[5] The Role of Disparity Gradient in Stereo Vision

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Abstract

Burt and Julesz experimentally demonstrated that, in addition to Panum's fusional area, a quantity defined by them and named disparity gradient also plays a crucial part in deciding whether the human visual system would be able to fuse the images seen by the left and right eyes. The physical meaning of this quantity remains obscure despite attempts to interpret it in terms of depth gradient. Nevertheless, it has been found to be an effective selector of matches in stereo correspondence algorithms. A proof is provided that a disparity gradient limit of less than 2 implies that the matches between the two images preserve the topology of the images. The result, which is invariant under rotations and under relative as well as overall magnifications, holds for pairs of points separated in *any* direction, not just along epipolar lines. This in turn can be shown to prevent correspondences being established between points which would have to be located in three dimensions on a surface invisible to one eye, assuming opaque surfaces.

1 Introduction

Binocular stereo vision entails reconstructing depth information from two images of a three-dimensional (3-D) scene taken from slightly different viewpoints. This involves establishing correspondences between points (solving the 'correspondence problem') and computing the depth by triangulation. In computer vision, provided that the camera geometry is known, this last step is trivial, and so solving the correspondence problem guarantees stereopsis.

In the case of human vision, there is also the concept of binocular fusion, which is when a stereoscopically presented image appears single, and this is not the same as stereopsis. It is well known that stereopsis can occur with fusion (eg depth perception despite diplopia), and fusion without stereopsis (eg in amblyopes). For computer vision, of course, there is no concept of fusion. Burt and Julesz (1980a, 1980b) conducted some interesting experiments to investigate the effect of nearby points on binocular fusion. They defined the disparity gradient (figure 1) between two nearby points as the difference in their disparities divided by their separation in visual angle, and demonstrated that fusion of at least one point fails when this gradient exceeds a critical value (approximately 1). Although their experiments were necessarily concerned with horizontal disparities only (since the arrangement of human eyes means that vertical disparities are usually very small), the definition of disparity gradient is

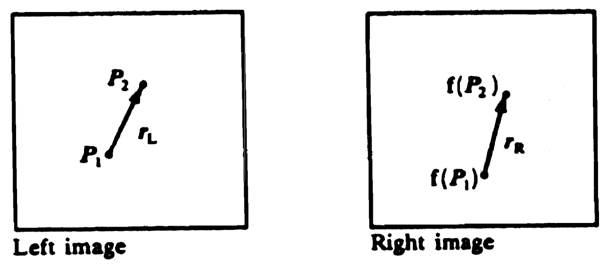

Figure 1. Vector r_L in the left image and the corresponding vector r_R in the right image. The disparity gradient is defined as $|r_L - r_R| / \frac{1}{2}|r_L + r_R|$.

not restricted to points with only horizontal disparities. There will be situations in computer vision (eg large vergence and/or gaze angles) where points will have both horizontal and vertical disparities, so it is useful to keep the concept of disparity general and not restrict the considerations to horizontal disparities only.

Despite the fact that this limit applies to fusion, which is meaningless in a computer vision context, the disparity gradient limit has provided a powerful disambiguator of correspondences in several stereo algorithms (Lloyd 1984; Pollard et al 1985). The success of these algorithms is perhaps surprising: why should a constraint on human fusion be a good constraint for computer stereopsis? Equally, why should the disparity gradient limit be a good constraint in human vision? There are certainly some empirical reasons: most false matches will cause the disparity gradient limit to be exceeded and so will be rejected by the algorithm, whereas most pairs of correct matches will satisfy the disparity gradient limit (Pollard et al 1985). Even on surfaces containing apirs of points exceeding the disparity gradient limited (consider, for example, a plane inclined away from the viewer so that it is close to horizontal), there will still be many pairs of points with disparity gradient less than the limit, and so the algorithm may still be able to solve the correspondence problem correctly. While not disputing in any way the success of these algorithms, we were, however, still unconvinced of the reasons for this success.

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We therefore propose an explanation of the disparity gradient limit based on the physics of image formation. This suggests a reason why it is sensible, both for human fusion and for computer stereopsis, to impose a disparity gradient limit. We shall show that imposing a disparity gradient limit of less than 2 ensures that no matches will be made between points that would have to be located in space on a surface invisible to one eye (assuming opaque surfaces). Observe that the converse (that all pairs of points on a non-self-occluding opaque surface satisfy the disparity gradient limit) does not follow; the example mentioned earlier of a plane inclined away from the viewer would be an obvious counter example. Imposing a disparity gradient limit is a conservative policy; it prevents false matches, but also disallows some correct ones. Burt and Julesz suggest a figure of 1, although some of their experimental data might suggest a higher figure. This is still safely below 2 and in perfect accord with our theory. It is interesting to note that, when restricted to pairs of points lying on the same epipolar line, a disparity gradient limit of less than 2 reduces to the familiar 'ordering constraint' used in many stereo algorithms, which demands that points on an epipolar line lie in the same order in both images.

We shall, in fact, show that imposing a disparity gradient limit of less than 2 ensures that the correspondence between the two images preserves their topology. In other words:

(a) each point in the left image corresponds to a unique point in the right image and vice versa;

(b) if we were given one image painted on a rubber sheet, we could, without tearing the shet or glueing it to itself, deform it so that we obtained the other image.

This may seem very obscure, but if we consider how the two images arise, things may become clearer. Let us suppose that we have a continuous non-self-occluding surface as in figure 2. Now, for each eye, the projection which transforms the surface into the image preserves the topology, and so each image must have the same topology as the surface does. Hence they must have the same topology as each other. Conversely, if we have a self-occluding surface as in figure 3, the appropriate matches will not preserve the topology. This can be seen quite easily by considering the point E, where the ray from D intersects the surface again. In the left eye the image of this point will appear in the same place as the image of D, that is D_L, whereas the two will be distinct in the right image. So D_L in the left image corresponds to two points (D_R and E_R) in the right image, and this implies that the topology cannot be preserved since condition (a) above would be violated.

2 Proof

Suppose that we have two images $\mathbf{I}_L$ and $\mathbf{I}_R$ and a correspondence f between them. So, for each point P in $\mathbf{I}_L$, we know the point or points in $\mathbf{I}_R$ corresponding to P. Suppose further that f obeys the disparity gradient limit of $2k$ (where $k< 1$), that is, if P_1 and P_2 are points in $\mathbf{I}_L$, then

$$\Gamma_D = \frac{\left|(P_1 - P_2) - \left[f(P_1) - f(P_2)\right]\right|}{\frac{1}{2}\left|(P_1 - P_2) + \left[f(P_1) - f(P_2)\right]\right|} \leq 2k < 2 .$$

In order to prove that the topology is preserved, we need to show that

(i) f is one-to-one (ie each point in $\mathbf{I}_R$ corresponds to a unique point in $\mathbf{I}_L$);
(ii) f^{-1} is one-to-one (ie each point in $\mathbf{I}_L$ corresponds to a unique point in $\mathbf{I}_R$);
(iii) f is continuous (roughly, nearby points in $\mathbf{I}_R$ correspond to nearby points in $\mathbf{I}_L$);
(iv) f^{-1} is continuous (roughly, nearby points in $\mathbf{I}_L$ correspond to nearby points in $\mathbf{I}_R$).

Since the disparity gradient limit is symmetric, we need only prove (i) and (iii), and then (ii) and (iv) will follow by symmetry.

In order to prove (i), suppose that a single point P_R in $\mathbf{I}_R$ corresponds to two points in P_{1L} and P_{2L} in $\mathbf{I}_L$. Consider the disparity gradient, Γ_D, between this pair of matches:

$$\Gamma_D = \frac{2|(P_{1L} - P_{2L}) - 0|}{|(P_{1L} - P_{2L}) + 0|} = 2 .$$

But this is not allowed because we are supposing that all the disparity gradients are less than 2. So P_R must correspond to a unique point P_L in $\mathbf{I}_L$: that is, f is one-to-one.

We now need to prove that f is continuous. Suppose that P_1 and P_2 are two distinct points in $\mathbf{I}_L$ and let us write $\mathbf{r}_L = P_1 - P_2$ and $\mathbf{r}_R = f(P_1) - f(P_2)$. In order to prove continuity we need to show that whatever smal positive number e we are given, we can always find another positive number d so that

$|\mathbf{r}_L| < d$ implies that $|\mathbf{r}_R| < e$.

The proof which follows is straightforward but technical. We simply suppose that $|\mathbf{r}_L| < d$ and apply the fact that f obeys the disparity gradient limit of $2k$ with $k < 1$, and deduce that

$$|r_R| < d\frac{2^{1/2}(1+k)^2}{1-k^2} .$$

Turning this round, we see that, given e, we simply choose

$$d = e\frac{2^{-1/2}(1-k^2)}{(1+k)^2}$$

and we are done. It is now clear why we need to stipulate that $k < 1$.

We now give the details of the proof.

Let us write $\mathbf{r}_L = (x_L , y_L)$ and $r_R = (x_R , y_R)$. Suppose that $|\mathbf{r}_L| < d$, then $|x_L|, |y_L| < d$. Now the disparity gradient limit can be rewritten

$$(x_L - x_R)^2 + (y_L - y_R)^2 \leq$$
$$k^2[(x_L + x_R)^2 + (y_L + y_R)^2]$$

or

$$x_R^2(k^2-1) + 2x_L x_R(k^2+1) +$$
$$x_L^2(k^2-1) + k^2(y_L + y_R)^2 - (y_L - y_R)^2 \geq 0.$$

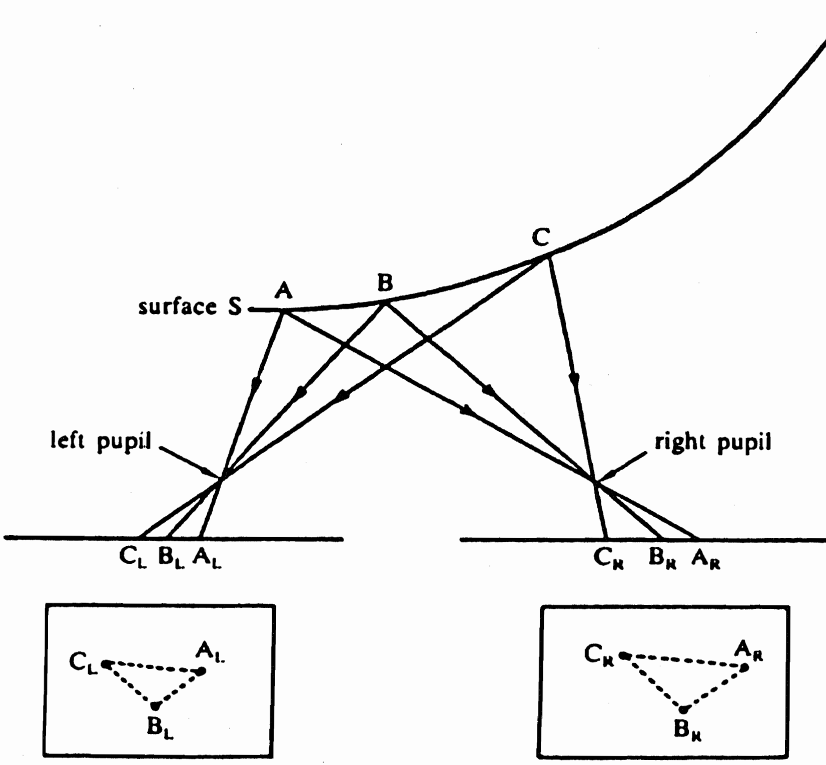

Figure 2 (Above) Points A, B, and C on a surface S (A, B, C not meant to lie on the intersection of the surface S and a plane) and their images. (Below) The two images are topologically equivalent.

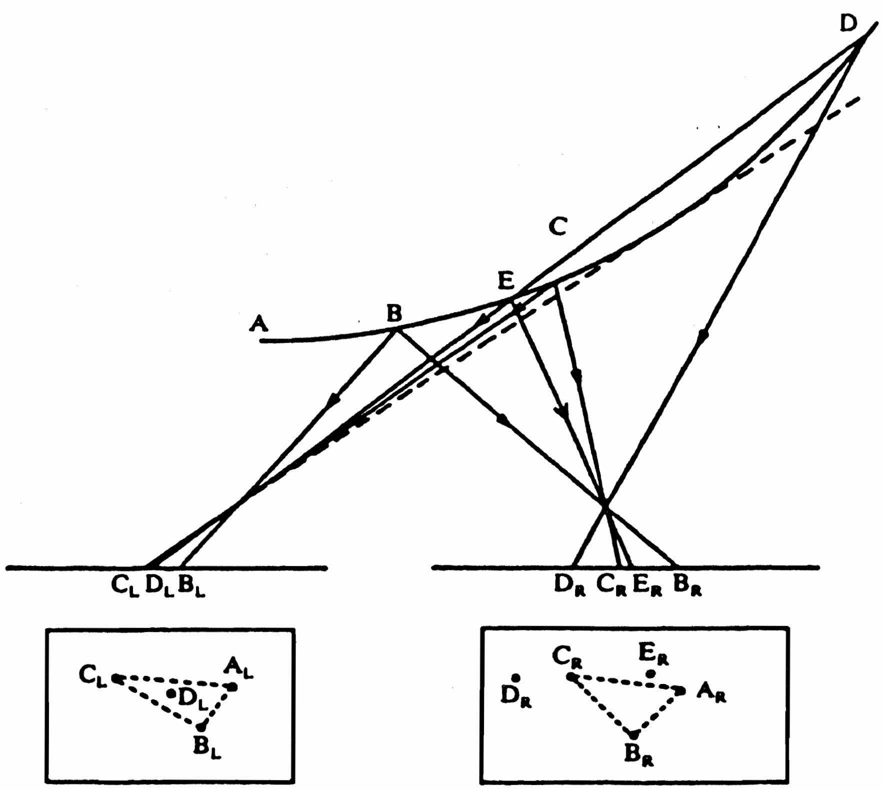

Figure 3 (Above) Surface S of figure 2. In addition, points D and E and a tangent ray passing through the left eye are also shown. (Below) The topologies of the two images are clearly not equivalent.

This is a quadratic in x_R, and the coefficient of x_R^2 is negative. The above inequality can only be satisfied when there are two real roots and x_R lies between them. The condition for the quadratic to have real roots is that

$$4x_L^2(k^2+1)^2 -$$
$$4(k^2-1)\left[x_L^2(k^2-1)+k^2(y_L+y_R)^2-(y_L-y_R)^2\right] \ge 0$$

or

$$-(1-k^2)^2 y_R^2 + 2y_L y_R(1-k^2)(1+k^2) +$$
$$4k^2x_L^2 - (1-k^2)^2 y_L^2 \ge 0.$$

Again, this is a quadratic in y_R with leading coefficient negative. It has two real roots, so the inequality is satisfied when

$$\frac{2y_L(1-k^4)-4k(1-k^2)|r_L|}{2(1-k^2)^2} \le y_R$$

$$y_R \le \frac{2y_L(1-k^4)+4k(1-k^2)|r_L|}{2(1-k^2)^2}$$

or

$$\frac{y_L(1+k^2)-2k|r_L|}{1-k^2} \le y_R \le \frac{y_L(1+k^2)+2k|r_L|}{1-k^2}$$

So certainly

$$|y_R| \le d\frac{(1+k)^2}{1-k^2}.$$

The two roots of the quadratic in x_R are

$$\frac{-2x_L(1+k^2)\pm 2\{4k^2x_L^2-(k^2-1)[k^2(y_L+y_R)^2-(y_L-y_R)^2]\}^{1/2}}{2(1-k^2)}$$

Now

$$k^2(y_L+y_R)^2-(y_L-y_R)^2 \le d\frac{4k^2y_L^2}{1-k^2}$$

(since the left-hand side is a quadratic in y_R with maximum value the expression on the right-hand side) so

$$|x_R| \le \frac{2d(1+k^2)+2\left[4k^2x_L^2+4k^2y_L^2\right]^{1/2}}{2(1-k^2)} \le d\frac{(1+k)^2}{1-k^2}$$

Hence

$$|r_R| < d\frac{2^{1/2}(1+k)^2}{1-k^2}$$

So, given $e > 0$, take $d = e[2^{-1/2}(1-k^2)/(1+k)^2]$, and then $|r_L| < d$ implies that $|r_R| < e$. Hence f is continuous. This is the end of the proof.

Note that the result is invariant to rotations and to relative as well as overall magnifications.

3 Discussion

In their attempt to incorporate the notion of disparity gradient limit in Panum's fusional area, Burt and Julesz (1980a) mention a "cone-shaped forbidden zone, symmetric about the line of sight" around a point object. This is readily understood in terms of our explanation if the cone is taken to be defined by the surface $k = 1$ passing through the point. (The extended lines of sight to each eye lie on this surface, for instance.) Then the matches between the left and the right images of the original point (at the tip of the cone) and other points - some inside and some outside the cone - will not, in general, preserve the topology of the images.

Our interpretation of the disparity gradient limit constraint in view of what we have said so far is that the human visual system expects to find *surfaces*, and that it expects the surfaces to be opaque. If a vision system is to solve the stereo correspondence problem *before* interpolating surfaces through the three-dimensional points so computed, the equivalent topology (or disparity gradient limit) constraint along with this interpretation would guarantee that no part of the surface will be obscured by another: a very powerful guarantee ensuring no conflict between stereo matching and surface interpolation at a subsequent stage of processing.

4 Conclusion

We have proved that a disparity gradient limit of less than 2 implies topological equivalence between left and right images. Further, we have shown that this guarantees that a group of three-dimensional points obeying the disparity gradient limit cannot lie on a surface which would have been obscured to one eye. Incorporation of the disparity gradient limit constraint in stereo matching makes it possible to proceed to surface interpolation in a bottom-up fashion.

References

Burt, P. and Julesz, B. (1980a) Modifications of the classical notion of Panum's fusional area. *Perception*, **9**, 671-682.

Burt, P. and Julesz, B. (1980b) A disparity gradient limit for binocular fusion. *Science*, **208**, 615-617.

Pollard, S.B., Mayhew, J.E.W., Frisby, J.P. (1985) PMF: A stereo correspondence algorithm using a disparity gradient limit. *Perception*, **14**, 449-470.

Lloyd, S.A. (1984) A new stereo algorithm. Memo CSRL/157, GEC Research Laboratories, Hirst Research Centre, Wembley, UK.