Fuzzy Intersection, Theory and Applications

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1 Background

After being proposed by L.A. Zadeh in 1965 [11][link] fuzzy set theory has been applied to a variety of different fields. While debate is still ongoing about the rigour of this theory, several studies have been conducted on algebraic foundations of many-valued logics (e.g. [5][link]) and many researchers have converged to agree that, keeping clear the conceptual distinction between many-valued logics and probability theory, the subject can be approached with mathematical rigour.
The idea behind fuzzy sets theory is that of solving some of the problems arising in the classical Western logic paradigm deriving by Aristotelian logic. This logic proposes a binary truth value which leads to the principle known as *law of excluded middle* which states that given any proposition either it or its negation must be true. In terms of set theory this translates into a binary inclusion of elements in a set so that given set A, either $a \in A$ or $a \notin A$ (and hence belongs to its complement). Such a principle, though, gives rise to several well known paradoxes generally grouped under the class of the *Sorites paradox* which suggests that if we accept the assumption that removing a grain from a heap of sand does not cause it to loose its state of being a heap, the successive removal of one grain at a time makes us categorise one grain as a heap of sand unless we allow the removal of a specific grain to trigger the change in classification of the quantity to "some grains of sand", which is unrealistic. Another important aspect which fuzzy set theory and fuzzy logic try to address is how to deal with the intrinsic vagueness of commonly used words such as *cold*, *high*, *big* which cannot be defined in classical (or crisp) set theory without introducing unreasonable jumps in proximity of the boundary of the classes. As an example, it is clear that setting a specific age (say h) as a boundary between the classes *young* and *old*, leads to the following

$$\begin{aligned} a &= h + \varepsilon \quad &\in Old \\ b &= h - \varepsilon \quad &\in Young \end{aligned}$$

even when $\varepsilon \to 0$, classifying two people with an infinitely small difference in age as belonging to different classes.
Fuzzy set theory, on the other hand, guarantees a smooth transition between adjacent classes allowing a continuous degree of membership ranging from 0, element completely out of the set, to 1, element completely in the set, so that two contiguous sets overlap over a certain region in which elements do not entirely belong to one of them. The concept of degree of membership is central to fuzzy set theory. For a given fuzzy set A defined over a space X, with x being a generic element, the *membership function* $f_A(x)$ is defined as a function assigning to each element x its degree of membership $\mu_A(x)$ to set A, so

$$f_A(x) : x \in X \to \mu_A(x) \in [0, 1]$$

The ideas of degree of membership and membership function will be widely used in the following sections. Further material on fuzzy set theory can be found in the websites proposed in the "Websites" section. It is important to understand, though, that fuzzy set theory and fuzzy logic are not trying to offer an alternative to probability theory since they are modelling vagueness rather then uncertainty. Keeping in mind this distinction is paramount to understand which one is more appropriate to model the aspects of interest (see [11][link] for further details).

2 Fuzzy Intersection

The concept of intersection in the fuzzy set framework relies upon that of **triangular norm** (generally referred to as t-norm). A t-norm is a generalisation of intersection for lattices and can be defined in different functional forms as long as the following fundamental properties are guaranteed:

1. **Commutativity**
   $t(A, B) = t(B, A)$

2. **Monotonicity**
   $t(A, B) \leq t(C, D)\, if\, A \leq C\, and\, B \leq D$

3. **Associativity**
   $t(A, t(B, C)) = t(t(A, B), C)$

4. **Neutrality of 1**
   $t(A, 1) = A$

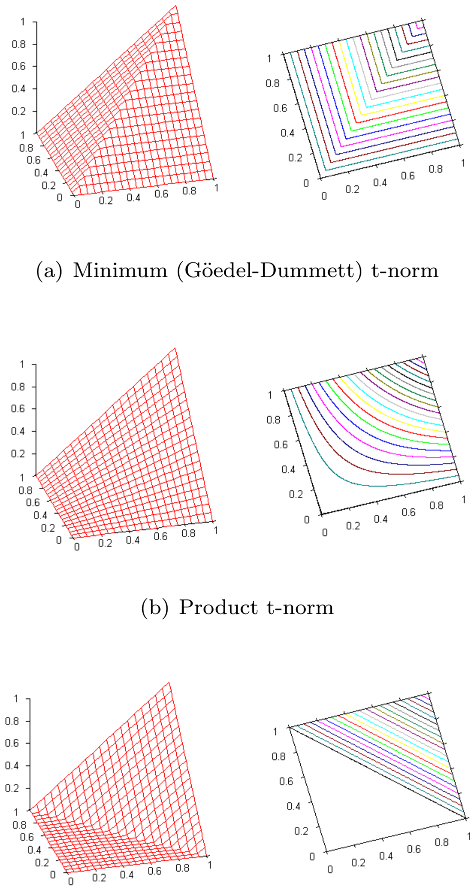
moreover in fuzzy logic the t-norm is also required to be a continuous function. Given these properties, different types of t-norms have been proposed to implement the intersection in fuzzy set theory and three of them are by far the most important. The definitions are here given in terms of membership functions of $x \in X$ w.r.t. fuzzy sets A, B and $C = A \cap B$:

1. **Göedel-Dummett t-norm**
   $f_C(x) = min(f_A(x), f_B(x))$

2. **Product t-norm**
   $f_C(x) = f_A(x) \cdot f_B(x)$

3. **Łukasiewicz t-norm**
   $f_C(x) = max(0, f_A(x) + f_B(x) - 1)$

The results obtained using different t-norms on the same application case can be found in some of the examples provided in a review paper by Isabelle Bloch [3][link] regarding the application of fuzzy set theory in the field of image analysis, while some visual examples of their effects are shown in figure 1 where the horizontal axes represent $f_A(x)$ and $f_B(x)$, and the vertical axis shows the resulting value of $f_C(x)$. Apart from continuity, which can be separated in left and right continuity, t-norms can have other properties. A t-norm is said to be Archimedean if $\forall x, y \in (0, 1)\, \exists\, n \in N$ such that $t_n(x) \leq y$ where $t_n(x)$ indicates the application of the t-norm t for n times on x itself. Continuous Archimedean t-norms, then, can be divided in two classes: strict and nilpotent t-norms. The former is different from the latter having 0 as the only nilpotent element. According to this classification Łukasiewicz t-norm is a nilpotent Archimedean t-norm, while the product t-norm is an Archimedean strict t-norm.



(a) Minimum (Göedel-Dummett) t-norm

(b) Product t-norm

(c) Łukasiewicz t-norm

Figure 1: Graph of different types of t-norms (Source: [1])

The t-norm originally proposed by Zadeh [11][link] is the Göedel-Dummett t-norm (also known as the minimum t-norm) and it is still the most commonly used in fuzzy sets theory. It is important to highlight that given this definition of intersection and adding the corresponding definitions of the t-conorm used for the union operation (keeping the Göedel-Dummett framework for $x \in X$ and fuzzy sets A, B and $C = A \cup B$ we obtain $f_C(x) = max(f_A(x), f_B(x)))$ and that of complement (if $c(A)$ is the complement of A in X we have $f_{c(A)}(x) = 1 - f_A(x)$) most of the properties defined for ordinary sets (such as De Morgan's Laws and

Distributive Laws of intersection w.r.t. union and *vice versa*) extend to fuzzy sets theory which is important for practical applications. Attention must be paid, though, since from the given definition of complementation follows that $A \cap c(A) \neq \emptyset$ since an overlap between the two sets exists in those regions where $0 < f_A(x) < 1$. This is a substantial difference from classical sets theory.
Interestingly some work have even been proposed to extend the concepts of intersection and union to fuzzy sets defined on different universes [4][link] and the authors claim that all the usual properties still hold.

2.1 Extension to Logic

As for classical set theory, fuzzy set theory has a natural extension in logic. Therefore, the various concepts have been generalised to guarantee the implementation of a structured logic. As an example the degree of membership is extended to be the degree of truth of a proposition, intersection and union become *and* and *or* conjunctions, the complement is turned into the negation of the corresponding proposition. According to the definition used for the t-norm, different propositional logics are derived. Listing applications of fuzzy logic would leave the focus of this work on fuzzy intersection, but plenty of literature can be found. It must be said, though, that papers rarely put much attention on the formal theory and focus more on the practical aspects of applications (another reason why they will not be reported here).

3 Applications

As said before most of the application found in literature are centred on fuzzy logic. Fuzzy set theory, though, is used in some interesting fields to extend the results obtained with crisp sets. What makes these applications even more interesting is the fact that they include far more attention to the theoretical basis of fuzzy sets and have better algebraic and mathematical grounding than those centred on fuzzy logic. The three applications presented are mathematical morphology, image analysis (mainly regarding spatial relationships between objects) and image processing. Given the complexity of the applications, and the necessity of a solid background to fully understand them, the following paragraphs are meant to give an idea of the practical use of the theoretical aspects previously introduced. The reader is encouraged to read the cited papers in order to get a detailed insight into the actual implementations.

3.1 Mathematical Morphology

Mathematical morphology is a well known theoretical approach, mainly based on sets theory, which allowed to create a series of techniques used in the analysis of images. Mathematical morphology had been originally formalised to work on binary (black and white) images, but has been subsequently extended to grey-scale images [7][link]. While initially this extension was based on crisp set theory, several authors proposed to use fuzzy sets to deal more effectively with the grey levels which can be seen as different degrees of membership to the black and white sets [2][link] [6][link] [8][link]. As a result, the original dilation and erosion in universe S of set X with structuring element SE and SE_x representing its translation to x

$$D_{SE}(X) = \{x \in S, X \cap SE_x \neq \emptyset\}$$
$$E_{SE}(X) = \{x \in S, SE_x \subseteq X\}$$

needed to be redefined, in fuzzy set theory, to be for all $x \in S$

$$D_{SE}(X)(x) = \sup\left\{t\left[f_{SE}(y-x), f_X(y)\right], y \in S\right\}$$
$$E_{SE}(X)(x) = \inf\left\{T\left[c(f_{SE}(y-x)), f_X(y)\right], y \in S\right\}$$

with t being the chosen t-norm, T the corresponding t-conorm, $c(\cdot)$ the complement operator and f_{SE} and f_X the membership functions of SE and X which are now fuzzy sets [3][link]. Once dilation and erosion have been defined, it is possible to derive the fuzzy version of the closing and opening operations (see [6][link] which includes extensive mathematical coverage of the topic) and then manipulate grey-scale images with the new functions derived.

3.2 Image Analysis

Starting from the definitions of fuzzy mathematical morphology and introducing more concepts based on fuzzy intersection (degree of inclusion, degree of intersection, fuzzy neighbourhood and fuzzy adjacency among the others) Bloch [3][link] shows how fuzzy spatial relationships can be used in the analysis of images. In her paper, through a case study based on MRI brain images, Bloch reports the good results obtained with fuzzy sets (which outperform crisp sets in capturing relevant aspects of the data), and highlights how choosing different t-norms affects the obtained result.

3.3 Image Processing

Shao et al. [9][link] [10][link] applied the aspects of fuzzy sets theory explained in the previous sections to optics, implementing optical image processors capable of performing any required analysis and processing on the received input. While summarising their work in few lines is impossible, it is worth mentioning since their papers show an uncommon and interesting application of fuzzy mathematical morphology.

4 Final Remarks

In this short work the most important aspects of the fuzzy intersection have been reported and commented on. While it is by no means exhaustive from the point of view of the possible applications, this report was meant to provide the theoretical basis which are necessary to proceed to more advanced topics, along with some examples on how these ideas have been used in research this far.

Recommended Readings

1. *Fuzzy Sets* by L.A. Zadeh [11][link]

2. *Fuzzy Spatial Relationships for Image Processing and Interpretation : a Review* by I. Bloch [3][link]

3. *Algebraic Foundations of Many-Valued Reasoning* my R.L. Cignoli et al [5][link]

Websites

Wikipedia: Fuzzy Set Operations
Wikipedia: Fuzzy Logic
Wikipedia: Sorites Paradox
Wikipedia: T-norm
Stanford Encyclopaedia of Philosophy: Fuzzy Logic

All the websites linked in the essay have been accessed on 30/07/2010.

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