Trans-dimensional MCMC

Frank Dellaert

References

- **Peter Green**, Reversible jump Markov chain Monte Carlo computation and Bayesian model determination, Biometrika, 1995
- **Peter Green**, Trans-dimensional Markov chain Monte Carlo, in "Highly Structured Stochastic Systems", 2003
- **David Hastie**, Ph.D. Thesis, 2005
- **Paskin & Thrun,** Robotic Mapping with Polygonal Random Fields, UAI 2005

Outline

- **Model Selection**
- **Reversible Jump MCMC**
- **Regression Example**
- **Rao-Blackwellized Sampling**
- **Polygonal Random Fields**

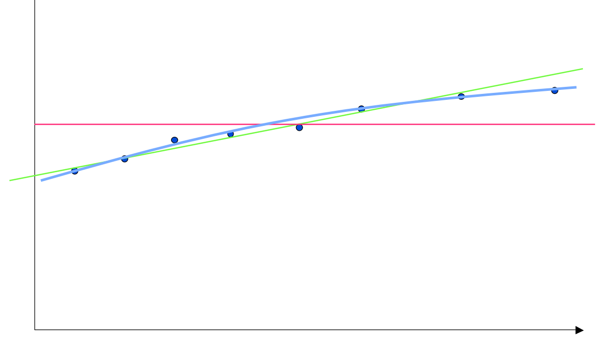
Model Selection

- Most common case: inference on p(x|z), x continuous parameters
- What if there are competing models ?

$$p(x_1 \mid z), x_1 \in R^3$$
$$p(x_2 \mid z), x_2 \in R^5$$
$$p(x_3 \mid z), x_3 \in R^7$$

Ex. 1: Regression

- Degree of polynomial ?



Ex.2: 3D Curve Fitting

- How many control points ?
- How many sharp corners ?

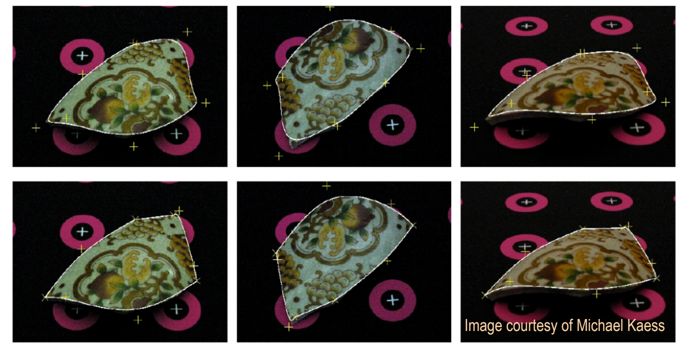
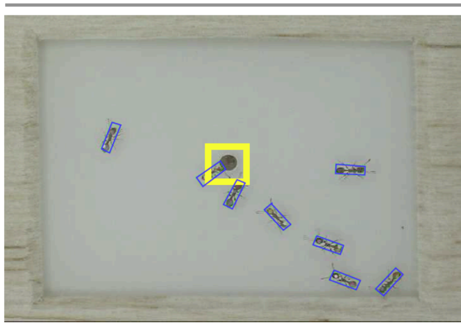


Image courtesy of Michael Kaess

Ex.3: Tracking Multiple Targets

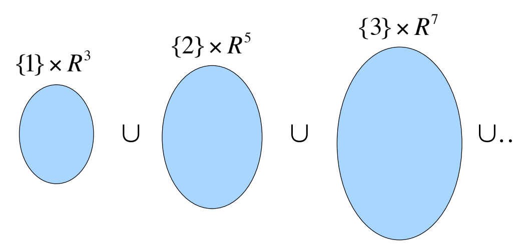


- With an unknown number of targets

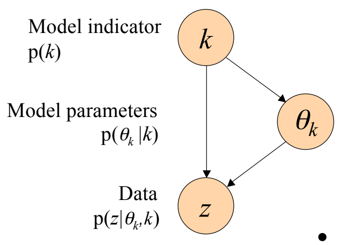
Image courtesy of Zia Khan

Union Space

- Model indicator k
- Parameter vector $\theta_k \in R^{n_k}$
- State space = union space $X = \bigcup_{k \in K} (\{k\} \times R^{n_k})$



Graphical Model



- Joint density factors as
  $p(z,\theta_k,k) = p(z|\theta_k,k)\ p(\theta_k|k)\ p(k)$

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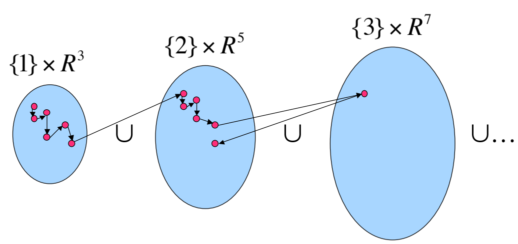
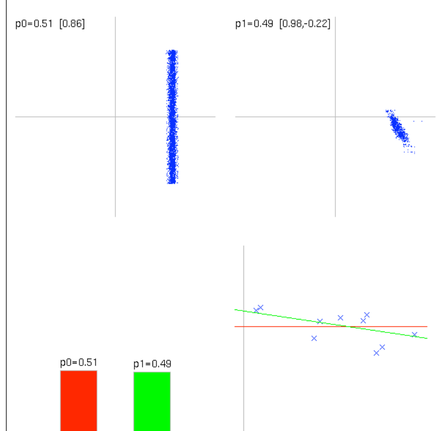
Trans-Dimensional MCMC

- set up Markov chain in union space X
- allow trans-dimensional jumps



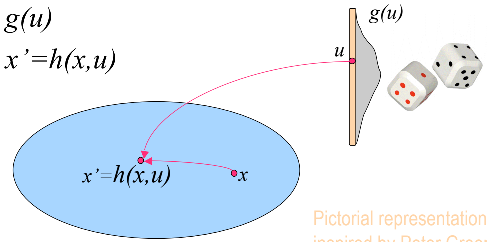
Regression Example



- Live Demo !

Explicit Jump Random Variables

- Re-state MCMC to avoid measure theory
- Explicit jump random variable u
- Proposal density *q(x,x')* replaced by
  - draw $u \sim g(u)$
  - calculate $x' = h(x,u)$



Example (single dimension)

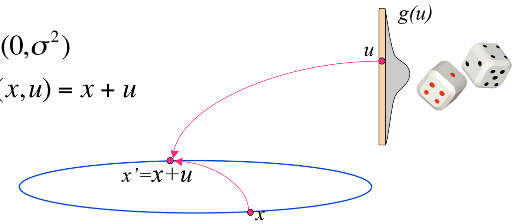
- Random walk on SO(2)

$$q(x,x') = N(x';x,\sigma^2)$$

$$\Rightarrow$$

$$u \sim N(0,\sigma^2)$$

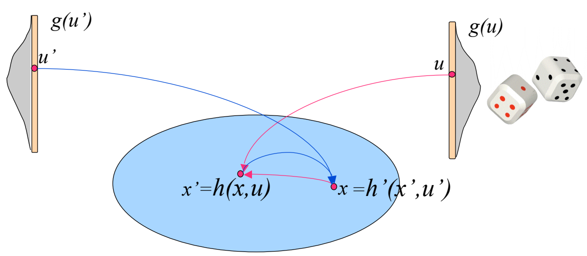
$$x' = h(x,u) = x + u$$



A New Proposal Ratio

$$a = \frac{\pi(x')q(x',x)}{\pi(x)q(x,x')} \quad \Rightarrow \quad a = \frac{\pi(x')g(u')}{\pi(x)g(u)}\left|\frac{\partial(x',u')}{\partial(x,u)}\right|$$

"Jacobian"



Jacobians

- Jacobian arises from change of variable
- Review from probability 101:

$$x \sim g(x)$$

$$y = t(x)$$

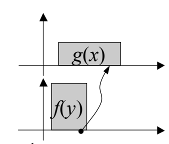
$$f(y) = \left|\frac{\partial t(x)}{\partial x}\right|^{-1} g(t^{-1}(y)) = \left|\frac{\partial x}{\partial y}\right| g(x)$$

Simple example:

$$x \sim g(x)$$

$$y = t(x) = x/2$$

$$f(y) = \left|\frac{\partial t(x)}{\partial x}\right|^{-1} g(t^{-1}(y)) = 2g(2y)$$



Detailed Balance in RJMCMC

- Change of variables as differential equality

$$f(y)dy = g(x)dx \text{ and } dx = \left|\frac{\partial x}{\partial y}\right|dy \quad \text{hence: } f(y)dy = \left|\frac{\partial x}{\partial y}\right|g(x)dy$$

Shows up in detailed balance equation:

$$\alpha(x,x')\pi(x)g(u)dxdu = \alpha(x',x)g(u)\pi(x')g(u')dx'du'$$

$$dx'du' = \left|\frac{\partial(x',u')}{\partial(x,u)}\right|dxdu$$

$$\alpha(x,x') = \min\left\{1, \frac{\pi(x')g(u')}{\pi(x)g(u)}\left|\frac{\partial(x',u')}{\partial(x,u)}\right|\right\}$$

Reversible -> Diffeomorphism

- We need to calculate both x' and u':
  (x',u') = t (x,u)
- Required that t is a diffeomorphism:
  - invertible
  - differentiable
- Jacobian =

$$\left|\frac{\partial(x',u')}{\partial(x,u)}\right| = \left|\frac{\partial t(x,u)}{\partial(x,u)}\right| = \begin{vmatrix} \frac{\partial t_{x'}(x,u)}{\partial x} & \frac{\partial t_{x'}(x,u)}{\partial u} \\ \frac{\partial t_{u'}(x,u)}{\partial x} & \frac{\partial t_{u'}(x,u)}{\partial u} \end{vmatrix}$$

A Constructive MCMC Recipe (Green)

- (Conventional) MCMC with jump variables:
  - draw u ~ g(u)
  - calculate proposal x'=h(x,u)
  - calculate reverse jump variable u' s.t. x=h'(x',u')
  - calculate acceptance ratio:

$$a = \min\left\{1, \frac{\pi(x')g(u')}{\pi(x)g(u)}\left|\frac{\partial(x',u')}{\partial(x,u)}\right|\right\}$$

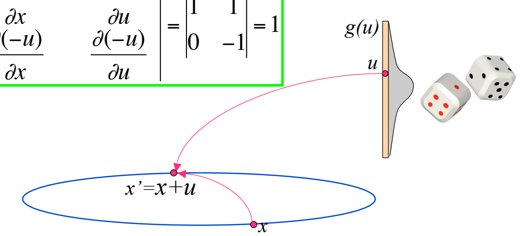
1D Example (continued)

- Random walk on SO(2)

$$u \sim N(0,\sigma^2) \qquad u' = x - x' = -u$$

$$x' = x + u \qquad \Rightarrow t(x,u) = (x+u, -u)$$

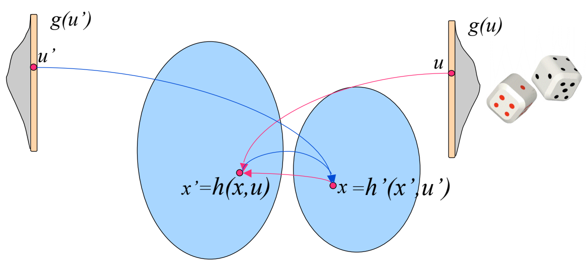
$$\left|\frac{\partial(x+u,-u)}{\partial(x,u)}\right| = \begin{vmatrix} \frac{\partial(x+u)}{\partial x} & \frac{\partial(x+u)}{\partial u} \\ \frac{\partial(-u)}{\partial x} & \frac{\partial(-u)}{\partial u} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = 1$$



Reversible-Jump MCMC

- Story holds if x and x' have different dimensions !

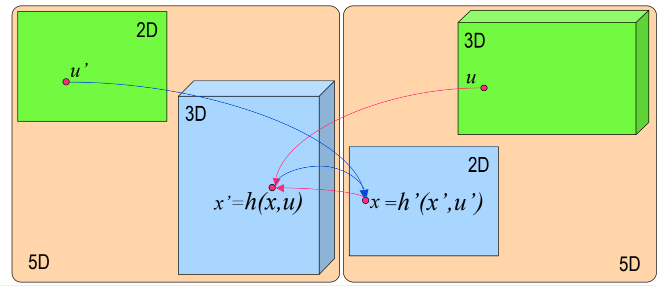
$$a = \frac{\pi(x')g(u')}{\pi(x)g(u)} \left| \frac{\partial(x',u')}{\partial(x,u)} \right|$$



Dimension Matching Constraint

- Requirement that (x',u')=t(x,u) remains a diffeomorphism

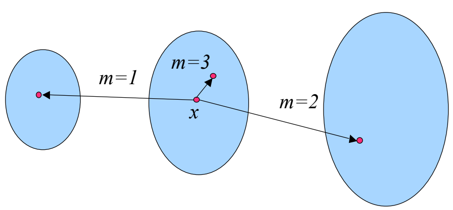
$\Rightarrow$ dim(x')+dim(u')= dim(x)+dim(u)



Move Types

- In each space, proposal is mixture of different move types
- Each move has probability $j_m(x)$

$$\Rightarrow \quad a = \frac{\pi(x')j_m(x')g_m(u')}{\pi(x)j_m(x)g_m(u)} \left| \frac{\partial(x',u')}{\partial(x,u)} \right|$$



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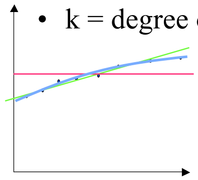
Regression Example

- k = degree of polynomial



$P(k{=}0)=0.5$

$P(\theta_0|0) = P(c_0|0) = N(c_0;\mu_0,\sigma_0^2)$

$P(z|\theta_0,0) = \prod_{i=1}^{N} N(y_i;c_0,\sigma^2)$

$P(k{=}1)=0.5$

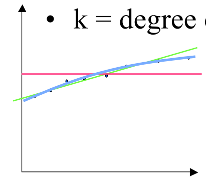
$P(\theta_1|1) = P(c_0,c_1|1) = N(c_0;\mu_0,\sigma_0^2)N(c_1;\mu_1,\sigma_1^2)$

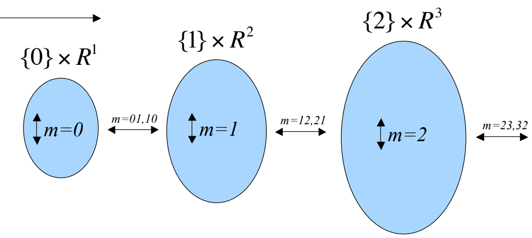
$P(z|\theta_1,1) = \prod_{i=1} N(y_i;c_0+c_1x_i,\sigma^2)$

Regression Example

- k = degree of polynomial

$$a = \frac{\pi(x')j_m(x')g_m(u')}{\pi(x)j_m(x)g_m(u)}\left|\frac{\partial(x',u')}{\partial(x,u)}\right|$$

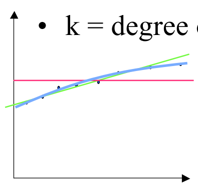


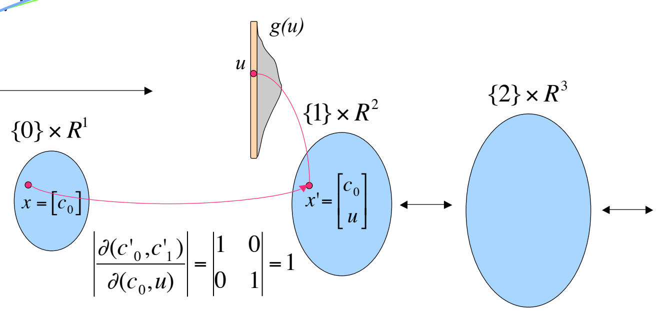


Regression Example

- k = degree of polynomial

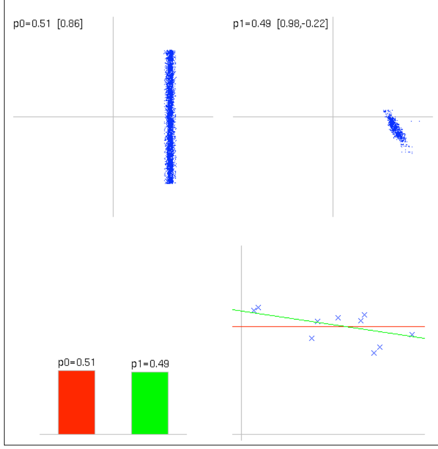
$$a = \frac{\pi(x')j_m(x')g_m(u')}{\pi(x)j_m(x)g_m(u)}\left|\frac{\partial(x',u')}{\partial(x,u)}\right|$$





$$\left|\frac{\partial(c'_0,c'_1)}{\partial(c_0,u)}\right| = \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} = 1$$

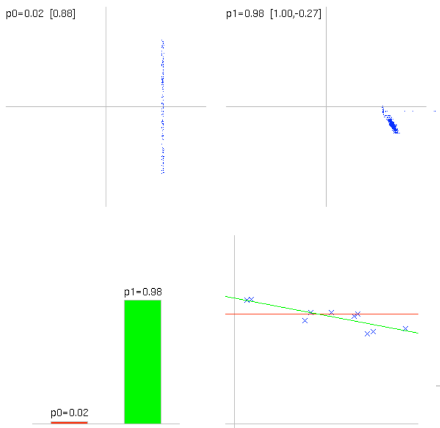
Regression Example



- Ambivalent data
- 10000 Samples

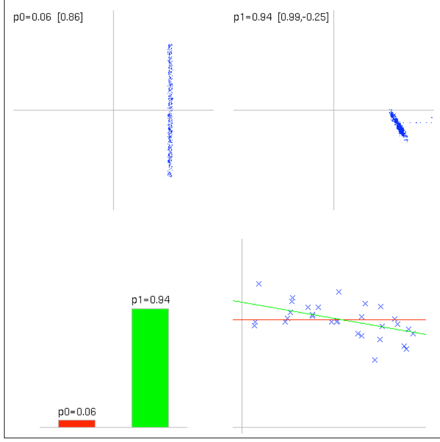
Regression Example

- Less Noise



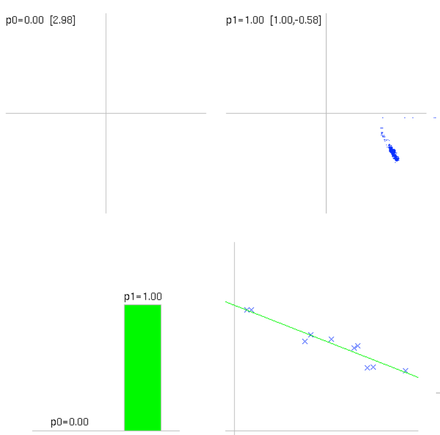
Regression Example

- More Data



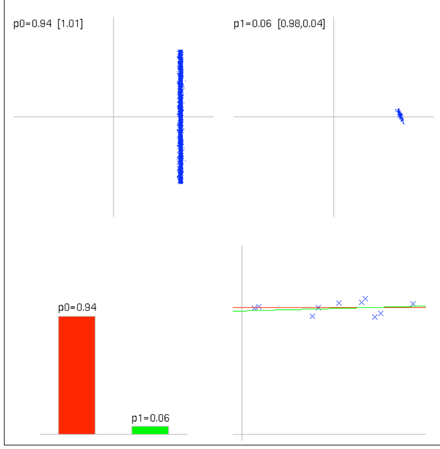
Regression Example

- No ambivalence

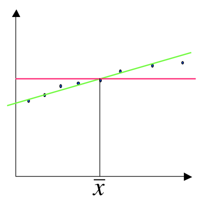


Regression Example

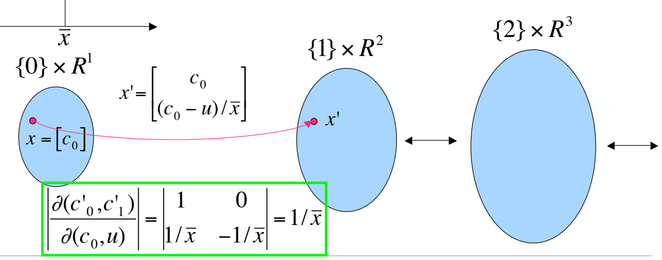
- No ambivalence for k=0 model



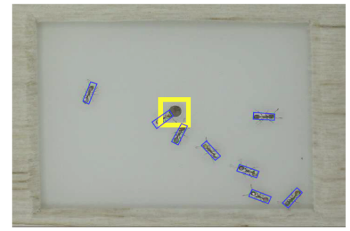
Careful Move Design



- E.g., move from constant to line such that reaches same height at mean(x)

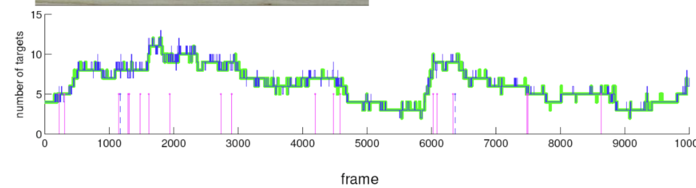


Tracking Multiple Targets



- Showing mean number of targets

Khan et al PAMI October



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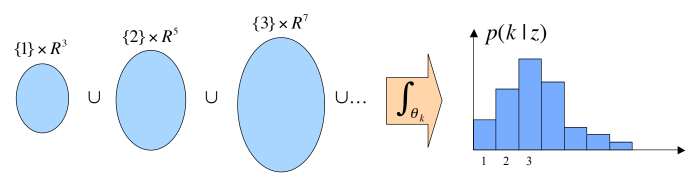
Rao-Blackwellized Sampling

- Second strategy:
  - integrate out model parameters θ_k
  - sample over marginal posterior

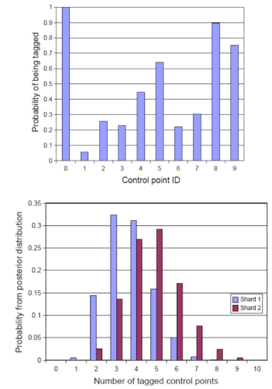
$$p(k \mid z) \propto p(k) \int_{\theta_k} p(z \mid \theta_k, k)\, p(\theta_k \mid k)$$



Rao-Blackwell Theorem

- Variance of sample approximation is reduced when part of state space is integrated out

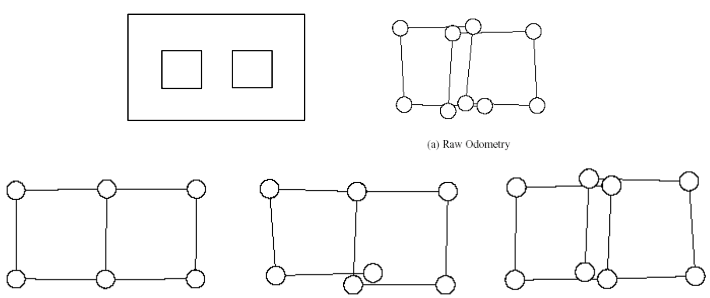
Curve Fitting Results



Images courtesy of Michael Kaess

Probabilistic Topological Maps

Sampling over the space of topologies

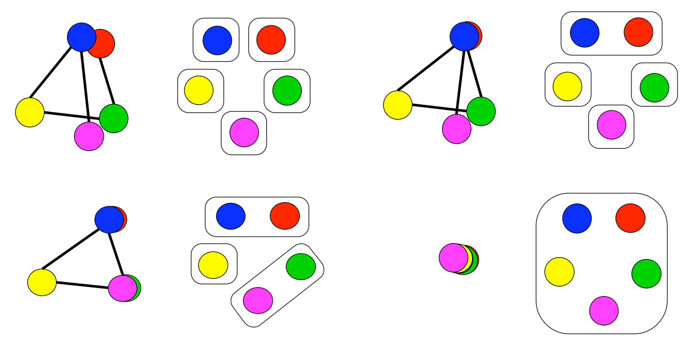


(a) Raw Odometry

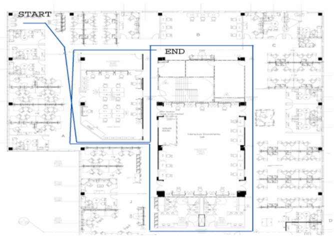
(b) 28.8 % (c) 23.0 % (d) 10.8 %

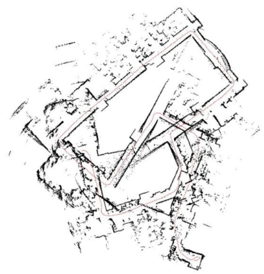
The space of topologies

Topologies ⇔ Space of set Partitions, very big !!

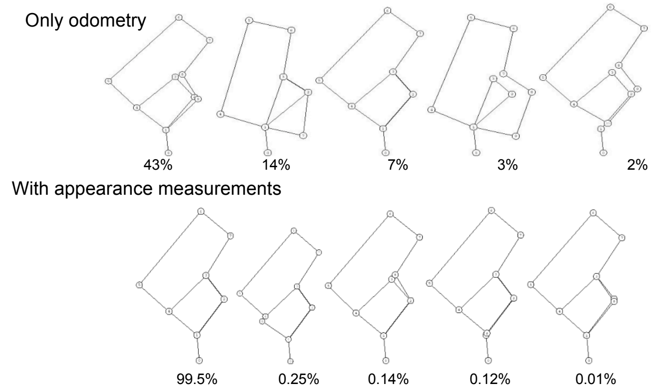


Experiments





Experiments (2)



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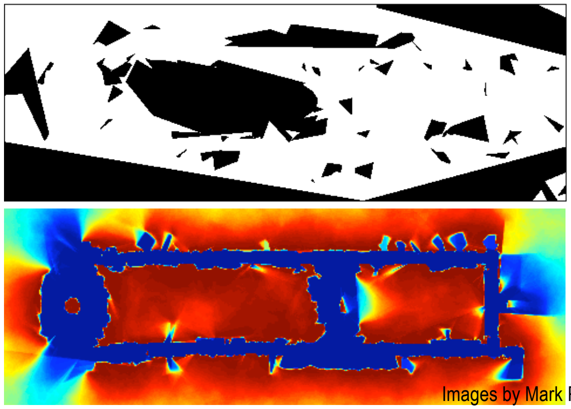
Polygonal Random Fields

- From field of spatial statistics
- Space = colorings of R^2 in window D
- Uses measure theory to build a (non-RJMCMC) sampler that satisfies detailed balance

Images by Mark Paskin

PRF in Robotics



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