[22] Dupin's Cyclide and the Cyclide Patch

John Knapman

IBM UK Scientific Centre
Winchester, SO23 9DR, UK

Reprinted, with permission of Butterworth Scientific Ltd, from *Image and Vision Computing*, 1987, 5, 167-173.

Abstract

To retrieve models from a data base for recognizing objects in stereo, a new formulation of patches of Dupin's Cyclide provides a succinct representation of surface shape.

The parameters can be extracted from the Weingarten Map and its derivatives at a point where a contour meets an extremal boundary.

Introduction

This work is part of a project to design and build a data base of object descriptions[1] for use together with a stereo vision system such as that proposed by Blake and Mayhew[2] in order to recognize objects in a scene.

There is here a requirement for a succinct representation of surface shape in order to keep the data base search reasonably simple and to ensure that shapes that are intuitively similar have similar search arguments.

Considerable effort has gone into exploring various representations for surface shape for the purposes of object recognition and also for geometric reasoning. Ikeuchi[3] advocates use of the extended Gaussian image. Pentland[4] has produced some impressive graphics based on super-quadrics enhanced by the use of fractals. Other representations include Bézier (bicubic) patches, B-splines, planar patches, quadrics, Coons patches and generalized cones.

The requirements are that a representation be economical, expressive, recoverable from real image data and stable under different conditions. Surface patches all suffer from lack of economy when the need arises to describe irregular surfaces precisely. Any formulation of surface patches can be made to fit to an actual surface shape by sufficiently fine sub-division but some require less sub-division than others at the expense of needing more parameters. Planar patches are at one extreme, requiring few parameters but fine subdivision. Cyclide patches - as formulated here - are near the middle of the range, having five numerical parameters plus patch size. Moreover, one of these parameters - the ratio of the principle curvatures at a point of symmetry - seems to capture something of the essence of shape to a remarkable degree.

Recoverability from real image data is perhaps the most stringent requirement for a representation. The extended Gaussian image requires knowledge of the surface normal at every point on the surface, something that even the human visual system is incapable of[5]. A super-quadric under a general translation and rotation requires 15 points to be known on a surface (see ref. 4, footnote 11, p.21). This may well be too many.

Koenderink and van Doorn[6] make a case for representing a surface shape qualitatively in terms of viewpoint catatrophes that appear as the observer moves about the object. For the problem of representing surface shape from a single stereo pair in order to recognize an object, the idea of qualitative representation can be thought of as deciding whether the surface is synclastic, anticlastic or developable (i.e. the Gaussian curvature is positive, negative, or zero, respectively) and distinguishing concave from convex (both synclastic) and recognising cylinders, cones and planes (all developable).

Such a representation would see a torus as composed of a convex outer patch with an anticlastic inner patch. However, it would not distinguish a rugby ball from a soccer ball. It is therefore recoverable and economical but not expressive. Problems may also arise in classifying surfaces that are on the borderline between two types (e.g. almost flat).

For these reasons the cyclide representation is preferred. This uses numerical parameters but can be related very simply to the qualitative description.

The Cyclide

Differential geometry teaches us that the shape of a surface in three dimensions is characterized by its lines of curvature, which form an orthogonal mesh upon it. (The tangents to a line of curvature are principal directions.) Hence at any point there are two orthogonal lines of curvature, one being the line of greatest curvature and the other being the line of least curvature. Working on applications to Computer Aided Design, Martin[7] and Nutbourne examined the class of patches having plane circular arcs as their lines of curvature.

Describing surfaces in terms of such patches is in some ways analogous to describing lines in terms of straight and circular segments as done by Pridmore et al[8].

The general class of surfaces having (planar) circles as their lines of curvature seems useful for object recognition because it provides a reasonable descriptive power but is based on simple geometric primitives. This class was first discovered by Dupin[9] in 1822 - whence the name Dupin's cyclide - and was studied by James Clerk Maxwell[10] (for its applications to optics), Cayley[11] and Darboux[12] in the last century.

The Dupin's cyclide is a surface of the fourth order having as special cases the torus, the cylinder, the cone and the sphere. Another interesting cyclide shape is the *spindle* which I call the *right spindle* when its axis is a straight line. It is the closest approximation to an ellipsoid of revolution when the lines of curvature are constrained to be circular arcs. It resembles a rugby ball or an American foot ball.

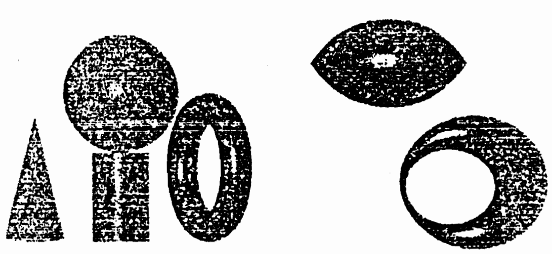

Figure 1. Special cases of the cyclide. Also the right spindle and ring cyclide

It is notable that these are the shapes proposed by Fisher[13] for modelling objects for purposes of recognition. There are obvious advantages in having a common parametrization to them, as furnished by the cyclide formulation. A more general case is the ring cyclide, sometimes called a "squashed torus". Essentially, a ring cyclide is to a torus as a cone is to a cylinder.

Cayley's Construction

Dupin defined the cyclide as the locus of a variable sphere touching three fixed spheres, but Cayley[11] gives a much simpler construction based on centres of symmetry of two circles.

Figure 2 shows the centres of symmetry S and T of two circles outside one another. It can be shown that S and T are at respective distances E_1 and E_2 from the centre of the smaller circle, where

$$E_1 = \frac{R_2 E}{R_1 + R_2} \text{ and } E_2 = \frac{R_2 E}{R_1 - R_2}$$

E being the distance between the centres. Similarly their respective distances from the centre H of the larger circle are

$$E'_1 = \frac{R_1 E}{R_1 + R_2} \text{ and } E'_2 = \frac{R_1 E}{R_1 - R_2}$$

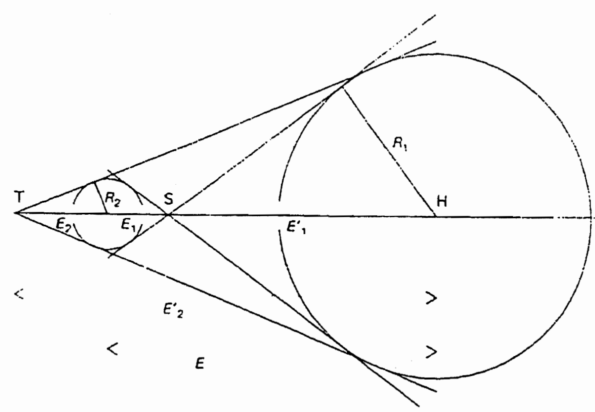

Figure 2. Centres of symmetry at S and T

These formulae apply also when one circle is inside another or when they are touching. Hence we can interpret Cayley's construction paraphrased as follows and illustrated in Figure 3.

> *Consider two circles in a plane. From either one of their centres of symmetry (S or T), draw a line cutting the circles at points A, B on the first circle and P, Q on the second. Now the tangent at A is parallel to the tangent at one of the two points P or Q. Let P be that point. On the line AQ construct a circle in the perpendicular plane having AQ as diameter. Do the same on BP. As the line is rotated about S or T, the locus of these two circles is a cyclide.*
>
> *A different cyclide results if the other centre of symmetry is taken.*

With the help of this construction, it is easy to visualize several cases of the cyclide. Consider the two circles in Figure 3. The centres of symmetry are marked.

Using the centre of symmetry S at distance E'_1 from the larger circle, the surface constructed is a ring cyclide like the one shown in Figure 1. If the circles become concentric the ring cyclide becomes a torus.

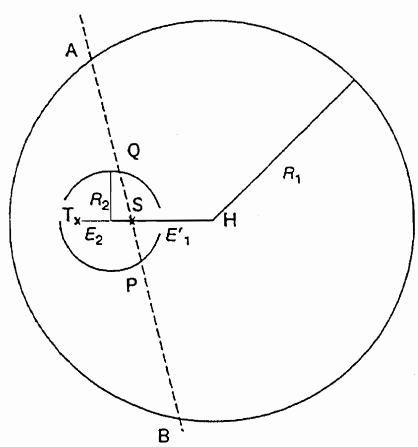

Figure 3. Example of Cayley's construction (ring cyclide)

As the radii of the circles tend to infinity while they remain concentric, the torus tends to a cylinder. If they are not concentric but their radii tend to infinity, the ring cyclide tends to a cone.

If the centre of symmetry at T instead of S is used, a spindle cyclide is constructed. The spindle itself is the central part, so that the small circle is its cross-section with its ends pointing upwards and downwards above and below the paper, as it were.

The locus of centres of the circles standing on AQ and PB is either an ellipse (see Figure 4) or a hyperbola. It can be shown[14] that the eccentricity of this conic is equal, in the respective cases, to

$$\varepsilon = \frac{E}{R_1 \pm R_2}$$

I call this quantity the *eccentricity* of the cyclide. It can also be shown[14] that the eccentricity in the limiting case of the cone is equal to the sine of the half angle, or slope, of the cone.

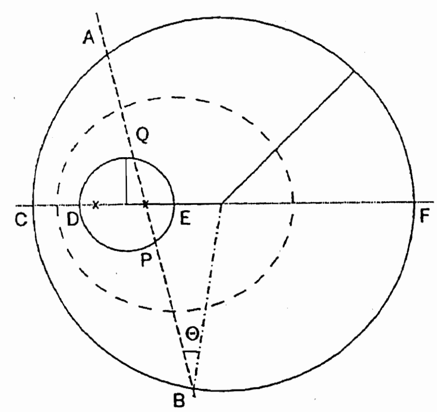

Figure 4. Locus of centres of circular lines of curvature. Also points of symmetry

If we continue the line of centres until it intersects the two circles, we find four points CDEF. These I call the *points of symmetry* of the cyclide.

The Cyclide Patch

Martin[7] and Nutbourne parametrize a cyclide patch in terms of the lines of curvature at a point. These can be continued to generate a complete cyclide if desired.

One of the difficulties in describing shape is that in order to give the complete differential geometry of a surface one needs the principal curvatures at every point. In machine vision, such information is not usually available. Sometimes it is difficult to extract even for one point. If it can be found at a point, it is useful to have some geometrical conventions on how the curvature might then be extrapolated over the surface.

The assumption that the surface has circles as its lines of curvature provides such conventions. Nutbourne and Martin show that once the two lines of curvature are given, the shape of the patch is constrained to within one further parameter. Their interpretation of this parameter is not suitable to our purpose and so we appeal to the underlying geometry of the cyclide for a more suitable formulation of this parameter. The eccentricity seems to be appropriate in this role.

Given a surface normal frame aligned with the principal directions of curvature at a point, a line of curvature is specified with three parameters.

1. Curvature κ of the line (reciprocal of radius)
2. Angle ϕ between the curve normal and the surface normal (so that the principal curvature of the surface in this direction is $\kappa \cos \phi$)
3. The arc length s

We adapt the notation of Forsyth[15,14] and name the two radii of principal curvature at a point on a cyclide R_θ, R_ψ and the two angles between the normals Θ, Ψ. Angle Θ is illustrated in Figure 4 and Ψ is analogous in a perpendicular plane. There are two position parameters θ, ψ related to the angles Θ, Ψ by

$$\tan\Psi = \frac{1}{\eta}\tan\psi \quad \text{and} \quad \tan\Theta = \frac{\varepsilon}{\eta}\sin\theta$$

where $\eta^2 = 1 - \varepsilon^2$. Following the usual parametrization of an ellipse, θ is the angle about the centre of the ellipse in Figure 4 on page 3. It is characteristic of the cyclide that Θ and R_θ are independent of ψ and that Ψ and R_ψ are independent of θ.

I parametrize the cyclide in terms of the ratio ρ of the principal curvatures at a point of symmetry, one of the principal curvatures there and the eccentricity ε. The ratio is defined as $\rho = \kappa_1/\kappa_2$ where κ_1 and κ_2 are the principal curvatures with $\kappa_1 < \kappa_2$ and therefore $\rho \leq 1$. It has the same sign as the Gaussian curvature with $\rho = 0$ indicating a developable surface (e.g. a cylinder or cone) while $\rho = 1$ indicates a spherical surface.

The second parameter is taken to be $\sigma = \kappa_2$, also at a point of symmetry. Then $\sigma = 1/r$ for a sphere or cylinder of radius r.

Taking the point of symmetry to be where $\theta = \pi$ and $\psi = \pi$ we can write the radii of curvature at an arbitrary point as[14]

$$R_\theta = \frac{1}{\sigma\rho(1-\varepsilon)}\left[\rho - \varepsilon - \varepsilon(1-\rho)\cos\theta\right] \qquad (1\theta)$$

and

$$R_\psi = \frac{1}{\sigma\rho(1-\varepsilon)}\left[\rho - \varepsilon - (1-\rho)\sec\psi\right] \qquad (1\psi)$$

Thus if the curvatures, angles and eccentricity can be found at some point in a scene, the parameters ρ, σ can readily be found. In particular, if we know the scale independent ratio $\mathrm{P} = R_\theta/R_\psi$ we can find ρ independently of σ. In fact, from equation (1),

$$\rho = \frac{\mathrm{P}(\varepsilon + \sec\psi) - \varepsilon(1+\cos\theta)}{\mathrm{P}(1+\sec\psi) - (1+\varepsilon\cos\theta)}$$

The next sections explain how to find at some points the curvatures, the angles Θ, Ψ and the eccentricity from stereo data so that the values of ρ and σ can be determined as described in this section.

Curvatures from a Profile and a Contour

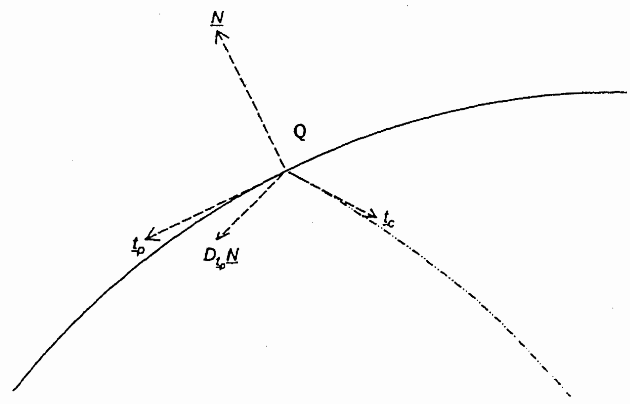

Figure 5. The Weingarten Map

A profile is the three dimensional curve at an extremal boundary, the line at which the surface normal is perpendicular to the line of sight. A contour is a line on the surface; it may be an intersection with another surface.

Proceeding purely from the local differential geometry of a point on a regular[16] surface in real Euclidean 3-space, it is possible to determine the principal curvatures and principal directions at a point where the profile meets a contour. This is because the surface normal (the *Gauss Map*) is known along a profile and can therefore be differentiated along it, yielding a vector that is tangent to the surface and therefore has two independent components. This is not generally true of contours.

The derivative of the Gauss Map is the *Weingarten Map*[17] (Figure 5) written $D_{t_p}\underline{N}$. This means the derivative of the surface normal $\underline{N}$ in the direction of the tangent vector $\underline{t}_p$ along the profile. $D_{t_p}\underline{N}$ is a tangent vector that differs from $\underline{t}_p$ unless $\underline{t}_p$ happens to be a principal direction. The Weingarten Map maps tangent vectors into tangent vectors. The eigenvectors of the Weingarten Map are the principal directions and its eigenvalues are the principal curvatures.

At a profile, the surface normal can be determined from a monocular image or a stereo pair, since it must be othogonal to the tangent of the curve and to the line of sight. In fact if $\underline{l}$ is the line of sight then $\underline{N}$ is given by the vector product.

$$\underline{N} = \frac{\underline{t}_p \times \underline{l}}{|\underline{t}_p \times \underline{l}|}$$

Note that the same analysis applies to a line of shadow on a surface if the position of the light source is known. We would simply replace $\underline{l}$ by a vector representing a ray of light.

The vision system[8] delivers a usable geometric description of a profile from a stereo pair, although strictly speaking the correspondences made are inexact. Since the surface normal is available along the profile, it is possible to calculate its derivative from a stereo pair as well. Differentiating the surface normal in the direction of $\underline{t}_p$ yields a vector

$$D_{\underline{t}_p}\underline{N} = -\underline{t}_1\kappa_1\cos\gamma - \underline{t}_2\kappa_2\sin\gamma \tag{2}$$

where $\underline{t}_1$ and $\underline{t}_2$ are principal directions, κ_1 and κ_2 are the principal curvatures and γ is the angle between $\underline{t}_p$ and $\underline{t}_1$ [16]. Hence, writing $\underline{t}_p = \underline{t}_1\cos\gamma + \underline{t}_2\sin\gamma$, the (normal) curvature of the surface along the profile is

$$\kappa_p = -\underline{t}_p.D_{\underline{t}_p}\underline{N} = \kappa_1\cos^2\gamma + \kappa_2\sin^2\gamma$$

which is Euler's relation. By setting $H = (\kappa_1 + \kappa_2)/2$ and $S = (\kappa_1 - \kappa_2)/2$ it can more conveniently be written

$$\kappa_p = H + S\cos 2\gamma \tag{3}$$

The vector $D_{\underline{t}_p}\underline{N}$ also has an orthogonal component which we will call τ.

$$\tau = -\underline{t}_o.D_{\underline{t}_p}\underline{N} = \kappa_1\cos\gamma\sin\gamma - \kappa_2\sin\gamma\cos\gamma$$

where $\underline{t}_o = \underline{t}_1\sin\gamma - \underline{t}_2\cos\gamma$. Therefore

$$\tau = S\sin 2\gamma \tag{4}$$

We have two equations, (3) and (4), in the three unknowns H, S and γ. Summing the squares to eliminate γ we obtain

$$(\kappa_p - H)^2 + \tau^2 = S^2 \tag{5}$$

If a contour - such as a surface intersection - meets the profile at a point Q, we can discover all three quantities at such a point. Let the tangent to the curve at this point be $\underline{t}_c$.

If α is the angle between $\underline{t}_c$ and the principal direction ($\cos\alpha = \underline{t}_c.\underline{t}_1$), the surface (normal) curvature in the direction of $\underline{t}_c$ is given by Euler's relation

$$\kappa_c = \kappa_1\cos^2\alpha + \kappa_2\sin^2\alpha \tag{6}$$

Here $\alpha = \gamma + \beta$ where β is the known angle between the profile and the intersection ($\cos\beta = \underline{t}_c.\underline{t}_p$) and γ is as in equations (3) and (4).

The curvature of the contour at Q is given by Meusnier's theorem as

$$\kappa = \frac{\kappa_c}{\cos\phi} \tag{7}$$

where $\cos\phi = \underline{n}.\underline{N}$ with $\underline{n}$ being the normal to the contour curve. The curvature κ_c is thus readily available by a linear equation (7). Then, rewriting equation (6) to be consistent with equations (3), (4) and (5),

$$\begin{aligned}\kappa_c &= H + S\cos(2\gamma + 2\beta)\\ &= H + S(\cos 2\beta\cos 2\gamma - \sin 2\beta\sin 2\gamma)\end{aligned} \tag{8}$$

Therefore, after substituting $\cos 2\gamma$ and $\sin 2\gamma$ from equations (3) and (4),

$$\kappa_c = H + (\kappa_p - H)\cos 2\beta - \tau\sin 2\beta$$

This is a linear equation in H. Equation (5) then gives a linear expression for S^2 and equation (4) determines γ.

A Surface Frame

We now have the principal curvatures at Q and a surface frame $[\underline{t}_1, \underline{t}_2, \underline{N}]$ there since

$$\underline{t}_1 = \underline{t}_p\cos\gamma + \underline{t}_o\sin\gamma \text{ and } \underline{t}_2 = \underline{t}_p\sin\gamma - \underline{t}_o\cos\gamma$$

Rate of Change of Curvature

Progress has been made[18] in estimating not only the curvature but also the rate of change of curvature from a stereo pair of images. It is therefore realistic to contemplate using these quantities to determine the cyclide patch parameters Θ and Ψ. To interpret the derivatives in this way assumes that the surface is a cyclide patch, whereas the derivation of the curvatures in the previous section makes no such assumption.

Differentiating all three quantities κ_p, τ and κ_c yields three equations in two unknowns, leaving some redundancy that might be used to estimate how close the patch is to being part of a cyclide.

Let v be the displacement in the direction of the tangent $\underline{t}_c$ to the contour at Q. We wish to find an expression for $\frac{d\kappa}{dv}$, the rate of change of curvature of the line, since this quantity should be obtainable from stereo images by discovering curvatures along the curve near Q.

Write $C = \cos\phi = \underline{n}.\underline{N}$. Then, differentiating equation (7),

$$\frac{d\kappa}{dv}\cos\phi = \frac{d\kappa_c}{dv} - \frac{dC}{dv}\kappa \tag{9}$$

Here

$$\frac{d\kappa_c}{dv} = 3(\kappa_1 - \kappa_2)\cos\alpha\sin\alpha\begin{bmatrix}\kappa_1\cos\alpha\tan\Psi\\ -\kappa_2\sin\alpha\tan\Theta\end{bmatrix} \tag{10}$$

and

$$\frac{dC}{dv} = \underline{N}.D_{\underline{t}_c}\underline{n} + \underline{n}.D_{\underline{t}_c}\underline{N} \tag{11}$$

where

$$D_{\underline{t}_c}\underline{N} = -\underline{t}_1\kappa_1\cos\alpha - \underline{t}_2\kappa_2\sin\alpha$$

similarly to equation (2) and the derivative of $\underline{n}$ must be obtained from the stereo image data.

The derivation of equation (10) is given elsewhere[14], as are those of equations (12) and (13) below, which are obtained by differentiating equation (2).

$$- \underline{t}_p . D_{\underline{t}_p}(D_{\underline{t}_p}\underline{N}) = 3(\kappa_1 - \kappa_2)\cos\gamma\sin\gamma\begin{bmatrix}\kappa_1\cos\gamma\tan\Psi \\ -\kappa_2\sin\gamma\tan\Theta\end{bmatrix} \quad (12)$$

and

$$- \underline{t}_o . D_{\underline{t}_p}(D_{\underline{t}_p}\underline{N}) = (\kappa_1 - \kappa_2)\begin{bmatrix}\kappa_1\cos\gamma(1 - 3\sin^2\gamma)\tan\Psi \\ -\kappa_2\sin\gamma(1 - 3\cos^2\gamma)\tan\Theta\end{bmatrix} \quad (13)$$

These are readily solved for $\tan\Theta$ and $\tan\Psi$.

We note that equation (12) represents the rate of change, in the direction $\underline{t}_p$, of the surface (normal) curvature in the direction $\underline{t}_p$. On the other hand, equation (13) represents the rate of change, in the direction $\underline{t}_o$, of the surface (normal) curvature in the direction $\underline{t}_p$.

The second derivative of $\underline{N}$ also has a component normal to the surface. This yields no new information but might be useful as a check on the accuracy of the differentiation. In fact,

$$- \underline{N}.D_{\underline{t}_p}(D_{\underline{t}_p}\underline{N}) = \kappa_1^2\cos^2\gamma + \kappa_2^2\sin^2\gamma \quad (14)$$

Note that this relationship is not unique to cyclides but follows directly from twice differentiating the equation $\underline{N}.\underline{N} = 1$ and substituting equation (2).

Eccentricity

This is a higher order quantity that can be obtained geometrically from considering two points P and Q at both of which a profile meets an intersection. In the most notable special case, the cone, geometrical methods are obviously applicable, as the eccentricity is the sine of the half angle (see "Cayley's Construction").

More generally, a local method is more satisfactory, not least because it affords, in principle, an exact determination at the point in question. This involves a further differentiation along a profile or an intersection and a somewhat lengthy calculation, so there must be some reservations about the accuracy of numerical methods from stereo images for this purpose.

The details appear in a fuller paper[14] where it is shown that differentiating equation (12) (or equation (10) along a contour) results in an equation of the form

$$P = Q\begin{bmatrix}\kappa_1\cos\gamma\dfrac{d}{du}(\tan\Psi) \\ -\kappa_2\sin\gamma\dfrac{d}{du}(\tan\Theta)\end{bmatrix} \quad (15)$$

This leads to a quadratic in ε^2 as follows.

$$\begin{bmatrix}(1-\varepsilon^2)P \\ -Q\kappa_1\kappa_2\cos\gamma\sin\gamma\begin{pmatrix}\sec^2\Psi + \tan^2\Theta \\ -\varepsilon^2(\sec^2\Theta - \tan^2\Psi)\end{pmatrix}\end{bmatrix}^2 = 4(\varepsilon^2\sec^2\Theta - \tan^2\Theta)(\sec^2\Psi - \varepsilon^2\tan^2\Psi) \quad (16)$$

The underlying cyclide may be derived by the methods of "The Cyclide Patch".

A Test

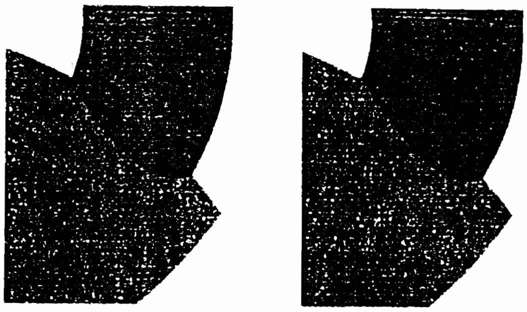

Figure 6. Right and left eye views of an intersection between a torus and a spindle

To verify the above principles, a test was conducted using the stereo vision system produced at the University of Sheffield[8] on stereo images prepared using Winsom[19], the IBM Winchester Solid Modeller. The test data is shown in Figure 6 where the left image is on the right and vice versa so that some readers will be able to obtain a stereoscopic impression by crossing the eyes. The scene is the intersection of a torus with a spindle.

Although internally the Sheffield system calculates curvature and rate of change of curvature at each point of every edge, this information was not used directly in this test but only indirectly, in that the vision system itself uses it to classify edges as straight, circular, planar or space curves. These classifications and the associated geometrical descriptions were used here to infer the parameters at the point at which the contour meets the profile.

The radius of the profile was found by the vision system to be 289 (in pixel units). Thus $\kappa_p = 1/289$ and $\tau = 0$. Note that the system is not able to distinguish profiles from other edges and this labelling was done manually. The radius of the circular approximation to the intersection curve was 53. The angle ϕ between the normal $\underline{n}$ to this curve and the surface normal $\underline{N}$ was 34°. The angle β between the contour and the profile was 97°. Substituting these values into equations (7) and (8) gives a value for H of 0.009648. Then from equation (5) we obtain S and thus κ_1 and κ_2.

The ratio κ_1/κ_2 was 0.22 as against the true ratio of the small and large radii of the torus which was 0.25. This discrepancy can be explained by the circular approximation made to the curve of intersection and by the vagaries of the stereo process itself, sensitive as it is to pixellation errors and the like, particularly when converting from disparity to depth. It is also true that the

Sheffield system takes no special account of the error in depth estimation at the profile caused by the fact that correspondence is made between slightly differing points on the surface in the two images, since the right eye sees a bit more of the surface, so to speak, than the left eye.

Equation (4) implies that $\gamma = 0$. Equations (12) and (13) both vanish, implying that both Θ and Ψ are zero. Also, in equation (16), $P = 0$ and $Q = 0$ implying that $\varepsilon = 0$.

Describing Surfaces

Once the geometry of a surface at some points is determined, a description of the surface as a whole is needed. Such a description has two purposes: recognition and geometric reasoning. As stated in the introduction, recognition in a data base of stored models is the primary focus of this work.

For this purpose, a stable description of the overall shape between edges is needed. If information is only available from one point, the surface may be assumed to be an extrapolation from there. Recognition, however, needs to proceed from a consistent point, not from the point that happens to be known. Appeal to the underlying geometry suggests the use of a point of symmetry of the cyclide. To be exact, we take a point at which the parameters Θ, Ψ and hence θ, ψ take the value π as stated in "The Cyclide Patch".

The two principal curvatures at that point then become the primary candidates to be arguments in searching a data base of object descriptions. The eccentricity can also be a search argument when available.

The ratio of the principal curvatures is probably more useful for recognition than the principal curvatures themselves. The ratio is all that is required to characterize the appearance of the profile.

Its utility is most apparent where only a plane intersection with an unknown surface is available with no other information about the surface. In such circumstances, it is possible to interpret the curve as an approximation to the Dupin indicatrix at a point P in the middle if the curve is close to a conic section. With suitable choice of co-ordinate direction, the equation of this curve is

$$\kappa_1 x^2 + \kappa_2 y^2 = 2\delta$$

where δ is a small (unknown) perpendicular displacement between the tangent plane at P and the plane of intersection, which are assumed parallel in this approximation. Hence the ratio ρ can be estimated even though κ_1 and κ_2 cannot.

If ρ is interpreted as the first parameter of a patch, the second parameter is most naturally taken (also at a point of symmetry) to be $\sigma = \kappa_2$. Thus a flat surface is indicated by $\sigma = 0$. In this case ρ is indeterminate.

The Gaussian curvature $K = \kappa_1\kappa_2$ does not seem to be useful in machine vision, a point noted also by Brady et al[20]. The mean curvature $H = (\kappa_1 + \kappa_2)/2$ does not appear suitable either, mainly because of its ambiguity when κ_1 and κ_2 have opposite signs.

More Information

When several points are known, a method for approximating a larger patch to fit a number of smaller patches is needed. The method chosen depends partly on how much is known. It may be that the principal curvatures at many points are available as would be the case using the "weak plate" method of surface interpolation described by Blake and Zisserman[21].

If, however, the differential geometry at just a few points is known, a possible approach would be to try to fit a large patch as close as possible to the smaller ones. A geometrical method would be to consider every patch's centre of symmetry and find the mean of their position, orientation and parameter values ρ and σ if some threshold is not breached. Some account would have to be taken of the reliability of the measurements of the patch parameters, based on the reliability of the data itself.

Further study of these possibilities is needed.

Conclusions

In stereo vision using normal lighting it is often impracticable to obtain a complete description of a surface in a scene. Rather, information about its curvature at some points can be gleaned from various sources, such as shading, specularities, extremal boundaries, and intersections.

Patches of surfaces from Dupin's cyclides have a parametrization that permits representation of surface shape with varying specificity. Given the principal curvatures at a point, the remaining three parameters of a patch, Θ , Ψ and ε, determine how the lines of curvature can be extended as circular arcs into the neighbourbood of the point.

The eccentricity ε relies on a higher order derivative than the other quantities and, in general, is likely to be available less often than the other parameters. The default assumption $\varepsilon = 0$ can conveniently be made when the data is unavailable, meaning that the surface patch is interpreted as part of a surface of revolution.

Once the parameters at certain points have been determined, they provide a basis for extrapolating the surface shape nearby. For purposes of geometric reasoning, methods for fitting and blending Cyclide Patches developed by de Pont[22] can be used.

For recognition, an overall shape description is needed, and this is perhaps best stated in terms of the scale independent ratio ρ of principal curvatures and the factor σ. These need to be given at a consistent point irrespective of viewpoint and a point of symmetry of the cyclide appears a particularly suitable place, the analysis then being at its simplest.

Among the various possible sources of information about shape, we have examined the surface normal and its derivatives along a profile (extremal boundary) and shown that, where a contour meets the profile, all five parameters can be determined locally. This has been tested on a pair of synthetic stereo images. Note that the use of the Weingarten Map along a profile to determine the local differential geometry is independent of the use of a cyclide representation. The assumption only affects the interpretation of the derivatives of this map.

Since the cyclide patch parameters are based on fundamental and well known quantities in differential geometry, other sources of estimation such as shading and specularity will also yield these parameters. Blake[23], for instance, derives equations for determining (under certain conditions) the Hessian of the height function of a surface in the neighbourhood of a point by using specular stereo. The principal curvatures and directions are readily obtainable from this Hessian[16].

Acknowledgements

I have had valuable discussions relevant to this topic with Tony Nutbourne, John Mayhew, Andrew Blake, John Porrill, Bob Fisher, John Woodwark and Alan Halbert. I am grateful to Steve Rake and Mike Gray for comments on an earlier version of this paper and to Simon Gee for checking the calculations.

References

1 **Knapman, J M** 'Vision Data Base: Requirements and Initial High Level Design' IBM UK Scientific Centre, Winchester, UK (April 1986)

2 **Blake, A and Mayhew, J E W** 'Alvey 2½D sketch project: Proposed structure for a development system' Department of Computer Science, University of Edinburgh, UK and A I Vision Research Unit, University of Sheffield, UK (November 1985)

3 **Ikeuchi, K** 'Recognition of 3-d objects using the extended Gaussian image' *Proc 7th IJCAI* Vancouver, B.C., Canada (1981) p 595

4 **Pentland, A P** 'Perceptual organization and the representation of natural form' Artificial Intelligence Center, SRI International, Menlo Park, CA, USA (1985)

5 **Langdon, P** Personal communication, A I Vision Research Unit, University of Sheffield, UK (1986)

6 **Koenderinck J J and van Doorn, A J** 'How the ambulant observer can construct a model of the environment from the geometrical structure of the visual inflow' in *Kybernetik* ed by Hausske, G and Butenandt, E, Oldenburg, Munich, W. Germany (1977)

7 **Martin, R R** 'Principle patches for computational geometry' Ph.D. thesis, Department of Engineering, University of Cambridge, UK (1982)

8 **Pridmore, T P, Bowen, J B and Mayhew, J E W** 'Geometrical description of the CONNECT Graph #2. The Geometrical Descriptor Base: A specification' A I Vision Research Unit ref no 012, University of Sheffield, UK

9 **Dupin, C.** 'Applications de géometrie et de méchanique' Bachelier, Paris, France (1822)

10 **Maxwell, J C** 1868 'On the cyclide' *Quarterly Journal of Pure and Applied Mathematics* vol IX (1868) p 111

11 **Cayley, A** 1873 'On the cyclide' *Quarterly Journal of Pure and Applied Mathematics* vol XII (1873) p 148

12 **Darboux, G** *La théorie générale des surfaces* Gautier - Villars, France (1887)

13 **Fisher, R** 'A proposal for a suggestive modeling system for object recognition' Working Paper, Department of Artificial Intelligence, University of Edinburgh, UK (November 1985)

14 **Knapman, J M** 'Dupin's Cyclide and the Cyclide Patch' UKSC Report number 156, IBM UK Scientific Centre, Winchester, UK (1986)

15 **Forsyth, A R** *Lectures on differential geometry of curves and surfaces*, Cambridge University Press, UK (1912)

16 **do Carmo, M P** *Differential Geometry of Curves and Surfaces* Prentice-Hall, Englewood Cliffs, NJ, USA (1976) pp 52,145,164

17 **Hicks, N J** *Notes on differential geometry*, van Nostrand, Princeton, NJ, USA (1965)

18 **Pridmore, T** Personal communication, A I Vision Research Unit, University of Sheffield, UK (1986)

19 **Quarendon, P** 'Winsom User's Guide' IBM U K Scientific Centre report no 123, St Clement St, Winchester, UK (August, 1984)

20 **Brady, M, Ponce, J, Yuille, A and Asada, H** Describing Surfaces' A I Memo 822, Massachusetts Institute of Technology, Artificial Intelligence Laboratory, Cambridge, Mass., USA (1985)

21 **Blake, A and Zisserman, A** 'Weak continuity constraints in computer vision' Internal Report CSR-197-86, Department of Computer Science, University of Edinburgh, UK (1986)

22 de Pont, J J 'Essays on the cyclide patch Ph.D. thesis, Department of Engineering, University of Cambridge, UK (1984)

23 Blake, A 'Specular Stereo' *Proc 9th IJCAI* Los Angeles, CA, USA (1985) p 973