Eigenspaces (Part 1)

CS 510

Lecture #14

Feb. 27, 2002

This material is also covered (too briefly) in Tr□co □ □rri□□□□

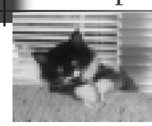
Image Eigenspaces

- Overview
- Background
  - What is the meaning of a Covariance Matrix.
  - Eigenspaces and rotation.
- Basics: Find template with Highest Correlation
  - Images as vectors and correlation for comparison.
- Eigenspace Theorem
  - Anatomy of the Covariance Matrix.
- Eigenspaces and Singular Value Decomposition
- Examples of Eigen Subspace Projection

Overview: Goal

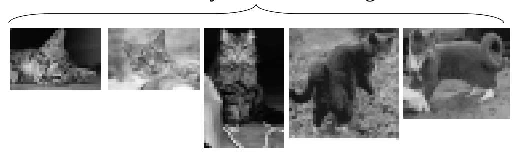
- Assume you have a gallery (database) of images, and a "probe" image.
- The goal is to find the database image that is most similar to the probe image.
  - "Similar" can be defined according to any measure
    - e.g. correlation

Example: Finding Cats



"Probe" image -- image to be matched

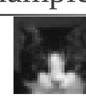
Gallery of database images



Registration

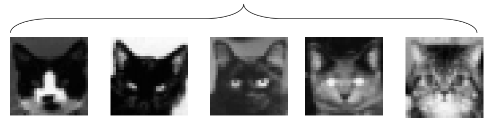
- For PCA, all images must be aligned
  - Images are points in N-dimensional space
    - Dimensions meaningless unless points correspond
  - In the minimum, comparisons undefined if sizes differ
- For scale, translation and in-plane rotation
  - Match three points
  - Apply an affine transformation
- For out-of-plane rotation:
  - Match four points
  - Apply a perspective transformation

Example: Finding Faces



Probe image, registered to gallery

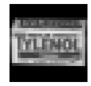
Registered Gallery of Images



Alternative: Multiple Images of One Object

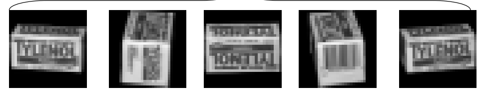
- Another reason for matching a probe against a gallery is that the gallery contains all possible views of an objects
  - Needs an image of the source object for all viewpoints
  - Needs an image of the source object under all lighting conditions

Alternate Example



Example Probe image

Five of 71 gallery images (COIL)



Eigenspace & Covariance.

- There are several ways to understand Eigenspaces.
- Related concepts include:
  - Principle Components Analysis.
  - Multivariate Random Variables.
- Supplementing Trucco, we begin with a simpler problem:
  - Label 2D points produced by two different processes.
  - Processes are multivariate normal random variables.
- The goal is to clarify the meaning of the covariance matrix.
  - The covariance matrix is a key concept.
  - Visualizing what it tells us in 2D will help with ND.

Background Concepts:

Variance

- Variance is a measure of *central tendency*, define as:

$$\frac{\sum (x-\bar{x})^2}{N}$$

- Note that the square root of the variance is the standard deviation

Background Concepts:

Covariance

- Covariance is a measure of whether two sets vary together:

$$\Omega = \frac{\sum (x-\bar{x})(y-\bar{y})}{N}$$

- How does this differ from correlation?

Background Concepts:

Covariance Matrices

- Covariance between two sets of vectors can be expressed as a matrix:
  - Let x = $\{x_1, \ldots, x_n\}$
  - Let y = $\{y_1, \ldots, y_n\}$
- Then:

$$\frac{1}{N}\Sigma = \begin{vmatrix} \sigma_{x_1,y_1} & \sigma_{x_1,y_2} & \rightarrow \\ \sigma_{x_2,y_1} & \sigma_{x_2,y_2} & \rightarrow \\ \rightarrow & \rightarrow & \ddots \end{vmatrix} \quad \sigma_{x_i,y_j} = \sum (x_i - \bar{x}_i)(y_j - \bar{y}_j)$$

Background Concepts:

Outer Products

- Remember that an outer product looks like:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} d & e & f \end{bmatrix} = \begin{bmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{bmatrix}$$

- Why? Because if I have two samples from a population of zero-meaned vectors, their covariance is their outer product

Multivariate Normal Random Variables & Covariance

- Covariance generalizes variance for multiple dimensions.
- The Gaussian Probability Distributions Function (pdf) in more than 1 dimension is:

$$f(\vec{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mu)^T \Sigma^{-1}(\mathbf{x}-\mu)\right]$$

- Consider the case of a 2D Gaussian.

$$f(\vec{x}) = \frac{1}{2\pi\begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix}^{1/2}} \exp\left[-\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}^{-1}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right]$$

Covariance Matrix

Special Case of 2D Gaussian

$$\sigma_{xx} = \sigma_x^2 \quad \sigma_{yy} = \sigma_y^2 \quad \sigma_{xy} = 0$$

- Let ...

$$f(\vec{x}) = \frac{1}{2\pi\left(\sigma_x^2\sigma_y^2\right)^{1/2}} \exp\left[-\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right]$$

$$= \frac{1}{2\pi\left(\sigma_x\sigma_y\right)} \exp\left[-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right]$$

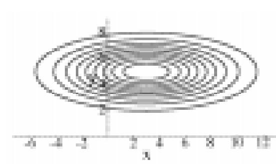
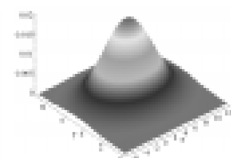
$$= \left[\frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right]\right]\left[\frac{1}{\sqrt{2\pi}\sigma_y} \exp\left[-\frac{1}{2}\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right]$$

Probability Level Curves

Consider the following 2D Gaussian.

$$f(x,y) = \frac{1}{2\pi\left(\sigma_x\sigma_y\right)} \exp\left[-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right]$$

$$\sigma_x = 4, \quad \sigma_y = 2, \quad \mu_x = 3, \quad \mu_y = 5$$

Quadratic Forms & Normal r.v.s

- Look at the exponent of the 2D Gaussian, it has the form:

$$f(x,y) = V^T M V = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= ax^2 + 2bxy + cy^2$$

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- Singular value decomposition tells us that:

$$M = R\Delta R^{-1}$$
$$= \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}$$

- R rotates coordinates such that matrix M is diagonal.

Quadratic Forms Rotated

Specify any quadratic form as rotation from axis aligned.

$$f(u,v) = V^T D\, V \qquad f(u,v) = [u \quad v]\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

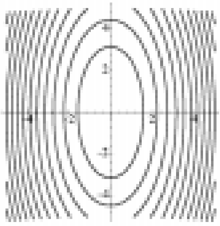
$$V = R\, X \qquad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$V^T = (R\, X)^T \qquad [u \quad v] = [x \quad y]\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

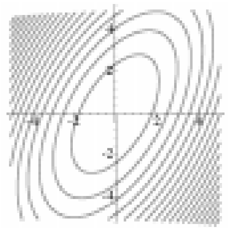
$$f(x,y) = X^T R^T D\, R\, X = X^T M\, X$$

A Rotated Form Example

$$R\left|\frac{1}{6}\pi\right| = \begin{vmatrix} .865 & -.500 \\ .500 & .865 \end{vmatrix}$$



$f_1(u, v) = 8\,x^2 + 2\,y^2$



$f_2(x, y) = 6.49\,x^2 - 5.20\,x\,y + 3.50\,y^2$

Putting It Together

- Take what we learned about rotated quadratic forms:

$$f(\vec{x}) = \frac{1}{2\pi \begin{Vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{Vmatrix}^{1/2}} \exp\left| -\frac{1}{2} \begin{vmatrix} x-\mu_x \\ y-\mu_y \end{vmatrix}^T \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix}^{-1} \begin{vmatrix} x-\mu_x \\ y-\mu_y \end{vmatrix} \right|$$

$$\Omega = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix}$$

$$= R^T D\, R$$

$$= \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} \sigma_x^{\,2} & 0 \\ 0 & \sigma_y^{\,2} \end{vmatrix} \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$$

- What about taking the inverse of the Covariance Matrix?

Take Reciprocal Down Diagonal

- A lovely little result about Diagonal Matrices.

$$\Omega = \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} \sigma_x^{\,2} & 0 \\ 0 & \sigma_y^{\,2} \end{vmatrix} \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$$

$$\Omega^{-1} = \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{vmatrix} \begin{vmatrix} \frac{1}{\sigma_x^{\,2}} & 0 \\ 0 & \frac{1}{\sigma_y^{\,2}} \end{vmatrix} \begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix}$$

SVD For Example

- The general form for Singular Value Decomposition.

$$M = U^T D\, V$$

- For our example

$$\begin{vmatrix} 6.49 & -2.60 \\ -2.60 & 3.50 \end{vmatrix} = \begin{vmatrix} -.866 & .501 \\ .501 & .866 \end{vmatrix} \begin{vmatrix} 8 & 0 \\ 0 & 2 \end{vmatrix} \begin{vmatrix} -.866 & .501 \\ .501 & .866 \end{vmatrix}$$

- Compare the U matrix to the original rotation matrix

$$R\left|\frac{1}{6}\pi\right| = \begin{vmatrix} .865 & -.500 \\ .500 & .865 \end{vmatrix}$$

Different Distributions

- Consider an observation **x** in R^3, and assume it must come from one of the following.

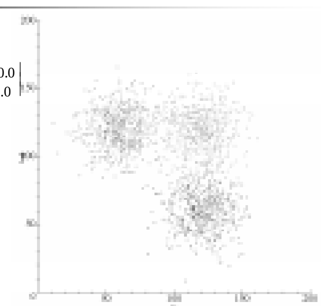
$$C_1 = N(\mu_1, \Omega_1),\; C_2 = N(\mu_2, \Omega_2),\; C_3 = N(\mu_3, \Omega_3),$$

- There are several cases.
  - Covariance matrices symmetric.
  - Covariance matrices are the same.
  - Covariance matrices aligned.
  - Each Covariance Matrix is distinct.
- More complex problem, means and covariance unknown.

Symmetric, Different Means

$$means = \begin{vmatrix} 120.0 & 60.0 & 120.0 \\ 120.0 & 120.0 & 60.0 \end{vmatrix}$$

$$sigmas = \begin{vmatrix} 15 & 15 & 15 \\ 15 & 15 & 15 \end{vmatrix}$$

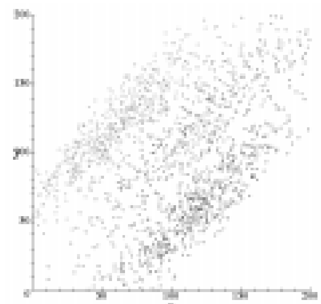


Common Alignment

$$means = \begin{vmatrix} 120.0 & 60.0 & 120.0 \\ 120.0 & 120.0 & 60.0 \end{vmatrix}$$

$$sigmas = \begin{vmatrix} 60 & 45 & 45 \\ 15 & 10 & 10 \end{vmatrix}$$

$$thetas = \begin{vmatrix} \frac{1}{4}\pi & \frac{1}{4}\pi & \frac{1}{4}\pi \end{vmatrix}$$

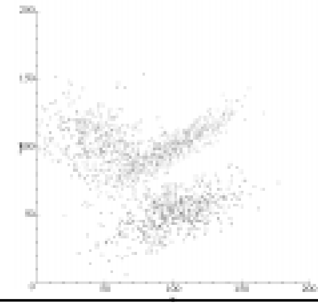


Different Covariances

$$means = \begin{vmatrix} 100.0 & 50.0 & 100.0 \\ 100.0 & 100.0 & 50.0 \end{vmatrix}$$

$$sigmas = \begin{vmatrix} 25 & 25 & 25 \\ 5 & 15 & 10 \end{vmatrix}$$

$$thetas = \begin{vmatrix} \frac{1}{6}\pi & -\frac{1}{6}\pi & \frac{1}{8}\pi \end{vmatrix}$$



Principle Components (Axes)

$$means = \begin{vmatrix} 100.0 & 50.0 & 100.0 \\ 100.0 & 100.0 & 50.0 \end{vmatrix}$$

$$sigmas = \begin{vmatrix} 25 & 25 & 25 \\ 5 & 15 & 10 \end{vmatrix}$$

$$thetas = \begin{vmatrix} \frac{1}{6}\pi & -\frac{1}{6}\pi & \frac{1}{8}\pi \end{vmatrix}$$