Physics Letters A 177 (1993) 102–106
North-Holland

PHYSICS LETTERS A

Rôle of a single scatterer in a multiple scattering medium

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Received 1 April 1993; accepted for publication 13 April 1993
Communicated by A. Lagendijk

The influence of one extra static scatterer on multiple scattered classical waves is considered. It is shown how to connect two "diffusons" or "ladders". Secondly, the interaction vertex of four diffusons ("Hikami box") is generalized to non-static situations and the presence of absorption. Beyond the second order Born approximation, eight diagrams are relevant.

1. Introduction

Consider the propagation of a classical scalar wave through a slab of thickness L with randomly placed point scatterers. This wave is for instance a sound wave or light in a scalar approximation. In the low density limit an independent scatterer approximation can be made. The system is then in the weak localisation regime, i.e. $kl \gg 1$, with l denoting the mean free path. First focus on the scattering of the wave by a single scatterer. This process is described by its t-matrix. We treat the simplest case: isotropic point scattering. In the weak scattering limit the t-matrix can be approximated by the first two orders of the Born approximation (zero- and one-loop diagrams). Near resonance, however, the second order Born approximation breaks down as higher powers of the potential become important. This typically happens in optical experiments with scatterers near resonance. The problem of going beyond the Born approximation for point scatterers is discussed by Nieuwenhuizen et al. [1]. The t-matrix is calculated in all orders of the scattering potential (i.e. all loop diagrams) and regularized [1]. This gives

$$t = \frac{u}{1 - u/u_0 - \mathrm{i}uk/4\pi}, \tag{1}$$

with $u \equiv vk^2$ being the strength of the scattering potential, and $u_0 \equiv vk_0^2$ being the resonance wavelength. v is a parameter, which, when dealing with a realistic scatterer, depends on the size and refractive index of the scatterer. The potential is proportional to k^2, causing the energy transport velocity in disordered systems to be different from both phase and group velocity [2]. With a scatterer density n, the mean free path is defined as $l \equiv 4\pi/nt\bar{t}$. It may become of the order of the wavelength near resonance and hence much smaller than the second order Born result $4\pi/nu^2$. Absorption is contained in the imaginary part of the scattering potential. The single scatterer albedo is given by $a = 1 - 4\pi\,\mathrm{Im}\,u/k|u|^2$. We assume that the absorption is weak, which is realistic in most interesting optical multiple scattering experiments.

We consider a slab geometry. Boundary effects due to internal reflections can be important, and are treated elsewhere [3,4]. Here, the diffuse intensity is supposed to vanish at one mean free path outside the sample. This approximate treatment of the skin layer is good enough for our present purpose. The most important contributions to the transmitted diffuse light are the so-called "ladders" or "diffusons". The ladders consist of a particle and a hole line sharing common scatterers. The particle Green function dressed with a self-energy nt is

$$G(p, \omega) = \frac{1}{p^2 - \omega^2/c^2 - nt}. \tag{2}$$

Suppose the particle line of the ladder propagates with a momentum $p + \frac{1}{2}q$ and a frequency $\omega + \frac{1}{2}\Delta\omega$. The hole line propagates with $p - \frac{1}{2}q$ and $\omega - \frac{1}{2}\Delta\omega$. p denotes an internal momentum to be integrated over. We call q the external momentum and $\Delta\omega$ the external frequency. The Green function is expanded in small external momentum and small external frequency,

$$G(p+\tfrac{1}{2}q,\omega+\tfrac{1}{2}\Delta\omega)=G(p,\omega)-(p\cdot q)G^2(p,\omega)$$
$$-\tfrac{1}{4}q^2G^2(p,\omega)+(p\cdot q)^2G^3(p,0)$$
$$+\frac{\Delta\omega k}{2c}G^2(p,\omega)+O(q^4,\Omega^2)\,. \qquad (3)$$

With this Green function the ladder is calculated. The ladder equation becomes equal to a diffusion equation on length scales large compared to the mean free path. In Fourier space it satisfies

$$(q^2+i\Omega+\kappa^2)L(q)=12\pi/l^3\,. \qquad (4)$$

Ω is proportional to the frequency difference of the ladders, $\Omega=-\Delta\omega/D$, with diffusion constant $D=\frac{1}{3}v_E l$, in which v_E is the energy transport velocity [2]. The inverse absorption length κ is related to the albedo as $\kappa^2=3(1-a)/l^2$. In the case of one single monochromatic plane wave both propagators in the ladder have the same momentum and frequency. The solutions of the diffusion equation (4) now decay exponentially due to absorption only; for $\Omega=\kappa=0$ the ladder decays linearly on this macroscopic level. The effect of going beyond the second order Born approximation is reflected in a different value of the mean free path and the occurrence of the energy transport velocity v_E.

2. Scattering from one extra, static scatterer

Following Berkovits and Feng [5], we consider the influence of one additional point scatterer on the diffuse intensity. This is a problem of practical importance as researchers try to locate an object in a strongly scattering medium by looking at the transmitted diffuse beam [6]. We treat this problem with a diagrammatic approach. The problem can be rephrased as: "*how can two ladders be tied together?*" In fig. 1 all relevant contributions to the scattering are presented. The two rightmost lower diagrams are self-energy diagrams which are not taken into account in the ladder L. These diagrams were overlooked in ref. [5]. However, their contributions are important as they cancel the leading term of the first r.h.s. (lower) diagram.

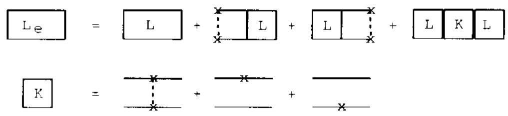

Fig. 1. Contributions from an additional scatterer to the ladder intensity. L denotes the ladder without the extra scatterer, L_e the resulting ladder with contributions from the extra scatterer taken into account. Thick lines denote particle propagators, thin lines denote hole propagators. The dashed line indicates that both the particle and the hole line interact with the extra scatterer. The t_0-matrix of the extra scatterer is denoted by a cross. Its total effect is denoted by K.

The extra scatterer is described by a t-matrix t_0, with an albedo a_0. It is placed at $r_0=(0,0,z_0)$, with z_0 the distance to the outgoing surface. Let the ladder propagators connected to the extra scatterer have momenta q_1 and q_2. Both momenta are defined pointing towards the extra scatterer. In lowest order of these momenta, we find for the lower diagrams of fig. 1

$$K=-\frac{t_0\bar{t}_0 l^2}{48\pi^2}[q_1\cdot q_2 l^2+3(1-a_0)]\,. \qquad (5)$$

Corresponding to the first line of fig. 1, the expression for the ladder with the extra scatterer included, reads in spatial coordinates

$$L_e(r,r')$$
$$=L(r,r')+\frac{t_0\bar{t}_0}{nt\bar{t}}[\delta(r'-r_0)+\delta(r-r_0)]L(r,r')$$
$$+\frac{t_0\bar{t}_0 l^2}{48\pi^2}\int dr_1\,dr_2\,[l^2\nabla_1\cdot\nabla_2-3(1-a_0)]L(r,r_1)$$
$$\times L(r_2,r')\delta(r_1-r_0)\delta(r_2-r_0)\,. \qquad (6)$$

We assume that the extra scattering is weak as compared to the total scattering $n_0 t_0\bar{t}_0 \ll nt\bar{t}$, with $n_0=1/V$, V is the slab volume. First we average the position of the extra scatterer over the whole slab. Behavior like formula (4) for a density of $n+n_0$ scatterers is expected. Indeed, this is recovered with a reduced mean free path $l_e=4\pi/(nt\bar{t}+n_0t_0\bar{t}_0)$. Starting from eq. (6) and substituting formula (4) we find to leading order in n_0

$$L_e(q)=L(q)+2\frac{n_0 t_0 \bar{t}_0}{nt\bar{t}}L(q)$$
$$+[q^2l^2-3(1-a_0)]\frac{n_0 t_0 \bar{t}_0 l^2}{48\pi^2}L^2(q)$$
$$\approx \frac{12\pi}{l_e^3}(q^2+\mathrm{i}\Omega_e-\kappa_e^2)^{-1}, \tag{7}$$

with $\Omega_e \equiv \Omega l/l_e$, consistent with our definition of Ω. Further,

$$\kappa_e^2 l_e \equiv 3[nt\bar{t}(1-a)+n_0 t_0 \bar{t}_0(1-a_0)]/4\pi$$

involves a weighted sum of albedos.

The situation where the scatterer is fixed is more interesting. We consider the transmission of a plane wave through a non-absorbing medium. The transmission is give by

$$T=l\left.\frac{\mathrm{d}L(L,z)}{dz}\right|_{z=0}.$$

L_e is calculated with a method of images [6]. Neglecting the second and third upper r.h.s. diagrams, the transmission in near field reads

$$T(\rho)=\frac{l}{L}+2ql\sum_{n=-\infty}^{\infty}\frac{z_0+2nL}{[\rho^2+(z_0+2nL)^2]^{3/2}}$$
$$-2pl\sum_{n=-\infty}^{\infty}\frac{\rho^2-2(z_0+2nL)^2}{[\rho^2+(z_0+2nL)^2]^{5/2}}, \tag{8}$$

in which $\rho=(x,y)$, and

$$q=-\frac{z_0}{L}\frac{3t_0\bar{t}_0}{16\pi^2 l}(1-a_0), \quad p=\frac{l}{L}\frac{t_0\bar{t}_0}{16\pi^2}. \tag{9}$$

Two cases are to be considered. If the extra scatterer does not absorb, only the p-term is present. This is analogous to a dipole in a static electric field between two capacitor plates. In the transmitted beam a wiggle is seen if $z_0<\frac{1}{2}L$, else a dip in the transmission is seen. If the extra scatterer absorbs, the q-term is dominant and the scatterer acts as a drain for the intensity. The result is a dip in the transmitted beam. This would be equivalent to a (negative) charge in electrostatics.

It is instructive to compare our approach with calculations within diffusion approximation. Den Outer et al. performed experiments where a pencil or glass fiber was located in a diffusive medium [6]. It was found that the experiments are well described by a diffusion approximation. Diffusion was assumed both in the medium and inside the scatterer. The scatterer was for instance a sphere with radius R, inverse absorption length κ_2, and different diffusion constant D_2. Consider the situation of weak uniform absorption and almost equal diffusion constants inside (D_2) and outside the sphere (D). Den Outer et al. find in this case $q=-\kappa_2^2 R^3 z_0/3L$ and $p=R^3(D-D_2)/3LD$. We replace our extra scatterer by a sphere of radius R with a density $n_0=3/4\pi R^3$ of extra scatterers. The extra scattering is weak as compared to the total scattering, if R is not too small, viz. $R \gg (lt_0\bar{t}_0)^{1/3}$. The diffusion constant inside the sphere is $D_2=\frac{1}{3}v_E l_e$. With this identification, our p and q values agree with the results of den Outer et al.

In ref. [5] the self-energy contributions presented in fig. 1 are not taken into account. As a result the unphysical result is found that scatterers without absorption act as a source of intensity.

3. Calculation of the generalized Hikami box

Going beyond second order Born approximation in the calculation of the ladder diagrams resulted in a simple replacement of the mean free path. However, sometimes the situation is more subtle. Two diffusons can interact by exchanging a hole or particle line. Within the second order Born approximation this interaction is described by a set of three diagrams, usually called the Hikami box. Hikami noted the importance of this type of diagrams in disordered electron systems [7]. Also in the calculation of long range correlations of multiple scattered waves the same interactions occur [4]. The diagrams are conveniently depicted as diamonds. To the legs of the diagrams ladders are to be connected, the incoming ladders to the vertical legs, the outgoing ladders to the horizontal ones. As can be seen from fig. 2, the box consists of just interchanging partners (the first r.h.s. diagram) and the effect of scattering from one extra scatterer (all other r.h.s. diagrams). It is important to take into account all corrections to the first term of the figure, as there is a cancellation of leading terms. These cancellations are imposed by energy conservation. Previous approaches only worked up to second order in the scattering potential. Therefore the box was calculated up to the same order, leading to the first three contributions of fig. 2. As we work in all orders of the potential all contributions have

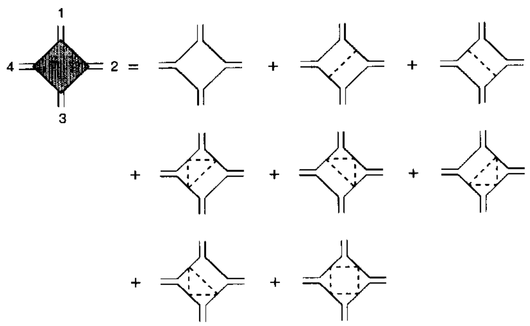

Fig. 2. The full Hikami box. In the second order Born approximation the upper three diagrams are dominant, beyond this approximation all eight are relevant. Thick lines denote particle propagators, thin lines denote hole propagators, and dashed lines indicate that scattering takes places at a common scatterer.

to be calculated. To leading order in $1/kl$, these are the eight diagrams depicted in fig. 2.

Let H denote the expression of the Hikami box, apart from a momenta conserving factor $\delta(q_1+q_2+q_3+q_4)$ (the numbers label the four legs of the Hikami box). Note that one also has $\Omega_1-\Omega_2+\Omega_3-\Omega_4=0$. After expansion in small external momentum and external frequency, we find to the lowest order in kl (all momenta are defined pointing into the box)

$$H(q_1,q_2,q_3,q_4) = \frac{l^5}{48\pi k^2}[-2q_1\cdot q_3-2q_2\cdot q_4 - (q_1+q_3)\cdot(q_2+q_4)] + \frac{l^5\kappa^2}{24\pi k^2}+\frac{il^5}{96\pi k^2}(\Omega_1+\Omega_2+\Omega_3+\Omega_4), \tag{10}$$

in which q_j, Ω_j denote the momentum and frequency of the ladder connected to the jth leg. This generalizes previous results of for instance refs. [8] and [7], as beyond the second order Born approximation the mean free path has a different value. Also effects of κ and Ω are taken into account. If the ladders are attached to the Hikami box, we may use momentum conservation and the ladder equation $q_j^2L_j=(-\kappa^2-i\Omega_j)L_j$. This simplifies the expression for the Hikami box. As a result the absorption part and the frequency dependent part cancel out. After the cancellations the general expression is

$$H(q_1,q_2,q_3,q_4)=-\frac{l^5}{48\pi k^2}(q_1\cdot q_3+q_2\cdot q_4). \tag{11}$$

This is the same expression as found by Hikami [7] and for instance Stephen and Cwilich [9] in the second order Born approximation. (We were quite surprised by the number of wrong expressions in the literature.) In our approach the definition of the mean free path is different. Correlation functions of multiple scattered waves are calculated elsewhere with this expression for the Hikami box [4].

4. Conclusion

We have presented two new diagrammatic calculations. First we considered the influence of one single scatterer on the diffuse propagation. It is shown that a non-absorbing object acts like a dipole, while an absorbing object acts like a charge. If the extra scattering is weak, agreement is found with results in the literature based on a diffusion approximation. Furthermore, we have presented a generalized expression for the Hikami box. It is seen that loop corrections beyond second order Born approximation result in a different definition of the mean free path and the transport velocity involved.

Acknowledgement

This work was started during a visit of Th.M.N. at MIT, Cambridge, USA. He would like to thank Pattrick Lee and Boris Altshuler for their hospitality and for many useful discussions. The authors are also grateful to Jean-Marc Luck and Peter den Outer for useful discussions. The research of Th.M.N. was made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences (KNAW).

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