Theorem 1. If \( E_{p_S}(\frac{p_T(f(S))}{p_T^*(f(S))}) < \infty \), then a constant \( c_\alpha < \infty \) exists such that 
\[
\int \tilde{p}_{S,\xi}(S,\xi) d\xi dS = 1, \text{ for any fixed } \alpha \in [0, 1].
\]

Proof. If \( \alpha = 1 \), then \( c_\alpha = 1 \). If \( \alpha = 0 \), then,
\[
\int p_S(S) \left( \frac{p_T(f(S))p(\xi)}{p_T^*(f(S))} \right) d\xi dS = \int p_S(S) \frac{p_T(f(S))}{p_T^*(f(S))} \frac{p_T(f(S))p(\xi)}{p_T^*(f(S))} d\xi dS = \int p_S(S) \frac{p_T(f(S))}{p_T^*(f(S))} dS < \infty
\]
Now we look at \( \alpha \in (0, 1) \). Firstly, for \( x > 0 \), if \( \alpha \in (0, 1) \), then \( g(x) = x^{1-\alpha} \) is a concave function, because \( g(x)' = -\alpha(1-\alpha)x^{-\alpha-1} < 0 \). Similarly, \( x^\alpha \) is also a concave function. Then we have
\[
\int p_S(S) \left( \frac{p_T(f(S))p(\xi)}{p_T^*(f(S))} \right)^{1-\alpha} d\xi dS
\]
\[
= \int p_S(S) \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right)^{1-\alpha} E_{p(\xi|f(S))} \left[ \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right)^\alpha \right] dS
\]
\[
\leq \int p_S(S) \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right)^{1-\alpha} \left[ E_{p(\xi|f(S))} \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right)^\alpha \right] dS
\]
\[
= \int p_S(S) \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right)^{1-\alpha} dS
\]
\[
\leq \left[ E_{p_S(f(S))} \left( \frac{p_T(f(S))}{p_T^*(f(S))} \right) \right]^{1-\alpha}
\]
\[
< \infty
\]
where Jensen’s inequality has been applied twice. Therefore, \( c_\alpha < \infty \) exists, satisfying 
\[
\int \tilde{p}_{S,\xi}(\xi, S) d\xi dS = 1. \]
Theorem 2. If \( \lim_{\delta \to 0} p_\delta(\tau) = p^*_\tau(\tau) \), and \( g_\delta(\tau) \) has bounded derivatives in any order, then
\[
\lim_{\delta \to 0} \int p_\delta(\tau|S) g_\delta(\tau) d\tau = g(f(S)).
\]

Proof. Since \( \tau \) is an Uniform distribution on \( [f(S) - \delta, f(S) + \delta] \) conditional on \( S \) and \( \delta \), we could draw \( N \) samples for \( \tau \) such that \( \tau_i = f(S) + (2u_i - 1)\delta \) where \( u_i \) is a sample drawn from the standard Uniform distribution, where \( i = 1, 2, \cdots, N \). By using Monte Carlo approximation and Taylor’s expansion, we have
\[
\lim_{\delta \to 0} \int p_\delta(\tau|S) g_\delta(\tau) d\tau
= \lim_{\delta \to 0} \int \frac{1}{N} \sum_{i=1}^{N} g_\delta(f(S) + (2u_i - 1)\delta) d\tau
= \lim_{\delta \to 0} \lim_{N \to \infty} \left\{ g_\delta(f(S)) + \frac{g'_\delta(f(S))\delta}{2} \frac{1}{N} \sum_{i=1}^{N} (2u_i - 1) + \frac{g''\delta(f(S))\delta^2}{2} \frac{1}{N} \sum_{i=1}^{N} (2u_i - 1)^2 + \cdots \right\}
= \lim_{\delta \to 0} g_\delta(f(S))
= g(f(S)).
\]

This holds, since \( |2u_i - 1|^k \leq 1 \) \((k = 1, 2, 3, \cdots)\) and \( \lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{N} |2u_i - 1|^k \leq 1 \), \( \sum_{i=1}^{N} \frac{1}{N} (2u_i - 1)^k \) converges absolutely when \( N \to \infty \). \( \square \)