

## Universal fluctuations in a simple disordered system

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The mapping  $\mathcal{Z}_i = 1 + x_i \mathcal{Z}_{i-1}$  is studied. The  $x_i$ ,  $i = 1, 2, 3$ , are independent random variables with common distribution. This mapping describes growth under fluctuating conditions, as may occur, e.g., in biology and economics. It also shows up in the grand canonical description of a directed polymer, bound to a wall of a random  $(1+1)$ -dimensional medium, in the limit where the polymer length goes to infinity. It is proven here that there is no self-averaging in this “thermodynamic” limit. Distribution functions which show this behavior explicitly are derived.

Disorder is an interesting subject as the behavior of systems can be strongly influenced by it. The disorder discussed here is called “quenched”. Disorder is quenched if it is related to structures that are static on the timescales that one is interested in. Examples of systems with behavior dominated by quenched disorder are: glasses, diluted antiferromagnets and conductor–insulator mixtures. Propagation of light through diffusive media is also a problem with quenched randomness, as the molecules essentially do not move during one mean scattering time of the light field. Quenched disorder leads to effects like sample-to-sample variations, the metal–insulator transition, the glass and spin glass phases, etc. Also universal conductance fluctuations belong to this class. Here, upon changing a control parameter like the magnetic field, a weakly disordered metal exhibits fluctuations in its resistance, of the order  $e^2/h$ , where  $e$  is the electron charge and  $h$  is Planck’s constant. These fluctuations are perfectly reproducible, but vary from sample to sample. For excellent introductions to disordered systems, see refs. [1–3]. The purpose of this Letter is to show that similar intriguing effects already arise in a simple system, where exact results can be obtained.

Consider a time-dependent growing quantity,  $\mathcal{Z}(t)$ . The growth of  $\mathcal{Z}$  follows from two competing effects. First, it increases due to an input flux, which we take constant. Second,  $\mathcal{Z}(t)$  decays with a fluctuating

rate. In discrete time, this system is described by the equation

$$\mathcal{Z}_i = 1 + x_i \mathcal{Z}_{i-1}, \quad (1)$$

with

$$x_i \equiv \exp(-\mu - \xi_i). \quad (2)$$

The first term in (1) is the steady input flux. The factor  $\mu + \xi$  is the fluctuating decay rate. To obtain a finite result, we fix  $\mu > 0$ . The  $\xi_n$  are the uncorrelated, random variables, with zero average,  $\bar{\xi} = 0$ . (A bar indicates averaging over random variables.) This system has been studied in the mathematical literature, see e.g. refs. [4–6]. The  $\xi_i$  have a Gaussian distribution, viz.  $\overline{\xi_i \xi_j} = \delta_{i,j} \times 2\Delta$ , but one could also study a binary or exponential distribution [7].

The long time limit ( $i \rightarrow \infty$ ) of (1) is given by

$$\mathcal{Z} = 1 + x_1 + x_1 x_2 + x_1 x_2 x_3 + \dots \quad (3)$$

Here we relabeled the  $x$ -variables. This sum converges as long as  $\mu$  is strictly positive. A measure for the number of terms effectively contributing to (3) is the “length”  $L$ ,

$$L = \frac{1x_1 + 2x_1x_2 + 3x_1x_2x_3 + \dots}{1 + x_1 + x_1x_2 + x_1x_2x_3 + \dots} = - \frac{d \ln \mathcal{Z}}{d\mu}. \quad (4)$$

The above equations occur in many systems. The model can describe a general growth equation in economics or biology. The same equation also occurs in

the discussion of directed polymers in disordered media in two dimensions (a directed polymer is a polymer stretched in one direction). Indeed consider the situation where the polymer is bound to a wall and where, in a grand canonical description, its length becomes larger and larger. It was shown in ref. [8] that in this limit the transverse excursions of the polymer become irrelevant and can be taken into account by eq. (1). The limit  $\mu \rightarrow 0$  of eq. (1) then describes the infinite length limit of the polymer. In this application the quantity  $L$  essentially is the thermal average polymer length in the stretched direction and the  $\mathcal{Z}$  is proportional to the grand canonical partition sum [8].

Now the system is studied in detail. First consider the case where the rates do not fluctuate,  $\xi=0$ . The long time limit of (1) is  $\mathcal{Z} = 1/(1-x) = 1/[1 - \exp(-\mu)] \approx 1/\mu$  when  $\mu$  is small. Clearly,  $\mathcal{Z}$  diverges when the decay rate  $\mu$  vanishes. Also  $L = 1/\mu$  diverges. This limit can be compared with the thermodynamic limit of many-particle systems.

One expects the divergence of  $\mathcal{Z}$  to be stronger in the presence of fluctuations. The moments of  $\mathcal{Z}$  can be calculated as follows [8].

$$\overline{\mathcal{Z}^n} = \sum_{k_1} \dots \sum_{k_n} \prod_{i_1}^{k_1} x_{i_1} \dots \prod_{i_n}^{k_n} x_{i_n}. \quad (5)$$

We assume that  $\mu$  and  $A$  are small with respect to unity. Then we can replace the sums by integrals and after performing the averaging over the disorder, we arrive at the result

$$\begin{aligned} \overline{\mathcal{Z}^n} &\approx n! \int_{0 < k_n < \dots < k_1} dk_1 \dots dk_n \\ &\times \prod_{\alpha=1}^n \exp[(-\alpha\mu + \alpha^2 A)(k_\alpha - k_{\alpha+1})] \\ &= \prod_{\alpha=1}^n \frac{1}{\mu - \alpha A} \prod_{\alpha=1}^n \frac{\alpha\mu - \alpha^2 A}{1 - \exp(\alpha\mu + \alpha^2 A)} \\ &= \frac{\Gamma(s-n)}{A^n \Gamma(s)} C(n). \end{aligned} \quad (6)$$

In the first equality we have defined  $k_{n+1} \equiv 0$ ,  $\Gamma$  is Euler's gamma function. The quantity

$$s \equiv \mu/A \quad (7)$$

is proportional to the distance to the critical point,

$\mu=0$ . The factor  $C(n)$  may be calculated using the infinite product representation of  $\sinh(x)/x$ . This leads to

$$\begin{aligned} C(n) &= \exp\left\{\frac{1}{2}n(n+1)\left[\frac{1}{2}\mu - \frac{1}{3}A(n+\frac{1}{2})\right]\right\} \\ &\times \prod_{k=1}^{\infty} \left(\frac{A}{2\pi k}\right)^{2n} \\ &\times \left| \frac{\Gamma(n+1 - \frac{1}{2}s + \omega)\Gamma(n+1 - \frac{1}{2}s - \omega)}{\Gamma(1 - \frac{1}{2}s + \omega)\Gamma(1 - \frac{1}{2}s - \omega)} \right|^2, \end{aligned} \quad (8)$$

where  $\omega = \sqrt{\frac{1}{4}s^2 + 2\pi i k/A}$ . Since  $C(n)$  is close to unity when  $n$  is small, it will be neglected from now on.

In the limit of vanishing disorder,  $A \rightarrow 0$ ,  $s \rightarrow \infty$ , such that  $\mu = sA$  is kept fixed, (6) reduces to  $1/\mu^n$ , in agreement with the solution  $\mathcal{Z} \approx 1/\mu$  discussed above. In the presence of noise, however,  $\mathcal{Z}$  fluctuates. The distribution function  $r(\mathcal{Z})$  which reproduces the moments given in eq. (6) reads

$$r(\mathcal{Z}) d\mathcal{Z} = \frac{\exp(-1/A\mathcal{Z}) d\mathcal{Z}}{(A\mathcal{Z})^s \Gamma(s)} \frac{d\mathcal{Z}}{\mathcal{Z}}. \quad (9)$$

For small  $s$ , eq. (9) has two very different regimes. Indeed, as long as  $\mathcal{Z}$  is not exponentially large in  $1/s$ , (9) may be approximated by  $s \exp(-1/A\mathcal{Z}) d\mathcal{Z}/\mathcal{Z}$ . This represents a broad distribution, the weight of which vanishes linearly when the transition  $s \sim \mu=0$  is approached. On the other hand, when  $\mathcal{Z}$  is very large we may set

$$\zeta = s \ln A\mathcal{Z}. \quad (10)$$

In the thermodynamic limit  $s \rightarrow 0$ , its distribution becomes

$$\exp(-\zeta) d\zeta \quad (11)$$

for  $\zeta > 0$  and vanishes for  $\zeta < 0$ . That this distribution strongly deviates from a narrow distribution is strange. Usually, in a thermodynamic limit where the number of particles goes to infinity, self-averaging occurs. It is not the case in the present situation where the number of elements of the directed polymer goes to infinity. In this limit,  $\mathcal{Z} = \exp(\zeta/s)/A$  diverges with probability one in an exponential way, but the rate  $\zeta$  has a broad (exponential) distribution. In fig. 1 we present the distribution of  $\zeta$  obtained by a numerical simulation and compare it to the theoretical curve. The reason why this distribution is broad is the following: the main contribution to  $\mathcal{Z}$  in (3)

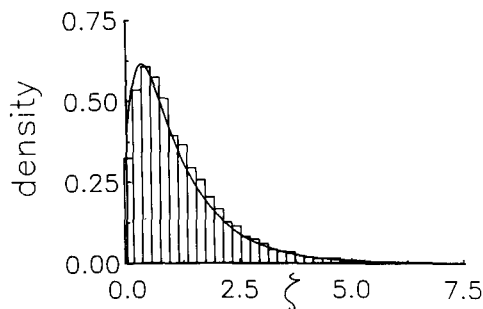


Fig. 1. The simulated density of  $\zeta$  for  $\Delta=1/24$ ,  $\mu=1/100$ . The solid line denotes the theoretical curve given in eq. (9). The distribution is broad, as there is no self-averaging.

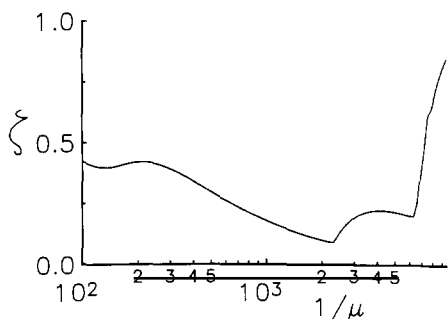


Fig. 2. The variable  $\zeta$ , defined in (10), as function of  $1/\mu$  of one specific sample.  $\Delta=1/24$ . It does not saturate, which implies the lack of self-averaging.

comes from terms with essentially  $L$  factors, where  $L$  is given in (4). When  $\mu$  becomes smaller and smaller,  $L$  diverges and regions of the random sequence  $\{x_i\}$  with larger and larger  $i$  are dominant. Since the sequence of  $x_i$  is random,  $\mathcal{Z}$  will indeed fluctuate in the limit  $\mu \rightarrow 0$ .

The non-self-averaging behavior is shown in fig. 2 for one specific random sequence. It is closely related to the phenomenon of universal conductance fluctuations, where reproducible fluctuations are observed, induced by disorder. In our case we have been able to give the full distribution function of the fluctuations. In fact, the situation is even more cumbersome and interesting. One can also calculate the joint distribution of  $\mathcal{Z}$  and  $L$ . It has a limit for  $s \rightarrow 0$  if  $\mathcal{Z}$  is scaled according to (10) and  $L$  according to

$$L = \frac{\lambda}{s^2 \Delta}. \quad (12)$$

If the second equality in eq. (4) is to be obeyed, it implies that  $\lambda = \zeta$ . In other words, it would follow that,

for given  $\zeta$ , the distribution of  $\lambda$  is narrow and centered around  $\zeta$ . However, in the limit  $s \rightarrow 0$ , the joint distribution of  $\zeta$  and  $\lambda$  is broad in both variables. The relation  $L = -d \ln \mathcal{Z} / d\mu$  becomes meaningless in the limit  $\mu \rightarrow 0$ , just as the function  $x \sin(1/x)$  has no well-defined derivative at  $x=0$ . As a result, the rates  $\zeta$  and  $\lambda$ , governing the divergence of the "polymer partition sum"  $\mathcal{Z}$  and of the "length"  $L$ , fluctuate independently in the limit  $\mu \rightarrow 0$ . Consider the function

$$M(n, t) = \overline{\mathcal{Z}^n \exp(-tL)} \\ = \Delta^{-n} \overline{\exp(n\zeta/s) \exp(-t\lambda/\mu s)}. \quad (13)$$

After one iteration of eq. (1), the values of  $\mathcal{Z}$  and  $L$  become

$$\mathcal{Z}' = x\mathcal{Z}(1 + 1/x\mathcal{Z}), \quad L' = \frac{L+1}{1 + 1/x\mathcal{Z}}. \quad (14)$$

Using this formula we have

$$\overline{\mathcal{Z}'^n \exp(-tL')} \\ \simeq \overline{\mathcal{Z}^n \exp(-tL) x^n (1 - t + n/x\mathcal{Z} + tL/x\mathcal{Z})}.$$

For small  $\mu$ ,  $\Delta$ ,  $n$ ,  $t$  this may be rewritten as

$$(n\mu - n^2\Delta + t)M(n, t) \\ \approx nM(n-1, t) - t\partial_t M(n-1, t), \quad (15)$$

where it has been assumed that the r.h.s. is a smooth function. In solving for  $M(n, t)$  we then find two poles, namely at  $n = \frac{1}{2}s \pm \sqrt{\frac{1}{4}s^2 + t/\Delta}$ . Since  $\mathcal{Z}$  is strictly larger than unity, it follows from definition (13) that there cannot be a singularity at negative  $n$ . Hence the r.h.s. of eq. (15) must have the form  $(n - \frac{1}{2}s - \sqrt{\frac{1}{4}s^2 + t/\Delta})M(n-1, t)$ . Next we observe that whenever  $\zeta$  is small, it is due to the fact that relatively few terms contribute to  $\mathcal{Z}$ . Hence the scaled length  $\lambda$  must be small whenever  $\zeta$  is small. In other words, at large negative  $n$  there should not remain an explicit  $t$ -dependence in (13). With these observations we have fully determined the r.h.s. of (15) and we arrive at the final result

$$M(n, t) = \frac{s}{\frac{1}{2}s - n + \sqrt{t/\Delta + \frac{1}{4}s^2}}. \quad (16)$$

(A more solid derivation of this result would be most welcome.) At  $t=0$  it has the same behavior near the pole as already present in eq. (6). Setting  $n=0$  and expanding in  $t$  one derives

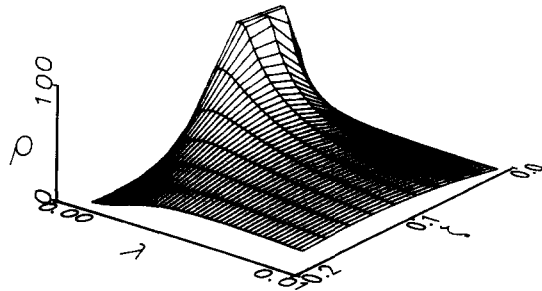


Fig. 3. The joint distribution  $\rho(\zeta, \lambda)$ .

$$\bar{L} = \frac{1}{4s^2}, \quad \overline{L^2} - \bar{L}^2 = \frac{2}{4^2 s^4}.$$

It is seen explicitly that also  $L$  is not self-averaging. The joint distribution of  $\zeta$  and  $\lambda$  is found by taking the inverse Laplace transform of  $M(n, t)$ . The result is

$$\rho(\zeta, \lambda) = \frac{\zeta}{2\sqrt{2\pi\lambda^3}} \times \exp(-\lambda/4 - \zeta/2 - \zeta^2/4\lambda). \quad (17)$$

It is plotted in fig. 3. Integrated over  $\lambda$  it gives back the previous result  $\exp(-\zeta)$ . Integrating over  $\zeta$  one obtains the density of  $\lambda$ ,

$$\rho(\lambda) = \int_0^\infty \rho(\zeta, \lambda) d\zeta = \frac{\exp(-\lambda/4)}{2\pi\sqrt{\lambda}} \int_0^\infty dt \frac{\exp(-t/4)\sqrt{t}}{\lambda+t}. \quad (18)$$

In fig. 4 we present a simulated versus the calculated distribution from (18). It is seen that there is a good agreement. From the above we know that eq. (18) describes the scaled horizontal length in the limit of small  $s$ . It is very remarkable that eq. (18) is identical to the scaled *height* distribution of an infinite polymer in the limit where the polymer undergoes a desorption transition from a wall. In other words, the directed polymer has the same scaling function both in the case where, while bound to a wall, its length goes to infinity and in the case where, having infinite length, it unbinds from a wall.

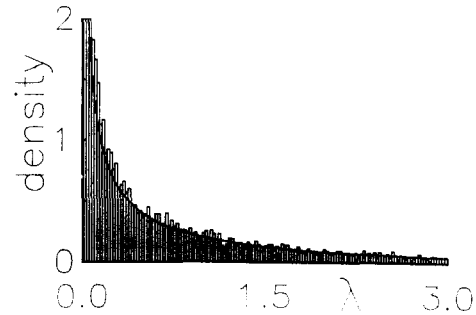


Fig. 4. The simulated distribution of  $\rho(\lambda)$  and the theoretical prediction (solid line).  $A=1/24, \mu=3/1000$ .

In conclusion, we have considered a simple system with disorder: a growth equation with fluctuating rates. It is related to many physical systems. The disorder strongly influences the behavior of the system. Interesting is that sample-to-sample fluctuations occur in the “thermodynamic” limit, very reminiscent to universal conductance fluctuations occurring in disordered metals. The relevant distribution functions have been derived explicitly.

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