IVR: Characterisation of a Control System

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informatics

Overview

- Modelling an control system
- Stationary behaviour and time constant of control systems
- Model identification

Differential Equations

$$\frac{dx(t)}{dt} = f\left(x\left(t\right)\right) \quad \text{subject to } x\left(t_0\right) = x_0$$

- Example from last lecture:

$$\frac{dx(t)}{dt} = a\, x\left(t\right) \quad \text{subject to } x\left(t_0\right) = x_0$$

- Solution

$$x\left(t\right) = x_0 \exp\left(a\left(t - t_0\right)\right)$$

- $a > 0$ fast increase for $x_0 > 0$
- $a < 0$ decay with time constant $\tau := -\frac{1}{a}$

Example: Electric motor

- Ohm's law & Kirchhoff's law

$$V_B = IR + e$$

- Motor generates voltage $e = k_1 s$
- Vehicle acceleration

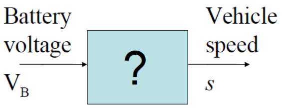
$$\frac{ds}{dt} = \frac{\tau}{M}$$

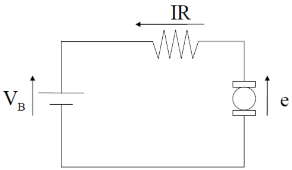
- Torque τ is proportional to current:

$$\tau = k_2 I$$

- Putting together:

$$V_B = \frac{M}{k_2} R \frac{ds}{dt} + k_1 s$$





Solving differential equations numerically

1. Given process model

$$V_B = k_1 s + \frac{M}{k_2} R \frac{ds}{dt}$$

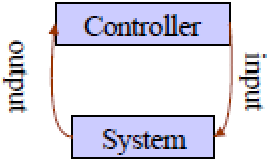
   Rewrite

$$\frac{ds}{dt} = -\frac{k_1 k_2}{RM} s + \frac{k_2}{RM} V_B$$

2. Choose a small step size Δt
3. Start at $t = 0$ and $s(0) = s_0$
4. Iterate $s(t + \Delta t) = s(t) + \Delta t \frac{ds}{dt}$ until $t = T$
5. How do control $s(t)$ towards a desired value by changing V_B?

Abstract form as a control system

$C = As + B\frac{ds}{dt}$	control system	plant and controller
$A + B\frac{d}{dt}$	process dynamics	operator applied to state
s	state variable	output: plant $\rightarrow$ controller
C	control variable	input: controller $\rightarrow$ plant



$$C = As + B\frac{ds}{dt}$$

$$z = As - C \quad \Longrightarrow \quad z = -B\frac{ds}{dt}$$

$$\frac{dz}{dt} = A\frac{ds}{dt} \quad \Longrightarrow \quad z = -\frac{B}{A}\frac{dz}{dt}$$

$$z(t) = z_0 e^{-\frac{A}{B}(t-t_0)} \quad \overset{z=As-C}{\Longrightarrow} \quad s\,(t) = s_0 e^{-\frac{A}{B}(t-t_0)} + \frac{C}{A}\left(1 - e^{-\frac{A}{B}(t-t_0)}\right)$$

Stationary state: $\frac{C}{A}$; time scale (decay by a factor of $\frac{1}{e}$): $\tau = \frac{B}{A}$

Back to the Example

$$V_B = k_1 s + \frac{M}{k_2} R \frac{ds}{dt}$$

$C = As + B\frac{ds}{dt}$ with $A = k_1,\ B = \frac{M}{k_2}R,\ C = V_B$

Stationary behaviour $s\left(t \to \infty\right) = \frac{C}{A}$

$$\frac{ds}{dt} = 0 \implies V_B = k_1 s \implies s\left(t \to \infty\right) = \frac{V_B}{k_1}$$

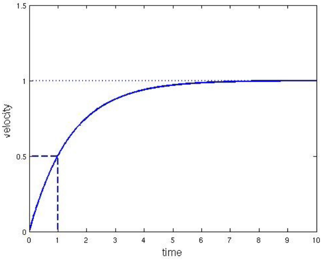
Time scale (decay by a factor of $\frac{1}{\mathrm{e}}$)

$$\tau = \frac{B}{A} = \frac{MR}{k_1 k_2}$$

Decay by a factor of $\frac{1}{2}$

$$s\left(t\right) = s_0 \mathrm{e}^{-\frac{k_1 k_2}{MR}\left(t - t_0\right)} + \frac{V_B}{k_1}\left(1 - \mathrm{e}^{-\frac{k_1 k_2}{MR}\left(t - t_0\right)}\right)$$

Starting from previous state $s_0 = 0$ towards new stationary behaviour at $s = \frac{V_B}{k_1}$



$$s\left(t\right) = \frac{V_B}{k_1}\left(1 - \mathrm{e}^{-\frac{k_1 k_2}{MR}\left(t - t_0\right)}\right)$$

$$\left(\frac{1}{2}\right)\frac{V_B}{k_1} = \frac{V_B}{k_1}\left(1 - \mathrm{e}^{-\frac{k_1 k_2}{MR}\left(t - t_0\right)}\right)$$

$$\frac{1}{2} = \mathrm{e}^{-\frac{k_1 k_2}{MR}\left(t - t_0\right)}$$

$$t_{\mathrm{half}} = -\ln\left(\frac{1}{2}\right)\frac{MR}{k_1 k_2} \approx 0.7\frac{MR}{k_1 k_2}$$

Example

Suppose $k_1 = 7$, $\frac{MR}{k_2} = 20$,

$$V_B = 7s + 20\frac{ds}{dt}$$

If the robot starts at rest and 7V are applied then steady state speed is

$$s = \frac{V_B}{k_1} = 1\text{m sec}^{-1}$$

$$\tau_{\frac{1}{2}} \approx 0.7\frac{MR}{k_1 k_2} = 0.7\frac{20}{7} = 2\text{sec}$$

Time taken to cover half the gap between current and steady-state speed.

Identification of a control problem

If $K = 5$, and $s_\infty = 8$, and the system takes 14 seconds to reach $s = 6$ starting from $s = 0$. What is the process equation?

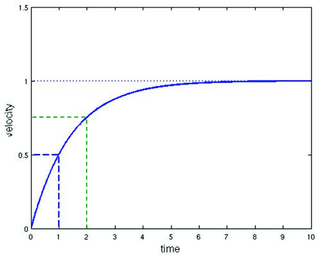
- Determine A and B in

$$Ks_\infty = B\frac{ds}{dt} + As$$

- General solution of the equation:

$$\tau = \frac{B}{A} \qquad s_\infty = \frac{C}{A}$$

where $C = Ks_\infty = 40$



Solution of the process equation (unit scale)

- Thus $A = \frac{C}{s_\infty} = K = 5, \quad B = \tau A$
- How do we obtain τ? (Hint $\tau = 7s$, i.e. $B = 35$)

Controlling the motor over time

Process model

$$V_B = k_1 s + \frac{M}{k_2} R \frac{ds}{dt}$$

Stationary behaviour (steady state)

$$s\,(t \to \infty) = \frac{V_B}{k_1}$$

Inverse model:

$$V_B = k_1\, s_{\mathrm{goal}}$$

Solution provides forward model:

$$s\,(t) = s_0 \mathrm{e}^{-\frac{MR}{k_1 k_2}(t - t_0)} + \frac{V_B}{k_1}\left(1 - \mathrm{e}^{-\frac{MR}{k_1 k_2}(t - t_0)}\right)$$

Summary

- In order to derive a process model, we need to understand the physics of the system
- For simple processes, the system can be characterised by stationary state and the time constant of the approach towards this state
- Given the model equation, a few measurements can help to derive the explicit model equation and thus to obtain the trajectory of the system