively $(BI)_C$ implies (BI) which implies $(BI)_D$; furthermore $(BI)_C$ is equivalent to (BI) together with the logical principle

(*) $\quad (\forall k \in \mathbb{N})\ [A \vee B(k)] \rightarrow A \vee (\forall k \in \mathbb{N})\ B(k)$.

To see that (BI) + (*) $\rightarrow$ $(BI)_C$, use (*) to show that the obvious monotone extension of C is inductive. To see that $(BI)_C \rightarrow$ (*) , suppose $\forall n\ (A(n) \vee B)$ define C by

$C(u)$ iff $u = < >$ and $B \vee \forall n\ A(n)$ or $u = <n>$ and $A(n)$.

Over most decent topological spaces, (*) and hence $(BI)_C$ fails. However, Dummett has shown that Kripke models with constant domain are characteristic for (*) (cf. [3]) ; so by Theorem 3.4 , $(BI)_C$ holds in (standard) Kripke models. In section 3 , we distinguish between (BI) and $(BI)_D$, and also show that extended principles of Bar Induction may fail in sheaf models where the other principles hold.

Bar Induction and the Fan Theorem are familiar principles of intuitionistic analysis. Clearly (BI) implies F.T. and $(BI)_D$ implies a decidable version of F.T. constructively. However, these principles do not have, in the context of sheaves, the significance attached to them in traditional intuitionistic analysis. For a useful analysis of Brouwer's view of bar induction the reader may consult Dummett [3] . Brouwer's view, which rests on the notion of a fully analysed (potentially infinite) proof, gives no argument for the fan theorem which does not also justify bar induction. We shall easily find cases where bar induction fails but the fan theorem still holds. In view of this it seems worth making some remarks on the significance of our three principles for the kind of mathematics which can be done in the logic of sheaves.

The compactness of the unit interval implies constructively that continuous real valued functions are uniformly continuous and have least upper bounds on bounded closed intervals. Thus Scott's transition from continuity to uniform continuity for Sh(T) mentioned above is a direct consequence of the fact that, as we shall show, in Sh(T) the unit interval is compact. The fan theorem which states the compactness of $2^{\mathbb{N}}$ evidently has similar consequences for Cantor space, and holds in sheaves over any topological space. Surprisingly, it also entails the local compactness of the Dedekind reals (Grayson [9]). The significance of bar induction is probably less well known. Its most immediate application is that from it one can prove constructively general forms of Souslin's Theorem that any two disjoint analytic sets may be separated by a Borel set. (Logicians may already be familiar with its use with the 2nd recursion theorem to show that there is a partial recursive functional giving a code for a separating Borel set in terms of codes for two analytic sets.)

3. MODELS OVER SPACES (Properties of elementary analysis).

If T is a topological space, the <u>standard model</u> of R over $\mathcal{O}(T)$ is R_T , the sheaf of continuous real-valued functions on T . It has continuous maps $a : U \rightarrow R$ as sections over $U \in \mathcal{O}(T)$ with equality given by

$$[\![a = b]\!] = \mathrm{In}\ \{ t \in Ea \cap Eb \mid a(t) = b(t) \} \quad ;$$

operations are defined pointwise. This representation is due to Tierney. The <u>standard model</u> of C over $\mathcal{O}(T)$ arises similarly; it is C_T , the sheaf of continuous complex-valued functions on T .

That the standard model of $\mathcal{R}$ is (even strongly) Cauchy complete is immediate as this fact is easily proved in higher-order logic for which our interpretations are sound (Fourman and Scott [7]). However it is instructive to see that here weak Cauchy completeness corresponds to the classical fact that a uniform limit of continuous functions is continuous: a section s of $\mathcal{R}^N$ over $U \in \mathcal{O}(T)$ is (weak) Cauchy iff the corresponding sequence $\langle s(n) \mid n \in \mathbb{N}\rangle$ of continuous functions converges in the sense that for $m > n$ and $t \in U$, we have

$$|s(m)(t) - s(n)(t)| < 1/n \quad .$$

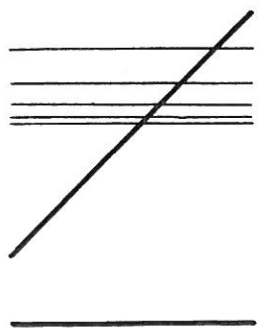
(The weak Cauchy condition is strictly local in the sense of Hyland [9] .)

Scott's model of the intuitionistic continuum (see [17]) is defined as an $\mathcal{O}(T)$-set where T is the Baire space $\mathbb{N}^{\mathbb{N}}$. It is equivalent to the standard model over $\mathbb{N}^{\mathbb{N}}$ as every section of R_T is locally the restriction of a global section, i.e. global sections generate.

If we take Baire space as our underlying space T , the countable axiom of choice:

$$\text{ACN} \quad \forall n \in \mathbb{N}\ \exists y \in A \quad \phi(n,y) \ \rightarrow\ \exists f \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad \phi(n,f(n))$$

holds: for in $\mathcal{O}(\mathbb{N}^{\mathbb{N}})$ any countable family $\{U_i \mid i \in \mathbb{N}\}$ of opens has a refinement by mutually disjoint clopen sets $\{W_i \mid i \in \mathbb{N}\}$ with $\bigcup U_i = \bigcup W_i$. Since $\text{ACN} \models R^c = \mathcal{R}$ (i.e. the Cauchy and Dedekind reals are the same) over Baire space both the Dedekind <u>and</u> the Cauchy reals are represented as the sheaf of germs of continuous real-valued functions. However, if we take the real numbers R as our underlying space ACN fails. It suffices to show that over $\mathcal{O}(R)$ we have $R^c \neq \mathcal{R}$. As the sections of Q in any spatial model are locally constant rational functions, a (Cauchy) sequence $s : N \longrightarrow Q$ defined over a <u>connected</u> open $U \in \mathcal{O}(R)$ corresponds by lemma 2.3 to a sequence of <u>constant</u> rational functions $\langle s(n) \mid n \in N\rangle$. No such sequence can converge to the <u>generic real</u> $\iota : U \longrightarrow R$ given by $\iota(t) = t$.



ι

$U \in \mathcal{O}(R)$

This shows that $Sh(R) \models \neg\, \iota \in R^c$. It follows that $Sh(R) \models \neg(R^c = \mathcal{R})$, so that $Sh(R) \models \neg(\text{ACN})$.

It is natural to ask whether $R^c = \mathcal{R} \to ACN$. That this is not so is shown by Cohen's model for $\neg AC$, since classically we do not need ACN to prove $R^c = \mathcal{R}$. An intuitionistic counterexample is even easier to construct. Topologise $T = R \cup \{*\}$ by letting $U \subseteq T$ be open iff $U \cap R$ is open in the usual topology and $\forall r \in R \ (r \in U \to * \in U)$.

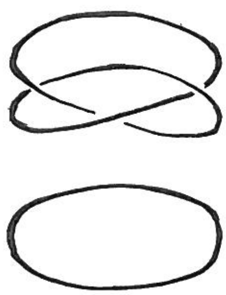
3.1 THEOREM $Sh(T) \not\models R^c = \mathcal{R} \to ACN$.

Proof. Since any continuous function from $U \in \mathcal{O}(T)$ to R is constant $Sh(T) \models R^c = \mathcal{R}$. It is easy to see there is no choice function defined on any neighbourhood of $t \in R$ for the family of subsets of Q given by

$$[\![q \in A_n]\!] = \{ r \mid |q - r| < 1/n \} \cup \{*\}$$

so $Sh(T) \not\models$ ACN. We shall use this idea of adding a generic point again in section 4.

We now look at some familiar properties of R and C which may fail in spatial models. In the standard model of $\mathcal{C}$ over the space $T = C$, the complex plane, we have a generic complex number, again the identity function, ι. We use this generic number to construct a polynomial which fails to have a root: $\not\models \exists x \ (p(x) = 0)$. The polynomial we consider is $x^2 - \iota$. Its set of roots may be pictured as the (sheaf of sections of) the Riemann surface of the square root function over C.



Over any circle about the origin, this looks like the double covering of the circle which has no global sections. Since any neighbourhood of the origin contains some such circle, $0 \notin [\![\exists x \ (p(x) = 0)]\!]$. Translating this example to $a \in C$ we see that $a \notin [\![\exists x \ (p(x) + a = 0)]\!]$ whence the result

$\models \neg \mathcal{C}$ is algebraically closed.

Similar considerations show that

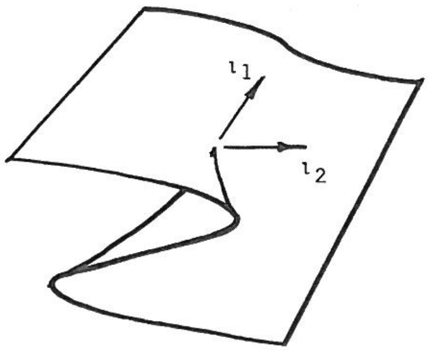
$$0 \notin [\![\exists r, \theta \in \mathcal{R} \quad \iota = re^{i\theta}]\!]$$

whence $\models \neg \forall z \exists r, \theta \quad z = re^{i\theta}$.

We use the same base space T, now considered as $R \times R$ to show that $\mathcal{R}$ need not be real closed. We have two generic reals ι_1, ι_2 : $R \times R \longrightarrow R$, the projections. The monic polynomial $p(x) = x^3 + \iota_1 x + \iota_2$ has odd degree but fails to have a root at the origin:

$$(0,0) \notin [\![(\exists x)(p(x) = 0)]\!] .$$

We leave the reader to construct a proof of this from the picture below of the set of roots of p .



If we take $\mathbb{N}^{\mathbb{N}}$ as our base space as we did earlier we find that not only does ACN hold, but also dependent choices

$$\text{DC} \quad \forall x \in A\ \exists y \in A \quad \phi(x,y) \rightarrow \forall a \in A\ \exists f \in N \rightarrow A \quad [f(0) = a \wedge \forall n \quad \phi(f(n),f(n+1))] ,$$

is satisfied. The proof is similar to that for ACN . Using DC one can prove the algebraic closure of C and the real closure of R . (The algebraic closure of C is treated in Bishop [1] where the real closure of R is an exercise: Dana Scott has shown us an elegant proof of the latter.) Thus over $\mathbb{N}^{\mathbb{N}}$, C is algebraically closed and R is real closed.

The failure in the standard models over $T = C = R^2$ of the algebraic closure of C , the representation of complex numbers in polar coordinates and the real closure of R , depended on a lack of continuity on parameters. This is part of a general phenomenon. In Hyland [11] , a notion of local continuity in parameters is described, and a class of formulae is given for which this notion is equivalent to truth in spatial models. In essence the class of formulae allows arbitrary quantification on a coherent sequent. Thus as alluded to in [11] , in Sh(T) , C is separably closed and R is separably real closed (in the sense of Wraith [19] and Kock [14] respectively).

It seems worth clearing up a possible confusion at this point. When intuitionists have talked about continuity in parameters, they have been referring to global continuity. A recurring example in their discussion of <u>choice principles</u> and <u>extensionality</u> is the cubic equation

$$x^3 - 3x + p = 0$$

with single real parameter p . There is no global solution continuous in the parameter p , but locally in p there is always a solution continuous in the parameter. This contrasts with the fact that if one regards p as varying over real number generators (= weak Cauchy sequences of rationals) then there is a global continuity (of a real number generator solution) in parameters. (In fact any monic polynomial of odd degree has a solution with this kind of continuity in parameters, as follows by [11] from the real closure of the reals in Bishop's Constructive Mathematics [1] .) However, the logic of sheaves calls for finer considerations. Local continuity in parameters is enough to show the truth in spatial models of a very large class of formulae (Hyland [11]). The problem for us arises when a polynomial has only repeated roots. Then it is not possible to find locally a solution continuous in the coefficients of the polynomial.

We now show that the unit interval, I and Cantor space $2^{\mathbb{N}}$ are compact in Sh(T) for any topological space T . Evidently I is represented by the projection $T \times I \to T$. As described in Fourman and Scott [7] , a covering of I_T corresponds to a covering of $T \times I$. In each fibre $\{t\} \times I$, the covering is finite, and as the projections are closed, this covering covers a uniform neighbourhood of the fibre, so $t \in [\![\exists \text{ finite subcover}]\!]$. Thus $Sh(T) \models I$ is compact . Replacing I by $2^{\mathbb{N}}$ in the above we see that $Sh(T) \models 2^{\mathbb{N}}$ is compact , i.e. $Sh(T) \models$ Fan theorem .

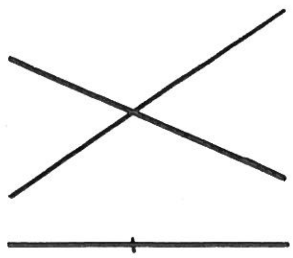
3.2 THEOREM In spatial topoi, R is locally compact and $2^{\mathbb{N}}$ is compact.

In fact the compactness of I and $2^{\mathbb{N}}$ in spatial models follows from general considerations of continuity in parameters, but we do not elaborate on this here.

Before considering how suitable choice of base space affects our models, we look at substructures of the standard models over more familiar spaces. We recall the remarkable fact (discovered by Scott) that there are topological models for Brouwer's theorem: "all functions from R to R are uniformly continuous on closed intervals". In fact this will hold in most of the topological models which we shall consider.

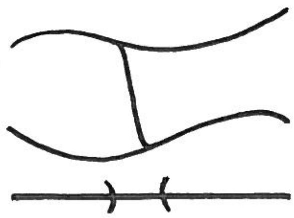
3.3 THEOREM (Hyland [12]). Suppose T is (locally) <u>either</u> first countable, zero-dimensional, Hausdorff, with no isolated points, <u>or</u> of the form $R \times S$ for some first countable and normal, topological space S , then $Sh(T) \models$ Brouwer's theorem.

It is instructive to note however that Brouwer's theorem is not in general inherited by subspaces of R . Taking R as base space (i.e. $\mathcal{O}(R)$ as cHa) let R_{diff} be the sheaf of C^{∞} functions on R . The map D taking a C^{∞} function $a : U \to R$ to its derivative is certainly a sheaf map but is not continuous:



functions which fail to be apart at a point $t \notin [\![a \neq b]\!]$ (i.e. $a(t) = b(t)$) may have wildly different derivatives at that point.

From the continuity theorem Scott derived the fact that " R is unzerlegbar " holds in his model. We now see that R_{diff} is unzerlegbar. Let A,B be two inhabited subsheaves of R_{diff} such that $\models \forall x \epsilon R_{diff} (x\epsilon A \vee x\epsilon B)$ then for every point $t\epsilon R$ there is an open interval U about t and sections a,b of R_{diff} over U such that $[\![a\epsilon A]\!] = [\![b\epsilon B]\!] = U$. Let V,W be non-empty open subintervals of U with disjoint closures construct a C^{∞} function f on U such that $f \restriction V = a \restriction V$ and $f \restriction W = b \restriction W$. Now $V \subseteq [\![f \epsilon A]\!]$ and $W \subseteq [\![f \epsilon B]\!]$:



Since $[\![f\epsilon A \vee f\epsilon B]\!] = U$ and U is connected we see that $[\![f\epsilon A \wedge f\epsilon B]\!] \neq \emptyset$. Thus there is no partition of R_{diff} into disjoint inhabited subsets.

The sheaf R_{diff} of C^{∞} functions is internally a subfield of the standard R but is not a model for analysis as it fails to be Cauchy complete.

As a final example in this section we consider the sheaf H of analytic functions on the base space C. The fact that H gives a model for complex analysis was first noticed and applied by Rousseau [16]. That H is a subfield of C is immediate. That H is Cauchy complete follows from Weierstrass' theorem. As a model for analysis, H has various remarkable properties which we mention briefly. Firstly not all functions are continuous; again this follows from the fact that the map D taking $a : U \longrightarrow C$ to $a' : U \longrightarrow C$ is a sheaf map. In this case there is more to be said. Weierstrass tells us that the derivatives of a uniform limit of analytic functions tend uniformly to the derivative of their limit:

$$\text{if } f_n \to f \text{ then } f_n' \to f' .$$

Thus internally the map D is <u>Cauchy continuous</u>:

$$\text{if } a_n \to a \text{ then } D(a_n) \to D(a) .$$

We repeat that D is not topologically continuous - it does not reflect apartness.

Further not every function on H is Cauchy continuous. This follows from the fact that the equality on H is decidable,

$$\models (\forall x,y)(x = y \vee \neg x = y) \quad .$$

(For if two analytic functions defined on an connected open U agree on some non-empty open subset of U then they agree on U .)

In the remainder of this section we consider bar induction in our models. We are mainly concerned with (BI) and $(BI)_D$.

First we give some conditions which ensure that (BI) (and hence $(BI)_D$) holds in sheaves over a space.

3.4 THEOREM Let T be locally countably compact. Then (BI) holds in Sh(T) .

Proof. We may restrict ourselves to the case when the value of the hypotheses of (BI) is T and attempt to show $[\![B(< >)]\!] = T$. Take any $\alpha \in \mathbb{N}^{\mathbb{N}}$; regarding α as the corresponding element in Sh(T) , we have

$$\bigvee \{ [\![B(\overline{\alpha(n)})]\!] \mid n \in N \} = T \quad .$$

So by the local countable compactness of T , we have a cover of T by V_i such that for each V_i we have

$$\bigvee \{ [\![B(\overline{\alpha(n)})]\!] \mid n \leq m \} \geq V_i \quad .$$

(These V_i are independent of α , coming from definition 2.6 .) Fix V_i . By monotonicity of B ,

$$[\![B(\overline{\alpha(m)})]\!] \geq V_i \quad .$$

Thus the predicate $[\![B(u)]\!] \geq V_i$ bars Baire space. It is clearly monotonic and inductive. So applying (BI) (in the real world) we obtain $[\![B(< >)]\!] \geq V_i$. Since the V_i cover T , this completes the proof.

Apart from the assumption of (BI) , the above proof is completely constructive.

In the proof of the next theorem we will use a natural principle of double bar induction implied by (BI) :

Suppose B(u,v) bars any pair of sequences of natural numbers, and B(u,v) is both monotonic and inductive in both its arguments (keeping the other fixed), then B(< >,< >) holds.

3.5 THEOREM Let T be (locally) a Lusin space (a bijective image of a closed subspace of Baire space); then (BI) holds in Sh(T) .

Proof. Without loss of significant generality we assume that the value of the hypotheses of (BI) is T and that $f : \mathbb{N}^{\mathbb{N}} \to T$ is a bijective continuous map. We claim that

$$\forall \alpha,\beta \ \exists n,m \ [\ f^{-1}([\![B(\overline{\alpha(n)})]\!]) \supseteq V(\overline{\beta(m)})]\ ,$$

where $V(u) \subseteq \mathbb{N}^{\mathbb{N}}$ is the clopen set of functions extending the sequence u, and α is regarded both as an element of $\mathbb{N}^{\mathbb{N}}$ and as a constant element of $\mathbb{N}^{\mathbb{N}}{}_T$, while β is in $f^{-1}(T)$.

Pick α and β. Then for some n,

$$f(\beta) \in [\![B(\overline{\alpha(n)})]\!] \quad ,$$

so that for some m,

$$\beta \in V(\overline{\beta(m)}) \subseteq f^{-1}([\![B(\overline{\alpha(n)})]\!]) \ .$$

This proves the claim. Now it is clear that the predicate of u and v,

$$f^{-1}([\![B(u)]\!]) \supseteq V(v) \quad ,$$

satisfies the hypotheses of the principle of double bar induction, so that we can conclude that $f^{-1}([\![B(\langle\ \rangle)]\!]) \supseteq V(\langle\ \rangle) = \mathbb{N}^{\mathbb{N}}$, i.e. $[\![B(\langle\ \rangle)]\!] = T$. This completes the proof.

The proof above is a trivial modification of Robin Grayson's constructive proof that (BI) holds over Baire space. (The simplification at the beginning of our proof is non-constructive, but plays an inessential role, and the proof is essentially constructive apart from the assumption of (BI).) Earlier there had been van Dalen's non-constructive proof in [2], and a non-constructive proof for complete metric spaces in unpublished notes of Hyland. This latter result is covered by our next theorem.

3.6 THEOREM Let T be (locally homeomorphic to) a complete metric space; then (BI) holds in $Sh(T)$.

Proof. Essentially we can follow the proof of Theorem 3.5, using the complete A-branching tree.

It is doubtful whether there is any useful constructive version of Theorem 3.6.

We next give a condition which ensures that $(BI)_D$ holds in $Sh(T)$.

3.7 THEOREM Let T be a locally connected space, then $(BI)_D$ holds in $Sh(T)$.

Proof. We may simplify matters by assuming that the value of the hypotheses of $(BI)_D$ is T and that T is connected. The internal decidability of the barring predicate P ensures that the predicate

$$[\![P(u)]\!] = T \ ,$$

bars Baire space externally. It implies the predicate

$[\![Q(u)]\!] = T$,

which is inductive. So applying $(BI)_D$ externally we find that $[\![Q(< >)]\!] = T$. This completes the proof.

Apart from the assumption of $(BI)_D$ externally, this proof is constructive.

It seems worth remarking that theorems 3.4 and 3.7 above are not essentially spatial. The conditions can be formulated and the results proved for cHa.

We now discuss some particular examples of the failure of bar induction.

3.8 THEOREM Let Q be the rationals with the usual topology. Then $Sh(Q) \models \neg(BI)$.

Proof. With any sequence $u = <u_o, \ldots ,u_{n-1}>$, we associate rationals [u] and [u]' by the continued fraction expansions

$$[u] = [u_o + 1, \ldots ,u_{n-1} + 1] \ ,$$

$$[u]' = [u_o + 1, \ldots ,u_{n-1} + 2] \ ,$$

where [,..,] is the usual notation for continued fractions (see e.g. Hardy and Wright [10]). Define B by

$$[\![B(< >)]\!] = \emptyset$$

$$[\![B(u)]\!] = \{ q \in Q \mid q \text{ is outside the closed interval between } [u] \text{ and } [u]' \} .$$

(Constructively we would write the above definition in a positive way.) It is easy to check that B bars $\mathbb{N}^{\mathbb{N}}$ is inductive and monotonic in Sh(Q) . However, $[\![B(< >)]\!] = \emptyset$. Hence $Sh(Q) \models \neg(BI)$.

In fact it is fairly easy to see that $Sh(Q) \models \neg(BI)_D$, but rather than give the details, we give another model, (independently discovered by Grayson) in which $\neg(BI)_D$ holds for a choice of P equal to Q .

3.8 THEOREM There is a space T such that $Sh(T) \models \neg(BI)_D$.

Proof. Let T be the collection $\mathbb{N}^{<\mathbb{N}}$ of finite sequences of natural numbers with the following topology:

$O \subseteq T$ is open iff whenever $u \in O$ then $\exists m \ \forall k > m \ \ u{*}k \in O$.

We define a predicate B by,

$$[\![B(u)]\!] = T \setminus \{ v \mid v \supseteq u \} .$$

It is easy to check that in Sh(T) , B is a decidable monotonic and inductive bar. However, $[\![B(u)]\!] = \emptyset$.

There are a number of models readily available in which $(BI)_D$ holds but (BI) fails. We could modify Q either by taking its cone or by connecting by adding a generic point $*$ (as we did in 3.1). However, for the strongest possible result, we take as our locally connected topological space T , the subspace of eventually zero points in $R^{\mathbb{N}}$ (with the product topology). By Theorem 3.7 $(BI)_D$ holds over T .

3.9 THEOREM $Sh(T) \models \neg BI$

Proof. For $u = (u_0, \ldots, u_{n-1}) \in N^{<N}$ define $[\![B(u)]\!] = \{ \alpha \mid \bigvee\!\!\!\bigvee_{i<n} \alpha(i) < 1/(u_i + 1) \}$ (so that $[\![B(< >)]\!] = \emptyset$). Then B is a monotone inductive bar in $Sh(T)$.

Finally we note that principles of extended bar induction generally fail in our sheaf models. For example, take $T = R$, enumerate Q as $\{ q_i \mid i \in N \}$. Let $A \subseteq N$ in $Sh(T)$ be defined by

$$[\![n \in A]\!] = R \setminus \{ q_i \mid i < n \} .$$

(Set theoretically, A can be viewed as an (unruly) ordinal.)

3.10 THEOREM $Sh(R) \models \neg BI(A)$.

Proof. Define B on $A^{<\mathbb{N}}$ by

$$[\![B(u)]\!] = [\![\exists\, i,j < \mathrm{length}(u) \ (i \neq j \text{ and } u(i) = u(j))]\!] .$$

B is obviously monotone. To see it bars $A^{\mathbb{N}}$ observe that $\models \forall \alpha \in A^{\mathbb{N}} \ \exists m \in \mathbb{N} \ \alpha \in m^{\mathbb{N}}$. That B is inductive follows from the fact that

$$\models \forall n \ \neg\neg n \in A .$$

Since $[\![B(< >)]\!] = \emptyset$, we are done.

4. KILLING POINTS

As we have seen (3.2) , over spaces $2^{\mathbb{N}}$ is compact and $\mathcal{R}$ locally compact. To find models where these principles fail we manufacture pointless cHa by taking $\wedge\bigvee$ quotients of spatial cHa. Pointless cHa are discussed in detail (under another name) by Isbell [13] . The process of taking quotients of cHa is well known (see e.g. Fourman and Scott [7]).

4.1 DEFINITION Let T be a T_1 space, the cHa $K(T)$ of <u>coperfect opens</u> of T is the least $\wedge\bigvee$ quotient of $\mathcal{O}(T)$ identifying T and $T \setminus \{t\}$ for each $t \in T$. We may identify $K(T)$ as the lattice of fixed points of $F : \mathcal{O}(T) \longrightarrow \mathcal{O}(T)$ where

$$F(W) = \bigcup \{ U \in \mathcal{O}(T) \mid \exists t \quad (W\backslash\{t\}) \subseteq U \} \quad .$$

As a first example of the strange things which go on over pointless cHa , let I be the unit interval and consider the simple sheaf $\underset{\sim}{R}$ with fibre R over $K(I)$. The subsheaf U of R defined by $[\![r\epsilon U]\!] = \{ t \mid r > t \}$ for $r\epsilon R$ is open in the internal interval topology on R as is L given by $[\![r\epsilon L]\!] = \{ t \mid r < t \}$. It is easy to see that

$$\models \forall r\epsilon\underset{\sim}{R} \quad \neg (r\epsilon U \wedge r\epsilon L) \quad .$$

Furthermore

$$\models \forall r\epsilon R \quad (r\epsilon U \vee r\epsilon L) \quad .$$

This is more surprising but simple to verify: given $r\epsilon R$ we have $[\![r\epsilon U \vee r\epsilon L]\!] = [\![r\epsilon U]\!] \vee [\![r\epsilon L]\!] = \top$. Taking the union of the two truth values gives the open $I\backslash \{r\}$. The least fixed point of F above this is I ($= \top$ in $K(I)$). Since $\underset{\sim}{R}$ is a simple sheaf $\models \forall r\epsilon\underset{\sim}{R} \ (r\epsilon U \vee r\epsilon L)$.

Thus we have shown that $\underset{\sim}{R}$ is not connected. We could use a similar argument to show that $\underset{\sim}{I}$ is not compact. By a slight refinement we instead show that the unit interval of the Dedekind reals may fail to be compact. Let Ω be the set of those $U \subseteq I \cup \{*\}$ such that $U \cap I \in K(I)$ and $\forall r \ (r\epsilon U \rightarrow *\epsilon U)$. Evidently Ω is a complete Heyting algebra:

all we have done is to add a generic point to $K(I)$.

4.2 LEMMA Over Ω the Dedekind reals $\mathcal{R}$ are given by the simple sheaf $\underset{\sim}{R}$.

Proof. We show that every $\wedge\bigvee$ map, a ,

$$\begin{tikzcd} \mathcal{O}(R) \arrow[rr, "a"] \arrow[dr] & & \Omega \\ & P(1) \arrow[ur] & \end{tikzcd}$$

factors uniquely through (1) as shown. That is, $\wedge\bigvee$ maps $\mathcal{O}(R) \longrightarrow \Omega$ correspond to points of $\mathcal{O}(R)$. Since $*$ is a point of Ω we have a point $a(*)$ of $\mathcal{O}(R)$ (by composition) such that

$$\forall U \quad (* \epsilon a(U) \rightarrow a(*) \epsilon U) \quad .$$

Now if $a(*)\epsilon U$ then $a(U \cup R\backslash\{a(*)\}) = \top$ whence $a(U) = \top$. So

$$a(U) = \top \leftrightarrow * \epsilon a(U) \leftrightarrow a(*) \epsilon U$$

and we have our factorization.

4.3 THEOREM In the sheaf model over Ω the Dedekind reals fail to be locally compact. $\not\models$ R is locally compact.

Proof. As R is a simple sheaf life is easy. Let

$$[\![r\in A_n]\!] = \{*\} \cup \{ t \mid |r-t| > 1/n \} \quad .$$

Certainly this gives an internal family of opens.

$$\models \ \forall r \ \exists n \quad r\in A_n$$

as, for $r\in R$, we have $\bigcup [\![r\in A_n]\!] = \{*\} \cup I\setminus\{r\}$ which is not a fixed point, $[\![\exists n \ \ r\in A_n]\!] = \bigvee [\![r\in A_n]\!] = \top$. On the other hand, given any k

$$[\![\forall f\in[0,1] \ \exists n\leq k \quad r\in A_n]\!] = \{*\}$$

as

$$r \notin [\![\ \exists n\leq k \quad r\in A_n]\!] \ .$$

We note that in the above model R has decidable equality. Because Ω has a generic point, the basic mathematical properties normally deduced from compactness hold. In fact, as far as elementary analysis goes, R looks completely classical in this model.

For our next model we use $\Omega = K(I\times I)$ where I is the unit interval. Basic opens of $I\times I$ form a basis for Ω . We leave the reader to check that Ω is locally connected.

4.4 THEOREM In the model over Ω , Cantor space is not compact.

$$\models \ \neg \ \forall B \ \ (\ \forall \alpha \ \exists n \ \ B(\overline{\alpha(n)}) \ \rightarrow \ \exists k \ \forall \alpha \ \exists n\leq k \ \ B(\overline{\alpha(n)}) \ .$$

Proof. It suffices for each $t\in I$ to find B_t such that

$$\models \ \forall \alpha \ \exists n \ \ B_t(\overline{\alpha(n)}) \ ,$$

but

$$[\![\exists k \ \forall \alpha \ \exists n\leq k \quad B_t(\overline{\alpha(n)})]\!] \cap I\times\{t\} = \emptyset \quad .$$

Map $2^{\mathbb{N}}$ continuously to the line $I\times\{t\}$ in $I\times I$ by binary "decimals". Identify $\alpha\in 2^{\mathbb{N}}$ with its image under this map. Define

$$[\![B_t(a)]\!] = \{ x\in I\times I \mid (\ \forall \alpha \supset a) \ \ d(x,\alpha) > 1/n \} \ ,$$

where n is the length of α and $d(\ , \ )$ is ordinary Euclidean distance in $I\times I$.

For $\alpha\in 2^{\mathbb{N}}$, we have by a now familiar argument that

$$[\![\exists n \ \ B_t(\overline{\alpha(n)})]\!] = \top \ ,$$

and since $2^{\mathbb{N}}$ is modelled by a simple sheaf we have,

$$\models \forall \alpha \, \exists n \;\; B_t(\overline{\alpha(n)}) \; .$$

Further, for any $k \in \mathbb{N}$, we see that for each $\alpha \in 2^{\mathbb{N}}$,

$$\alpha \notin [\![\exists n \leq k \;\; B_t(\overline{\alpha(n)})]\!] \; ,$$

as a perfect neighbourhood of α is omitted. So

$$[\![\exists k \, \forall \alpha \, \exists n \leq k \;\; B_t(\overline{\alpha(n)})]\!]$$

does not intersect $I \times \{t\}$. This completes the proof.

As with Theorem 4.3 , the above model is completely classical from the point of view of elementary analysis.

A postscript added in proof follows the references.

REFERENCES

1. Bishop, E.: Foundations of Constructive Analysis. McGraw-Hill, 1967.

2. van Dalen, D.: An interpretation of intuitionistic analysis. Annals of Math. Logic, 13 (1978), 1-43

3. Dummett, M.A.E.: Elements of Intuitionism. Oxford, 1977

4. Fourman, M.P.: Comparaison des réelles d'un topos; structures lisses sur un topos élémentaire. Cah. Top. Géom. Diff., 16, 233-239

5. Fourman, M.P.: The logic of topoi. *In* Handbook of Mathematical Logic (Ed. Barwise, J.), 1053-1090. North-Holland 1977

6. Fourman, M.P.: Formal spaces. *In preparation*

7. Fourman, M.P., Scott, D.S.: Sheaves and logic. *This volume*

8. Grayson, R.: A sheaf approach to models to set theory. M. Sc. Thesis. Oxford, 1975

9. Grayson, R.: Heyting valued models for intuitionistic set theory. *This volume*

10. Hardy, G.M., Wright, E.M.: An Introduction to the Theory of Numbers. Oxford, 1938

11. Hyland, J.M.E.: Aspects of constructivity in mathematics. *In* Oxford Logic Colloquium '76. North-Holland 1977

12. Hyland, J.M.E.: Continuity in spatial topoi. *This volume*

13. Isbell, J.R.: Atomless parts of spaces. Math. Scand., 31 (1972), 5-32

14. Kock, A.: Formally real local rings and infinitesimal stability. Math. Scand. *To appear*

15. Moschovakis, J.R.: A topological interpretation of second-order intuitionistic arithmetic. Comp. Math., 26, 261-276

16. Rousseau, C.: Topos theory and complex analysis. *This volume*

17. Scott, D.S.: Extending the topological interpretation to intuitionistic analysis. Comp. Math., 20 (1968), 194-210

18. Scott, D.S.: Extending the topological interpretation to intuitionistic analysis, II. *In* Intuitionism and Proof Theory (eds. Myhill, J., Kino, A., Vesley, R.), 235-255. North-Holland 1970

19. Wraith, G.C.: Generic galois theory of local rings. *This volume*

POSTSCRIPT

Since this paper was written, Andre Joyal has shown that a model in which $[0,1]$ is not compact (strengthening our result of 4.3) may be obtained using the construction of generic models for geometric theories. Specifically, Joyal introduces a generic ideal I of the partial lattice of rational open intervals and forces the following conditions:

(i) $r \in I$ for each $r \in R$

(ii) $\mu(\bigcup F) < \frac{1}{2}$ for each *finite* $F \subseteq I$ (where μ is Lebesgue measure).

Joyal's most surprising insight is to see that in this model R is represented by the constant sheaf $\underset{\sim}{R}$. This tells us that a cover of the old reals is a cover of the new reals. Using the same method, Joyal shows also that it is consistent to have a *partial* function from $\mathbb{N}$ *onto* $\mathbb{N}^{\mathbb{N}}$; that is, $\mathbb{N}^{\mathbb{N}}$ is subcountable. Unfortunately these results are yet to appear in print. Joyal's approach leads us to view much of this paper in a new light. In particular, the isomorphism between formal and actual opens mentioned after 2.14 should be viewed as a completeness theorem for a particular theory.

The *ad hoc* models we used in §4 are primitive in comparaison with Joyal's elegant constructions. We hope to exploit his methods in a future paper. However, it seems that this general method is not applicable to the problems we discuss in §3 .