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NOTIONS OF CHOICE SEQUENCE

Michael P. Fourman

Department of Mathematics
Columbia University
New York, New York 10027
U.S.A.

We use sheaf models to undertake a constructive analysis of the effects of admitting non-constructive choice sequences to mathematics.

§1 PREAMBLE

> "A choice sequence is an infinite sequence of natural numbers whose terms are generated in succession; inthe process of generating them, free choices may play a part. At one extreme, the selection of each term may be totally determined in advance by some effective rule: a sequence generated by such a rule is a lawlike sequence. At the other extreme, we have a sequence the selection of each term of which is totally unrestricted: these are the lawless sequences. In between are those choice sequences the selection of whose terms is partially restricted in advance, but not completely determined." Dummett (Elements, p. 418)

The general notion of choice sequence allows that future choices need not be entirely free: they may be subjected to effective restrictions laid down at any stage. For example, we may impose the condition that the finite initial segments of our sequence should belong to some subtree S of the tree of all finite sequences, or that they should be generated by applying some function to the results of some other generating process. Different notions of choice sequence arise from differing types of restriction. Following Troelstra (CS) and Dummett, we shall consider in turn various different notions.

Our analysis starts, not from a conception of a particular notion of choice sequence but rather, from a particular conception of the data which may be available at some stage. This then determines a notion of choice sequence. Our aim is to provide a formalization for various notions of data which will allow the informal but rigorous discussions found in Troelstra and Dummett to be replaced by calculations. For those who understand the jargon, we say immediately that sites codify particular conceptions of data. Formally, a type of data is a site; a category equipped with a Grothendieck topology. The forcing definition for these sites formalizes the type of conceptual analysis described by Troelstra. We give the basic definitions below. For the general theory we refer the reader to Makkai & Reyes (1977) and Kock (1981).

1.1 Representation of data by a site. A category is a collection of states (objects) and arrows representing incoming data:

$$\begin{tikzcd} U & & V \arrow[ll, "f"'] \arrow[loop right, "h"] \arrow[dl, shift left, "g"] \\ & W \arrow[ul, "f*e"] \arrow[ur, shift left, "e"] & \end{tikzcd}$$

Note that, incoming data may transport one from one state to another - backwards along the arrow, or may leave one in the same state. In a given state only certain items of data may be received. Data may be received cumulatively (provided the associated states match up) so we have a partial operation $*$, obviously associative. The identities required by the definition of a category correspond to giving no information $\langle\rangle: V \to V$. We normally call the arrows *morphisms*.

By introspection, we may recognize a truth whose verification seemingly requires more data by observing that, no matter which of an *exhaustive collection of possibilities* for extra data transpires, the verification will occur. We call such exhaustive collections of possibilities for future data *covering families*. Formally we require that these satisfy the axioms for a *Grothendieck pretopology*:

1) $\{\langle\rangle\}$ covers U for each state U,

2) If $K = \{ f: V_f \to U \mid f \in K \}$ covers U, and $e: W \to U$ then $\{ g \mid e*g$ factors through some $f \in K \}$ covers U.

These are clearly valid for the intuitive notion to hand.

1.2 Remarks. Our models are similar to the familiar Beth and Kripke models. Formally, the forcing definition for sites is that for Beth models with bars replaced by abstract covers, and passage to a later stage, $\leq$, replaced by arrows representing incoming data, $\to$.

We make a distinction between a *constructive* explanation of meaning, given in terms of a notion of proof, and an *intuitionistic* explanation of meaning which we shall give in terms of data. Unlike Dummett, (Elements p. 403) we do not view these as rival accounts. We *assume* a basic conception of the mathematics of lawlike or constructive objects. (We discuss later the minimum demands we make on this *metatheory* - which may be classical.) We introduce various notions of data and representations for non-contructive objects based on these notions. Our analysis of the *meaning* of predicates involving non-constructive parameters leads us to the justification of various intuitionistic principles; it does not affect the mathematics of lawlike objects. Technically the theory of non-constructive objects is a *conservative extension* of our metatheory.

Our project is not novel: Beth models were introduced to formalize just such an explanation of meaning. Our use of categories in place of posets arises from a basic *philosophical difference*. We aim, not to give a model of the activities of a single idealized mathematician, but rather, to analyse the *objective* mathematical truths which may be justified on the basis of a particular conception of data. Thus, for us, the information to hand at a particular time is not, in general, part of the state, but part of the representation of a particular non-constructive object.

Two extreme examples of categories are posets and monoids. In a poset, there is, for each pair of objects $\langle p,q \rangle$, at most one morphism $p \leq q$ from p to q. Posets represent a totally subjective conception of data, which identifies the state with the information to hand. A monoid is a category with only one object, only the morphisms and the (total) operation of composition matters. We use them to represent objectively conceptions of data for which we can recognize that future possibilities for data are independent of the information to hand.

Some types of restriction on future data compel us to consider separate states to represent the differing data which is acceptable. In these cases an objective viewpoint amounts to being able to juxtapose two states given independently to give a single process. Formally, given two states A and B, there is a state $A \times B$ accessible from A and B by morphisms

$$A \xleftarrow{\pi A} A \times B \xrightarrow{\pi B} B$$

which are covers and for each pair $e: C \longrightarrow A$ and $f: D \longrightarrow B$ there is a unique $e \times f$ making the diagram commute

$$\begin{tikzcd} A \arrow[r] & C \\ A \times B \arrow[u] \arrow[r, "e \times f"] \arrow[d] & C \times D \arrow[u] \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

Note that, $A \times B$ is not a categorical product since we have no pairing in general. The requirement that the "projection" maps cover says that, we can always introduce a new process independent of that under consideration.

1.3 Definitions. A spread is a subtree $S \subseteq N^{<N}$ with every branch infinite: if $a \varepsilon S$ then $a * n \varepsilon S$ for some $n \varepsilon N$. For $\alpha : N \longrightarrow N$, we say $\alpha \varepsilon S$ iff $\forall a.(\alpha \varepsilon \bar{a} \longrightarrow a \varepsilon S)$. If S and T are spreads, a neighbourhood function $F: S \rightarrow T$ is a monotone function such that for each $n \varepsilon N$, the set of nodes $a \varepsilon S$ such that $\mathrm{Lth}\ (F(a)) \geq n$ is an inductive bar of S. Given $\alpha \varepsilon S$ and $F: S \longrightarrow T$ we define $\overline{F(\alpha)}$ by $\alpha \varepsilon a \longrightarrow F(\alpha)\ \varepsilon\ F(a)$.

Spreads represent subsets of $\mathbb{B}$, neighbourhood functions represent continuous functions. Composition of neighbourhood functions gives the composition of the associated functions. Some functions are canonically represented: in particular, the open inclusion corresponding to a finite sequence e is represented canonically by $\lambda a.e*a$ (and in many other ways by merely deferring the information); more generally, if f is an open map it has a canonical representation

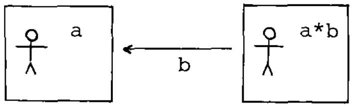
$$F(a) = \wedge\{\ b \mid a \subseteq f^{-1}(b)\ \}.$$

Countable dependent choice and a suitable form of Bar Induction imply that every continuous function has a neighbourhood function. We write B for the universal spread of all sequences.

§2 TYPES OF DATA

The examples which follow should make clearer the translation from the informally rigorous description of a notion of choice sequence to the appropriate site.

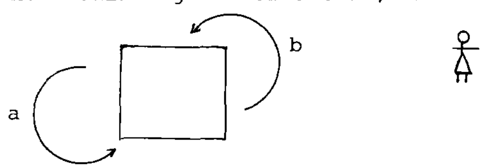
2.1 Open Data. The simplest data we shall consider, open data, consists of finite sequences a of natural numbers. We can construct a sequence α from such data in many ways, the simplest of which is to consider the information thus far received as an initial segment $\alpha \varepsilon a$. We have decided in advance that, having received the information a we will treat subsequent data (another finite sequence, b) in a particular way: we concatenate $a*b$. Thus there are various states which in this example may be identified with the information to hand. Incoming information takes us from one state to another.



Abstractly, we have a category whose objects are the states and whose morphisms represent finite amounts of information. In our present example this structure is represented abstractly as the tree of finite sequences or, more concretely, as the category of basic opens of Baire space and open inclusions between them.

Since, to recognize that $\alpha \varepsilon a*n$ for some $n \varepsilon N$ it suffices to recognize that $\alpha \varepsilon a$, we must let $\{ a*n \mid n \varepsilon N \}$ cover a for each finite sequence a. These covers generate the open cover topology for formal Baire space (see Fourman & Grayson (this volume)). Our analysis has merely served to reconstruct the Scott (1968) - Moschovakis (1973) model.

From a more objective view of mathematics the distinction between various states seems unjustified: it portrays the activity of a particular idealized mathematician rather than the mathematics which results from reflection on the general nature of such activity. A more satisfactory model (from this point of view) is given by the monoid of finite sequences. We can picture this concretely as the monoid of neighbourhood functions canonically representing open inclusions. Again the appropriate topology is the open cover topology. This model corresponds to the liberation of the idealized mathematician: realizing her situation, she can transcend it and is free.



This conception of open data allows that all possible data can be coded up in a single choice sequence.

2.2 Independent Open Data. We modify our model to consider not a single generating process but a potentially infinite collection. It is essential to distinguish this from a potentially infinite sequence of processes, which we could code as a single choice sequence. The force of this distinction is that, at any stage, the information we have is just the collection of initial segments to hand. They are not taken in any particular order.

The subjective states for this notion consist of finitely much information about finitely many sequences. Concretely we represent such a state by a basic open $U \subseteq B^n$ modulo any action by a permutation of n.

Actually it is more convenient to consider basic opens as states. At any stage we may introduce finitely many independent generating processes as well as obtaining more information about those already considered. We represent such information by a map

$$\begin{array}{ccc} U & \longrightarrow & V \\ \cap\!\shortmid & & \cap\!\shortmid \\ B^m & & B^u \end{array}$$

which is induced by the projection corresponding to some injection $n \hookrightarrow m$. The morphisms induced by permutations of n have the effect of identifying states which represent different orderings of the same collection. In addition to allowing open covers as before, we stipulate that the projection

$$\begin{array}{ccc} U & \longrightarrow & \pi(U) \\ \cap\!\shortmid & & \cap\!\shortmid \\ B^m & \longrightarrow & B^n \end{array}$$

is a cover. This reflects the possibility of adding finitely many independent processes to any discussion. In general, a family of morphisms covers iff the union of the images is an (open) cover of V. All the morphisms here are open maps and we shall later consider them as represented canonically by neighbourhood functions.

Again, we have arrived at a well-known model: according to Hyland (personal communication), the forcing definition for this model corresponds to the Kreisel-Troelstra elimination of lawless sequences.

A more objective representation of this type of data is obtained by identifying states in which the same number of generating processes are considered. The site we use to represent this type of data has as objects, the various B^n, and as morphisms, compositions of projections $B^m \longrightarrow B^n$ induced by $n \hookrightarrow m$ and open inclusions $B^n \longrightarrow B^n$ induced by n finite sequences. Once more, all the morphisms are open maps canonically represented by neighbourhood functions and we use the topology in which a family of morphisms covers iff its images cover.

<u>2.3 Lawless Data</u>. This conception of data was motivated by the following passage from Troelstra (CS p. 16):

> "Suppose we have started two lawless sequences α and β, alternately selecting values: $\alpha 0$, $\beta 0$, $\alpha 1$, $\beta 1$, $\alpha 2$, $\beta 2$, ... Now we may also regard this as a <u>single</u> process γ, with $\gamma(2n) = \alpha n$, $\gamma(2n+1) = \beta n$. <u>However</u>, we cannot regard α,β,γ as all being lawless within the same context: either we have to decide α and β to be lawless, and then γ is a sequence . . . which is not itself lawless . . .; or we consider γ as lawless, in which case α,β are sequences (not lawless ones) constructed from γ."

This discussion cannot be expressed in the Kreisel-Troelstra formalisation of choice sequences. This is because, their notion of lawlessness is not an objective one. In our previous example, all the states are, in fact, remarkably similar: B^n is homeomorphic to B. Essentially, the differences between the states arise because we have chosen a coding $B \simeq B^n$ in term of which we choose which maps to put in our category. We now take the point of view that different codings simply reflect different ways of considering the same reality. Thus the controversy as to which of α, β and γ are lawless, in our example above, arises from the difference in viewpoint formalized by the pairing $B \simeq B \times B$.

We now consider an abstract view of the same kind of data, which is independent of such codings. We call this lawless data, it consists of those endomorphisms $e: B \to B$ which can be decomposed as "projections modulo some coding":

$e = j^{-1}{\bullet}\pi{\bullet}i$
for some homeomorphisms i and j.

$$
\begin{tikzcd}
B^m \arrow[r, "\pi"] & & B^n \\
U \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, "\pi"'] & & V \arrow[u, phantom, sloped, "\subseteq"] \\
& B \arrow[ul, "i"] \arrow[ur, "j"'] &
\end{tikzcd}
$$

As usual, all our morphisms are open maps and we use the topology in which surjective families cover.

We shall see that lawless sequences for this conception of data behave more sociably than is traditional. For example, two views of the world may at some stage turn out to be the same so equality is not decidable. To formalize our discussion of this type of lawlessness, we shall introduce a notion of independence: basically, α and β are independently lawless iff $\gamma = \langle\alpha,\beta\rangle$ is lawless. Returning to Troelstra's example, α, β and γ are lawless α and β are independent and γ is independent of neither of them.

2.4 Spread Data. Here we attempt to formalize Brouwer's description of the generation of a free choice sequence.

> " . . . the freedom of proceeding, without being completely abolished, may at some time p, undergo some restriction, and later on further restrictions."
>
> Brouwer (Cambridge p. 13)

The restrictions discussed by Brouwer demand that future choices belong to some spread. Spreads correspond to certain sublocales of B. We consider such sublocales $F \subseteq B^n$ and morphisms between them induced by projections. We take as covers projections and open covers. This gives us (in this example) the topology in which a family covers iff the interiors of its images cover.

This topology involves no new insights, many stronger topologies (more covers) are conceivable: It is certainly plausible that we might justify the conclusion that every member of a spread S belongs to one of the spreads T_i without showing that the interiors of the T_i cover S, by appealing to particular properties of S. This would be reflected in our models by adopting a stronger topology. What we will show is that it is consistent to assume that the only covers are those we have built into the definition of the topology.

The main insights justified by this conception of data are the relativisation of $\forall\alpha\,\exists\beta$ choice and continuity for lawless α to lawless elements of some spread and the extension of Bar Induction to give induction over arbitrary spreads.

Brouwer's conception of choice sequence has been criticized for not being closed under continuous operations. The spreads we have introduced are blank or naked spreads, which, for Brouwer, simply provide a framework for the generation of mathematical entities. By attaching "figures" to some nodes of a spread S we produce new objects. In particular, any neighbourhood function $F: S \to T$ produces for each choice sequence $\alpha \in S$ a sequence $F(\alpha) \in T$. The infinite sequences generated in this way are clearly closed under those continuous operations which have neighbourhood functions. We shall see that (in our models) all lawlike functions have neighbourhood functions. Furthermore using such dressed spreads (with $\xi \in \langle S,F \rangle$ interpreted as, for some $\alpha \in S$, $\xi = F(\alpha)$), we shall see that an axiom of "spread data" is valid for these sequences.

2.5 Continuous data. We start from Brouwer's 1933 description of a dressed spread as reported by van Dalen (Cambridge p. 17). Here A generates a lawless sequence α and B applies to it a neighbourhood function to obtain a sequence $F(\alpha)$ as described earlier. We modify this picture by no longer requiring that A's sequence be lawless: it may in fact be generated as a continuous function of some sequence generated by X who, in turn, refers to Y, and so on. We require that although this chain of dependence may be potentially infinite, all that B can be aware of at any given stage is a finite chain of dependencies, resulting in the knowledge that $\beta = \Gamma(\alpha)$ for some α generated by someone down the line, and some neighbourhood function Γ.

We represent such data by a neighbourhood function $\Gamma: S \to T$ between spreads. Note that, although in principle we should want to consider dependence on more than one sequence, such data reduces to dependence on a single sequence by means of the pairing $B \times B \simeq B$. We give this category the "open cover topology" in which the canonical representatives of a covering family of open inclusions form a cover. Of all our models we believe that this one best represents the notion of choice sequence. Nevertheless, we discuss two variants.

Firstly, if we are concerned only with extensional properties, we can use continuous functions in place of neighbourhood functions. Secondly, if instead of using arbitrary spreads we consider the monoid of continuous functions $B \to B$, with the open cover topology, we obtain a model for Kreisel and Troelstra's theory CS. (This was observed independently by Moerdijk & van der Hoeven (1981), Grayson (1981) and the author (1981)). The forcing definition for this model corresponds to the elimination mapping for choice sequences of Kreisel and Troelstra (1970).

In these models we verify full $\forall \alpha \exists \beta$ choice and continuity principles. The advantage of the extended model in which we allow arbitrary spreads as domains is to justify restricted versions of these and extended Bar Induction as for spread data.

2.6. Other types of data. In our paper Continuous Truth (1982) we consider more general types of data; in particular, data represented by continuous maps between opens of R^n. We also give a general treatment of the "elimination mappings" associated with each type of data and the relationships between various types of data mediated by geometric morphisms between the corresponding topoi.

§3 NON-CONSTRUCTIVE OBJECTS

We now embark on the analysis promised in 2.1. An understanding of a collection of objects is merely an understanding of what it is to be presented with such an object and of what it is to show that two such objects are equal. This does not automatically give rise to a determinate collection of predicates; rather we must introduce predicates by explicitly giving their meanings. Other predicates may, of course, be compounded from ones previously understood using the logical connectives.

We suppose the meanings of statements involving lawlike parameters, quantification over lawlike objects and the meanings of the logical connectives applied to such statements, to be understood. Traditionally, an explanation is given in terms of an informal notion of construction (for example, Dummett (Elements p. 12ff.)). Our explanation of the meaning of statements involving non-constructive objects is independent of this (and, to a large extent, of its results), similar to it in form, and different from it in content. The meaning of a statement involving non-constructive objects is given in terms of a constructive understanding of which items of data justify a given assertion.

3.1 Non-constructive Objects. Our archetype is given by Brouwer's notion of a dressed spread: A partial function ϕ assigning lawlike objects to the nodes of some spread S. The idea is that any choice sequence α of the spread S generates successive approximations, $\phi(a)$ for $\alpha\varepsilon a$, to a non-constructive object $\phi(\alpha)$. Abstractly, we assume that the constructive objects $\phi(a)$ have a preorder, $x < y$ if x contains "more information" than y, and that ϕ is monotone, $a < b$ implies that $\phi(a) \leq \phi(b)$. For example, any neighbourhood function $\bar{F}$ represents such a non-constructive object ξ in that it says what information $F(a)$ about ξ can be justified on the basis of the data $a\varepsilon B$. Once the data e has been assimilated, further data will be treated differently; then, on the basis of data a, we may justify $\phi(e(a))$. The assimilation of the data e causes us to change our representation of ξ; we call this change restriction along e and use $\phi \bullet e$ to represent $\phi\upharpoonright e$, as $(\phi\bullet e)(a) = \phi(e(a))$. This is why data should be represented concretely by neighbourhood functions.

In general then, a non-constructive object given in state U is (represented by) a monotone map $\xi: U \to P$ where P is some partially-ordered domain of lawlike objects, and the assimilation of data is represented by composition of functions. Abstractly, the non-constructive objects based on a given poset of lawlike objects form a presheaf. (For more examples of suitable domains of lawlike objects, see Fourman & Grayson (this volume).) Constructive objects are represented by themselves, as sections of constant presheaves.

3.2. Meaning. For a mathematician in state U, the meaning of a statement ϕ involving non-constructive parameters is given by saying which items of data e justify ϕ. We write this relation $e \Vdash \phi$ and write $U \Vdash \phi$ for $\langle\rangle \Vdash \phi$. The definition is inductive.

Firstly, we have two basic properties of this notion of justification:

if K is a cover of U and $e \Vdash \phi$ for each $e\varepsilon K$, then $U \Vdash \phi$,

if $V \Vdash \phi\upharpoonright e$ then $e \Vdash \phi$,

where $e: V \to U$ and $\phi\upharpoonright e$ is the result of restricting each parameter.

Secondly, we give the meaning of basic (atomic) predicates. The basic statements we may make concerning non-constructive objects are few (two):

<u>intensional equality</u>: if $\alpha\upharpoonright e = \beta\upharpoonright e$ then $e \Vdash \alpha \equiv \beta$

<u>finite information</u>: if $\alpha(e(\langle\rangle)) \leq p$ then $e \Vdash \alpha \varepsilon p$

For basic predicates involving only constructive parameters,

if ϕ then $e \Vdash \phi$.

Finally, we explain the meanings of the logical connectives:

$\wedge$ if $U \Vdash \phi$ and $\Vdash \psi$ then $\Vdash \phi \wedge \psi$

$\vee$ if $U \Vdash \phi$ or $\Vdash \psi$ then $\Vdash \phi \vee \psi$

$\exists$ if $U \Vdash \phi(\alpha)$ then $U \Vdash \exists x.\phi(x)$

$\rightarrow$ if for all e, if $e \Vdash \phi$ then $e \Vdash \psi$, then $U \Vdash \phi \rightarrow \psi$

$\forall$ if for all e: $V \rightarrow U$ and each ξ given at V we have $V \Vdash \phi\upharpoonright e(\alpha)$, then $U \Vdash \forall x.\phi(x)$

$\perp$ is never <u>explicitly</u> justified, although it may become so at some nodes by virtue of the general clauses above (this eventuality does not arise in our present models).

We claim that this definition reflects the intended meaning of statements involving non-constructive parameters. It coincides with the standard forcing definition for sites. We also define quantification over independently generated free choice sequences: for e: $V \rightarrow \mathbb{B}$.

if $U \times V \Vdash \phi\upharpoonright\pi_1(e\upharpoonright\pi_2)$ then $U \Vdash \underline{\forall}\alpha\varepsilon e.\phi$

if $U \times W \Vdash \phi\upharpoonright\pi_1(\beta\upharpoonright\pi_2)$ then $U \Vdash \underline{\exists}\alpha\varepsilon e.\phi$

(for any β: $W \rightarrow \mathbb{B}$ which factors through e)

(In §3.4 we see how to regard e in general as a subset of $\mathbb{B}$.)

<u>3.2.1 Lemma</u>. 1) $f \Vdash \phi\upharpoonright e$ iff $e * f \Vdash \phi$

2) if $e \Vdash \phi$ then $e * f \Vdash \phi$

3) $e \Vdash \phi$ iff ϕ, if ϕ has no non-constructive parameters or quantifiers.

<u>3.3 Basic Types</u>. We have represented non-constructive objects abstractly as (local) sections of certain separated presheaves. We now see that this representation gives us the higher types defined formally in sheaf models. To save space we then treat these models more or less formally. We ask the reader to bear in mind that the models are intended to reflect the meaning of non-constructive mathematics as well as its formalism.

<u>3.3.1 Lemma</u>. If F, G: $U \rightarrow T$ represent non-constructive objects at U then

$U \Vdash F \varepsilon N^N$ iff F is a neighourhood function

$U \Vdash F = G$ iff F and G represent the same function

Where = is <u>extensional</u> equality.

In any sheaf model the discrete spaces N and $N^{<N}$ are represented by constant presheaves. In our present models, the (set of points of) Baire space is represented by the separated presheaf of continuous

Baire - valued functions (see CT). Thus the non-constructive sequences we have defined correspond extensionally to the abstract sequences given by the higher-order logic in sheaf models. The pre-sheaf $\mathbb{B}(U) \simeq (U, N^{\leq N})$ of neighbourhood functions represents Baire space intensionally. The representable presheaf $\mathcal{B}(U) \subseteq \mathbb{B}(U)$ represents the free choice sequences for the notion of data concerned. The idea is that nothing more is known of them than can be given directly by data.

Non-constructive objects are represented extensionally as continuous maps to formal spaces (see Fourman & Grayson). Restriction is given by composition. The basic predicates are given by

equality	if $\alpha \upharpoonright e = \alpha \upharpoonright e$ then $e \Vdash \alpha = \beta$
finite information	if $(\alpha \upharpoonright e)^{-1}(p) = T$ then $e \Vdash \alpha \varepsilon p$.

For consideration of extensional properties it suffices to consider our sites extensionally as categories of continuous maps.

3.4 Lawlike Objects In general, we think of constant presheaves as representing lawlike objects. In most sheaf models there is no canonical way in which to define "the" collection of lawlike elements of a given sheaf. In our present models we define the collection of lawlike elements of a sheaf X to be the subsheaf $L(X) \subseteq X$ generated by global sections of X. We claim that L(X) is a constant sheaf: Suppose a and b are global sections if $A \Vdash a=b$ then $A \times B \Vdash a=b$ whence $B \Vdash a=b$ (as projections are covers). (For monoid models of course the topology is irrelevant and we always have a notion of "lawlike" given in this way.)

Of course, the discrete spaces N and $N^{<N}$ are lawlike. We now give some other examples:

3.4.1. The lawlike elements of $\mathbb{B}$ are given by the constant functions in $\mathbb{B}(U)$, which are canonical representatives of the constant functions .

3.4.2. The collection K_X of lawlike operations $\mathcal{B} \to X$ is given by the Yoneda Lemma: $\Gamma(K_X) \simeq X(\mathcal{B})$ with the action

$$U \Vdash F(\alpha) \boxminus \xi \qquad \text{iff } F \upharpoonright \alpha = \xi$$

Note that these are given intensionally. However, every lawlike operation with non-constructive objects as values is given as a neighbourhood function and thus acts extensionally.

3.4.3. Lawlike data; Morphisms with codomain B may be viewed as subobjects of $\mathbb{B}$ as follows: For $e: U \to B$ and $\xi: V \to B$

$$\text{if } \xi \text{ factors through } e \text{ then } V \Vdash \xi \varepsilon e$$

In each model $\xi \varepsilon e$ is in fact definable: e may be viewed as an element F_e of $K_{\underline{U}}$ and

$$V \Vdash \forall \xi . (\xi \varepsilon e \leftrightarrow \exists \alpha \varepsilon \underline{U} . F_e(\alpha) = \xi).$$

(Where $\underline{U}$ is the representable presheaf.)

If $u: U \rightarrowtail B$ is monic then $\xi \varepsilon X(U)$ may be viewed as a lawlike function defined for all $\alpha \varepsilon u$ by

$$U \Vdash F(u \upharpoonright \beta) = \xi \upharpoonright \beta.$$

§4 PROPERTIES OF THE MODELS

All sheaf models provide interpretations of HAH. These ones have special properties. We view this as justifying certain intuitionistic principles on the basis of particular conceptions of data. We assume choice principles for lawlike objects, (which is constructively unexceptionable). This allows us to reduce existence on a cover of U to existence on U itself for lawlike objects (as every open cover of U has a disjoint clopen refinement). For continuous data, existence on a cover always reduces to existence on U, as covering families of monomorphisms generate the topology.

Below, ξ, ζ range over non-constructive objects, α, β, γ over free choice sequences, e, f over data a, b over B and n, m over N. We use x, y as variables for lawlike objects in general, and write $\xi\varepsilon L$ to signify that ξ is lawlike.

4.1 Choice Principles. We obtain countable lawlike choice with non-constructive parameters.

ACN* $\Vdash \forall A(\forall n \exists x.A(n,x) \rightarrow \exists f \forall n.A(n,f(n)))$.

From the remarks above, this is standard. For continuous data, the same proof gives

ACN$\mathcal{B}$* $\Vdash \forall A(\forall n \exists \xi.A(n,\xi) \rightarrow \exists f \forall n.A(n,f(n)))$.

It is perhaps surprising that for open data (for example) this is not justified. As $\mathcal{B}$ is representable, we obtain, for those models in which projections are covers, forms of $\forall\alpha\,\exists x$ choice:

AC$\mathcal{B}$ $\Vdash \forall A\varepsilon L(\forall \alpha\varepsilon u. \exists x.A(\alpha, x) \rightarrow \exists f\varepsilon L. \forall \alpha\varepsilon u.A(\alpha,f(x)))$

AC$\mathcal{B}$* $\Vdash \forall A(\underline{\forall}\alpha\varepsilon u.\exists x.A(\alpha,x) \rightarrow \exists\bar{\gamma}\exists f\varepsilon L\,\underline{\forall}\alpha\varepsilon u.A(\alpha,f(\alpha,\bar{\gamma})))$

For AC$\mathcal{B}$, suppose A is lawlike and U $\Vdash \forall\alpha\exists xA(\alpha,x)$, then UxB $\Vdash \forall\alpha\exists x.A(\alpha,x)$ whence B $\Vdash A(id,a)$ for some $\varepsilon X(\mathcal{B})$. This a represents the required function as in 3.4. For AC$\mathcal{B}$* if $U\times V \Vdash \exists x.A\upharpoonright\pi_1(u\upharpoonright\pi_2,x)$ then $U\times V \Vdash A\upharpoonright\pi_1(u\upharpoonright\pi_1,a)$ for some $a \in X(U\times V)$. Now introduce parameters $\bar{\gamma}$ for U and view a as a function of these and $\alpha\varepsilon u$ as in 3.4. Note that, for continuous data, $\underline{\forall}$ and $\forall$ coincide.

For continuous data, the same proofs give $\forall\alpha\exists\xi$ choice principles:

AC$\mathcal{B}\mathcal{B}$ $\Vdash \forall A\varepsilon L(\forall \alpha\varepsilon u,\exists\xi.A(\alpha,\xi) \rightarrow \exists F\varepsilon L\ \forall\alpha\varepsilon u.A(\alpha,F(\alpha)))$

AC$\mathcal{B}\mathcal{B}$* $\Vdash \forall A(\forall \alpha\varepsilon u.\exists\xi.A(\alpha,\xi) \rightarrow \exists F\varepsilon L.\exists\bar{\gamma}.\forall\alpha\varepsilon u.A(\alpha,F(\alpha,\bar{\gamma})))$

For the monoid model for continuous data, in which we do not allow restrictions to arbitrary spreads, we obtain a stronger form:

AC$\mathcal{B}\mathcal{B}$** $\Vdash \forall A(\forall\alpha\exists\xi.A(\alpha,\xi) \rightarrow \exists f\varepsilon L.\forall\alpha.A(\alpha,F(\alpha)))$

Suppose $B \Vdash \forall\alpha\exists\xi.A(\alpha,\xi)$ then $B \Vdash A(id,\xi)$, for some $\xi\varepsilon X(B)$ which represents the required function.

In each case above, we may replace $\forall\alpha$ by $\forall\alpha_1,\ldots\alpha_n$ and $\underline{\forall}\alpha\varepsilon u$ by $\underline{\forall}\alpha_1\varepsilon u_1,\ldots,\alpha_n\varepsilon u_n$ and obtain the corresponding choice principles with the same proofs. The same remark applies to our principles below.

4.2 Continuity Principles. The first principle, we call Existence of Data.

ED $\Vdash \forall\alpha.\forall A\varepsilon L(A(\alpha) \rightarrow \exists e\varepsilon D.(\alpha\varepsilon e \wedge \forall\beta\varepsilon e.A(\beta)))$

Given $\alpha\varepsilon\mathcal{B}(U)$ with $U\Vdash A(\alpha)$, view α as an element of D to justify ED. We also obtain a stronger form for data in which <u>independent generating processes</u> may be introduced:

ED* $\Vdash \underline{\forall}\alpha_1,\ldots,\alpha_p(A(\alpha_1,\ldots,\alpha_p) \to \exists u_1,\ldots,u_p$

$$(\bigwedge \alpha_i\varepsilon u_i \wedge \underline{\forall}\beta_1\varepsilon u_1,\ldots,\beta_p\varepsilon \ddot{u}_p.A(\varepsilon_1,\ldots\beta_p)))$$

Suppose $B^p x U$ $W\times V\Vdash A\restriction\pi(u_1,\ldots,u_p)$ where W is a basic open and the u_i are components of the inclusion. These u_i do the trick: $W\times W\times V\Vdash A(u_1,\ldots,u_p)$ as the two possible orders for considering W are isomorphic.

The second general principle is that every lawlike function defined on $\mathcal{B}$ is given by a neighbourhood function. We have already see (3.4) that elements of $\mathbb{B}$ (B) represent lawlike functions. By 3.2.1.(3) these may also be viewed as neighbourhood functions in the model. The application defined in 3.4 is just standard application of neighbourhood functions carried out internally. Thus,

K $\Vdash \forall F\varepsilon L\colon \mathcal{B} \to \mathbb{B}$. "F is given by a neighbourhood function."

The same remark holds for functions to other non-constructive domains and for those defined on subspaces of $\mathcal{B}$ given by monic data u. Note that the F above are <u>a priori</u> given intensionally, it is a consequence of K that they act extensionally. This remark is not deep, it shows how little intensional information we have taken into account. We finish this section with the remark that all objects are lawlike functions of a finite number of free choice parameters

U $\Vdash \forall\xi\,\exists n\,\exists F\varepsilon L.\exists\,\alpha_1,\ldots,\alpha_n.\xi = F(\alpha_1,\ldots,\alpha_n)$.

(U is for Uniformization).

This enables us to make good an earlier promise. Dressed spread data holds in all our models as a consequence of the existence of data for free sequences because if $\exists\overline{\alpha},F.\xi = F(\overline{\alpha})$ then for any A we have $A(\xi)\leftrightarrow A(F(\alpha))$.

<u>4.3 Brouwer's Dogma</u> of Bar Induction is also built into our models. We show that

$\Vdash \forall K\subseteq N^{<N}$ (K monotone $\wedge$ K inductive $\to (\forall\alpha\,\exists n\varepsilon K.\alpha\varepsilon n \to \langle\rangle\,\varepsilon K))$

Note that K is <u>not</u> required to be lawlike. Suppose that $U\Vdash$ "K is a monotone, inductive bar". Consider

$$\mathbb{K} = \{\langle V,n\rangle \mid V\Vdash \underline{n}\ \ K\restriction V \text{ with } V\varepsilon\ \mathcal{O}(U),\ n\varepsilon N^{<N}\}$$

$\mathbb{K}$ is monotone and doubly inductive (i.e. $\mathbb{K}$ is a closed crible in $\mathcal{O}(U\times B)$). Thus if $\mathbb{K}$ covers $U\times B$ then $\langle U,\langle\rangle\rangle\ \varepsilon\ \mathbb{K}$ and we are done.

Now $U\times B \Vdash \forall m\varepsilon K\restriction\pi_1.\pi_2\varepsilon m$, by persistence of forcing so

$\mathbb{K}^* = \{\langle V,n\rangle \mid Vxn \Vdash \underline{m}\varepsilon\ K\restriction\pi_1 \wedge \pi_2\varepsilon\underline{m}$ for some m$\}$ covers $U\times B$. But $\mathbb{K}^* \subsetneq \mathbb{K}$ because

$$V\times n \Vdash m\varepsilon\ K\restriction\pi_1 \quad\text{iff}\quad V \Vdash m\varepsilon K \text{ as projections cover}$$

and

$$V\times n \Vdash \pi_2\varepsilon m \text{ iff } n \geq m.$$

Thus we are done. Clearly, the same proof justifies bar induction over any spread which occurs as a state in our representation of data.

<u>4.4 Equality</u>. We begin by considering open data, which models the theory LS, lawless data and spread data, which give another kind of lawlessness. For all of these, extensional and intensional equality

coincide:

$$\Vdash \forall \alpha,\beta\ (\alpha = \beta \longleftrightarrow \alpha \equiv \beta).$$

The major difference between them is that, for open data free sequences have decidable equality

$$\Vdash \forall \alpha,\beta\ (\alpha = \beta \vee \neg\alpha = \beta)$$

Whereas for lawless and spread data,

$$\Vdash \neg \forall \alpha,\beta\ (\alpha = \beta \vee \neg\alpha = \beta)\ .$$

In these models we can represent the notion of non-constructive objects generated by independent processes. Given $x\varepsilon X(U)$ and $y\varepsilon Y(V)$ let

$$U\times V \Vdash x{\restriction}\pi_1 \ \ddagger\ y{\restriction}\pi_2 \qquad \text{"x is \underline{independent} of y".}$$

Then $W \Vdash x \ddagger y$ iff locally there is a cover $p\colon U\times V \to W$ with $x{\restriction}p = x{\restriction}\pi_1$ and $y{\restriction}p = y{\restriction}\pi_2$ for some $x\varepsilon X(U)$ and $y\varepsilon Y(V)$.

For free sequences, independence is definable:

$\Vdash \forall \alpha,\beta\ (\alpha \ddagger \beta \leftrightarrow \neg\alpha = \beta)$ for subjective open data,

$\Vdash \forall \alpha,\beta\ (\alpha \ddagger \beta \leftrightarrow \exists\gamma.\gamma = \langle\alpha,\beta\rangle)$ for lawless and spread data.

Note that for other types of data such a predicate becomes trivial since a priori independent processes might later be found to be related.

For these types of data, the quantifiers $\underline{\forall}$, $\underline{\exists}$ satisfy the axioms and rules given by Troelstra (CS, p. 35). They have the effect of quantifying over lawless sequences generated by processes independent of any under consideration

$$\Vdash \underline{\forall}\alpha_1,\ldots,\ \alpha_n(\ \ddagger\ (\alpha_1,\ldots,\alpha_n))$$

For lawlike predicates we have

$$\Vdash \forall A\varepsilon L\ (\underline{\forall}\,\bar{\alpha}.A(\bar{\alpha}) \leftrightarrow \forall\,\bar{\alpha}\ \ (\ddagger(\bar{\alpha}) \to A(\bar{\alpha})))$$

Thus ED* coincides with the usual form of open data.

We conclude our discussion of these variations on open data by listing some simple properties whose verification is left to the reader.

$\Vdash \forall a\,\exists\,\alpha.\alpha\varepsilon a$ Free sequences are dense.

$\Vdash \forall\,\xi\neg\forall\,\alpha(\neg\xi = \alpha \to \xi \neq \alpha)$ Free sequences are very dense.

For open and lawless data, we also have,

$\Vdash \forall\alpha.\neg\alpha\varepsilon L$ Free sequences are not lawlike.

For continuous data, extensional and intensional equality do not coincide

$$\Vdash \exists\ \alpha,\beta\ (\alpha = \beta \wedge \neg\alpha \equiv \beta)$$

every sequence is free

$$\Vdash \forall\ \xi\ \exists\,\alpha.\xi \ \equiv\ \alpha$$

<u>4.5 Identification of Data.</u> All the data we have used is given by neighbourhood functions. As we remarked in 4.2, these just act internally as lawlike neighbourhood functions. Thus data is given in general as analytic data with a class of functions restricted appropriately for each notion of data. For open and lawless data, this reduces internally to open data. For spread data it is just spread

data in the tradional sense. For continuous data it is analytic data. We know the internal and external characterization coincide by 3.2.1 (3). For each type of data, we have the property of density

$$D \qquad \Vdash \forall e \varepsilon D. \exists \alpha . \alpha \varepsilon e \quad ,$$

trivially, as α can be taken equal to e.

§5 CONCLUDING REMARKS

5.1 Metatheory. We have been somewhat cavalier in our use of higher types. In particular, we have stated many principles in universally quantified form, using quantifiers which are constructively unacceptable. The corresponding schemata may be justified by our methods using a metatheory equivalent to IDB_1 (CS p. 31). For a discussion of this see v.d. Hoeven & Moerdijk (1982).

5.2 The Other Kind of Lawlessness. The model we presented at the Brouwer Symposium, we now view as ad hoc. We considered the monoid of local homeomorphisms of Baire space with the open cover topology, and picked out a domain of "lawless sequences" represented by the local projections. We now view the lawless data presented in 2.3 as a better representation of the notion we had in mind. The free sequences for this notion of data have all the properties we mentioned in our abstract.

5.3 Other Models. Once this project was well-advanced, the author realized that few of the ideas here are really new. The elimination translations of Kreisel and Troelstra coincide with our models for LS (open data) and CS (continuous data) as remarked above. Thus, it cannot be said that there is anything more than a difference in viewpoint distinguishing our approach. More concretely, Dragalin (1974) uses essentially the same ideas as us to construct essentailly the same type of model. We hopethat the presentation of these ideas as sheaf models will at least aid progress by providing a mathematically apt setting for comparing various notions of choice sequence. It is on the level of philosophical analysis that we hope to have provided something novel. For example, we claim that the verification of the axioms of CS in the monoid model for continuous data, provides an adequate conceptual basis for these axioms: acoherent notion of choice sequence and a verification of the axioms based on this notion.

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