The Reachability Problem for 2-dimensional Vector Addition Systems with States

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Reachability in Two-Dimensional Unary Vector Addition Systems with States is NL-Complete

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Abstract
Blondin et al. showed at LICS 2015 that two-dimensional vector addition systems with states have reachability witnesses of length exponential in the number of states and polynomial in the norm of vectors. The resulting guess-and-verify algorithm is optimal (PSPACE), but only if the input vectors are given in binary. We answer positively the main question left open by their work, namely establish that reachability witnesses of pseudo-polynomial length always exist. Hence, when the input vectors are given in unary, the improved guess-and-verify algorithm requires only logarithmic space.

1. Introduction
To quote from Bojańczyk's preface to Schmitz's very recent survey [8], the reachability problem for vector addition systems with states (VASS) is one of the most celebrated decidable problems in theoretical computer science, though, is not only theoretical. The headline result of Blondin et al. is that 2-VASS reachability is PSPACE-complete, but that is provided the input to the problem is succinct, i.e. the integers that specify the action, source and target vectors are given in binary. When the encoding is unary, a considerable complexity gap has remained, between NL hardness and NP membership, and that is what we close.

We believe this is noteworthy at least for the following reasons:

- To make progress on the challenge of the complexity of the general problem, it is natural to fix some parameters, especially the dimension. Bypassing the border between dimensions 2 and 3, which is where there is a jump beyond semi-linearity, seems to be very difficult with current techniques [cf. 1]. For dimension 1, the complexities were determined as NP-complete in the binary case [4] and NL-complete in the unary case [9].

- The unary encoding is frequently enough, e.g. the classical modeling of concurrent systems by VASS [3] produces integers that are proportional to how many processes may interact in a state transition. Also, VASS given in unary can be translated into VASS with many counters in a unary encoding [10].
The Turtle Problem

Fix $S \subseteq \mathbb{Z}^2$. A point $p \in \mathbb{Z}^2$ is visited by $\pi = v_1 v_2 \ldots v_k$ if $p = v_1 + v_2 + \cdots + v_j$ for some $j \leq k$. 
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Let $\Lambda \subseteq S^*$ be a language of the form

$$\Lambda = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \ldots \beta_K^* \alpha_K$$

where $\alpha_i, \beta_i \in S$. 
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1. it visits only points in \( \mathbb{N}^2 \)
2. \( \sum_{i=1}^{k} v_j = 0 \).
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1. it visits only points in $\mathbb{N}^2$
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Theorem

All points $p$ visited by shortest witnesses have $\|s\| \leq (K \cdot \|S\|) O(1)$. 

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Rational Cones

Definition
The cone spanned by $A \subseteq \mathbb{Z}^2$ is the smallest set satisfying

- $\text{Cone}(A) \supseteq A$
- $\text{Cone}(A) = \text{Cone}(A) + \text{Cone}(A)$
- $\text{Cone}(A) = \text{Cone}(A) \cdot \mathbb{Q}_{>0}$. 

Property 1
If a cone is not contained in a half-plane then it contains 0.

Property 2
If 0 $\in \text{Cone}(A)$ then 0 is a nonempty linear combination of at most three vectors from A and with coefficients in \{1, \ldots, 2 \parallel A \parallel_2 \}.

Property 3
If $v \in \text{Cone}(A)$ and $w \in \mathbb{Z}^2$ is such that $\parallel w \parallel \leq \parallel A \parallel$ and $v \cdot w$ is maximal, then $w \in \text{Cone}(A)$. 

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The cone spanned by $A \subseteq \mathbb{Z}^2$ is

$$\text{Cone}(A) \overset{\text{def}}{=} \left\{ \sum_{i=1}^{l} a_i v_i \mid a_i \in \mathbb{Q}_{>0}, \ v_i \in A, \ l > 0 \right\}$$

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Cones and Paths

For \( \pi = \alpha_0 \beta_1^{n_1} \alpha_1 \beta_2^{n_2} \ldots \beta_K^{n_K} \alpha_K \in \Lambda \) write

\[
\text{Cycles}_B(\pi) \overset{\text{def}}{=} \{ \beta_i \mid n_i \geq B \} \subseteq S
\]

for those cycles \( \beta_i \) occurring \( n_i \geq B \) times and let

\[
\text{Cone}_B(\pi) \overset{\text{def}}{=} \text{Cone}(\text{Cycles}_B(\pi)).
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for those cycles $\beta_i$ occurring $n_i \geq B$ times and let

$$Cone_B(\pi) \overset{\text{def}}{=} Cone(\text{Cycles}_B(\pi)).$$

Monotonicity

For any path $\pi \in \Lambda$ and $B \in \mathbb{N}$

$$\sum \pi = c + s$$

where $c \in Cone_B(\pi)$ and $\|p\| \leq |\Lambda| \cdot B \cdot \|S\|.$
A Cut Lemma

Observation
If $s\pi$ visits only points in $\mathbb{N}_{\geq B} \times \mathbb{N}_{\geq B}$ and $\pi'$ is a subword of $\pi$ such that $B \geq (|\pi| - |\pi'|) \cdot \|S\|$ then $s\pi'$ visits only points in $\mathbb{N}^2$. 
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**Cut Lemma**
If \( \rho\sigma\tau \in \Lambda \) is a shortest witness and \( (\sum \rho)\sigma \) visits only points in \( \mathbb{N}^2_{\geq B} \), then \( 0 \notin \text{Cone}_B(\sigma) \).
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Cut Lemma
If $\rho\sigma\tau \in \Lambda$ is a shortest witness and $(\sum \rho)\sigma$ visits only points in $\mathbb{N}^2_{\geq B}$, then $0 \notin \text{Cone}_B(\sigma)$.
This works for any $B \geq 6\|S\|^3$. 
Proving the Turtle Theorem

Theorem
All points $p$ visited by shortest witnesses have $\|p\| \leq (|\Lambda| \cdot \|S\|)^{O(1)}$. 
Proving the Turtle Theorem assuming the Magic Lemma

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Magic Lemma
Shortest witnesses do not visit points outside $\mathbb{N}^2_{\leq c} \cup \mathbb{N}^2_{\geq b}$ for some bounds $b, c \in \mathbb{N}$ polynomial in $|\Lambda|$ and $\|S\|$.
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All points $\mathbf{p}$ visited by shortest witnesses have
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Rough Idea:

- Assume that a minimal witness visits some large point $\mathbf{p}$
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- Assume that a minimal witness visits some large point $p$
- derive a contradiction to the minimality assumption
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Rough Idea:
- Assume that a minimal witness visits some large point $p$
- derive a contradiction to the minimality assumption
- estimate a polynomial bound on $\|p\|$ for this to work.
So?
Vector Addition Systems with States

Definition
A \(d\)-VASS is a finite automaton \((Q, T)\), where \(Q\) is a finite set of control states, and \(T \subseteq Q \times \mathbb{Z}^d \times Q\) is a finite set of transitions.
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A pair $(q, v) \in Q \times \mathbb{N}^d$ is called configuration; The step-relation between configurations is

$$(q, v) \rightarrow (q', v')$$

if there is $(q, a, q') \in T$ s.t. $v' = v + a$. $\rightarrow^*$ denotes the transitive and reflexive closure of $\rightarrow$. 
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Reachability Problem: does $(q, v) \rightarrow^* (q', v')$ hold?
Vector Addition Systems and Control Languages

Definition
A $d$-dimensional VAS is a finite set of vectors $A \subseteq \mathbb{Z}^d$.
For $\mathbf{v}, \mathbf{v}' : \mathbb{N}^d$ it has a step $\mathbf{v} \xrightarrow{a} \mathbf{v}'$ if $\mathbf{v}' = \mathbf{v} + \mathbf{a}$. 
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Let's lift this to words and languages over $A$:

\[
\begin{align*}
\varepsilon \xrightarrow{a} & \overset{\text{def}}{=} \text{Id}_{\mathbb{N}^d} \\
aw \xrightarrow{a} & \overset{\text{def}}{=} w \circ a \\
L \xrightarrow{a} & \overset{\text{def}}{=} \bigcup_{w \in L} w
\end{align*}
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where $\varepsilon$ is the empty word, $a \in A$, $w \in A^*$ and $L \subseteq A^*$. 
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- VASS reachability: $s \xrightarrow{L} t$ for regular $L$
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- Pushdown-VASS reachability: $s \xrightarrow{C} t$ for context-free $C$. 
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- VASS reachability: \( s \xrightarrow{L} t \) for regular \( L \)
- Pushdown-VASS reachability: \( s \xrightarrow{C} t \) for context-free \( C \).
- Turtle Problem: \( 0 \xrightarrow{\Lambda} 0 \) for \( \Lambda = \alpha_0\beta_1^* \ldots \beta_k^*\alpha_k \).
Some equivalent Problems

Formal languages
The nonemptiness problem of a language $\sqcup \sqcap (R_1) \cap R_2$, where $R_1, R_2$ are regular and $\sqcup \sqcap : \Sigma^* \times \Sigma^* \rightarrow 2^{\Sigma^*}$ is the shuffle operator.

Logic
The validity problem for the $!$-Horn Fragment of Linear Logic.

Program Verification
fair-PTL model checking of communicating processes $C \times U^n$. 
The Reachability Problem – Milestones
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1962 · · · Petri: “Kommunikation mit Automaten”.
The Reachability Problem – Milestones

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1976 · · · • Lipton: The reachability problem requires exponential space.
What is the size of the input?

Define the size of a $d$-VASS $A = (Q, T)$ as

$$|A| \equiv d \cdot |Q| \cdot |T| \cdot \|T\|$$

where $\|T\|$ is the maximal absolute value of any integer in $T$. 

▶ The encoding of the input is irrelevant for the complexity of the general VASS Reachability problem.

▶ Not so for the subproblems with dimensions fixed!
What is the size of the input?

Define the size of a $d$-VASS $A = (Q, T)$ as

$$|A|_2 \overset{\text{def}}{=} d \cdot |Q| \cdot |T| \cdot \log_2(\|T\|)$$

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A $d$-VASS can be translated in logspace into an equivalent $(d + n)$-VASS $A'$, where $|A'| = |A'|_2$. 

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This is not true for dimensions $d \geq 3$: 

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(0, 0, 0) ———-(0, 1, -1) ———-(0, 0, 0)
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$\rightarrow$

$d$-VASS can be simulated by $(d + 3)$-VAS.
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(Binary) reachability is $NP$-hard for 2-VASS

- The Hopcroft and Pansiot algorithm works in $2^{2^O(|T| \cdot \|T\|)}$ time.
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Linear Path Schemes and Flat Systems

A language $\Lambda = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \ldots \beta_k^* \alpha_k \subseteq A^*$, where $\alpha_i, \beta_i \in A^*$ is called a linear path scheme. Finite unions of LPSs are called semilinear path schemes.
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Leroux and Sutre ’04: Every 2-VASS is flat.

- acceleration techniques terminate for 2-VASS.
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Polynomial Flattability Lemma

Let \((Q, T)\) be a 2-VASS representing the language \(L \subseteq \mathbb{Z}_2^*\).

There exist finitely many LPSs \(\Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq L\) such that

1. \(L \rightarrow = \bigcup_{i=1}^{k} \Lambda_i \rightarrow\)

2. \(|\Lambda_i| \leq (\|T\| + |Q|)O(1)\) for all \(1 \leq i \leq k\).

▶ Small solutions lemmas for linear programming lead to a \(2^{(|Q| \cdot \|T\| \cdot \|T\|)}O(1)\) bound on the length of shortest witnesses.

▶ PSPACE-completeness for binary encoded 2-VASS.

▶ NP upper bound for unary encoded 2-VASS.

▶ NL lower bound only!
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Let $(Q, T)$ be a 2-VASS representing the language $L \subseteq (\mathbb{Z}^2)^*$. There exist finitely many LPSs $\Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq L$ such that

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Blondin et al. ’15
“Reachability in 2-VASS Is PSPACE-Complete”

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Let \(\Lambda = \alpha_0 \beta^* \alpha_1 \beta^* \cdots \beta^* \alpha_k \subseteq S^*\) be a simple LPS, where \(S \subseteq \mathbb{Z}^2\). If \(0 \Lambda \rightarrow 0\) then there exists \(\pi \in \Lambda\) of length \(|\pi| \leq (k \cdot \|S\|)O(1)\) such that \(0 \pi \rightarrow 0\).

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Polynomial Flattability Lemma (v2)

Let \((Q, T)\) be a 2-VASS representing the language \(L \subseteq (\mathbb{Z}^2)^*\). There exist finitely many simple LPSs \(\Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq L\) such that:

1. \(L \Rightarrow = \bigcup_{i=1}^{k} \Lambda_i \Rightarrow\)
2. \(|\Lambda_i| \leq (\|T\| + |Q|)^O(1)\) for all \(1 \leq i \leq k\).
3. For all \(\pi \in \Lambda_i\) exists \(l \in L\) with \(\pi \Rightarrow \subseteq l \Rightarrow\) and \(|l| \leq |\pi|^{O(1)}\)
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Finally, our Contribution

Polynomial Flattability Lemma (v2)
Let \((Q, T)\) be a 2-VASS representing the language \(L \subseteq (\mathbb{Z}^2)^*\).
There exist finitely many simple LPSs \(\Lambda_1, \Lambda_2, \ldots, \Lambda_k \subseteq L\) such that

1. \(L \rightarrow = \bigcup_{i=1}^{k} \Lambda_i \rightarrow\)
2. \(|\Lambda_i| \leq (\|T\| + |Q|)^{O(1)}\) for all \(1 \leq i \leq k\).
3. For all \(\pi \in \Lambda_i\) exists \(l \in L\) with \(\pi \rightarrow \subseteq \rightarrow l\) and \(|l| \leq |\pi|^{O(1)}\)

Turtle Lemma
Let \(\Lambda = \alpha_0 \beta_1^* \alpha_1 \beta_2^* \ldots \beta_k^* \alpha_k \subseteq S^*\) be a simple LPS, where \(S \subseteq \mathbb{Z}^2\).
If \(0 \overset{\Lambda}{\rightarrow} 0\) then there exists \(\pi \in \Lambda\) of length \(|\pi| \leq (k \cdot \|S\|)^{O(1)}\)
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- \(PSPACE\)-completeness for binary case
- \(NL\)-completeness for unary case.
Conclusion

2-VASS have \((|Q| \cdot |T| \cdot \|T\|)^{O(1)}\) long reachability witnesses.

- The reachability problem is \(NL\)-complete (unary) and \(PSPACE\)-complete (binary).
- Our proof uses effective (polynomial) flattability and small solutions lemmas for linear equations.
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Outlook

- Does this generalise to \((d > 2)\)-dimensional flat VASS?
- regular separability of 1-VASS languages
- 1-dim. pushdown VASS, or 2-dim. branching VASS?
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