Eigenspaces (II)

CS510
Lecture #15
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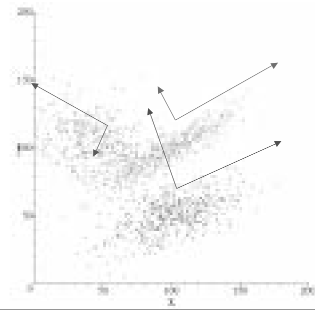
Review:
Principle Components (Axes)

The principle components are the axes of maximum variation.

There are N-1 independent axes in an N dimensional space



Covariance →Principle Components

- The principle components can be computed from the covariance matrix.
- The exponent of the 2D Gaussian has the form: $f(x,y) = V^T M V = \begin{vmatrix} x & y \end{vmatrix} \begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}$
  $= ax^2 + 2bxy + cy^2$
- Singular value decomposition tells us that:

$$M = R \Delta R^{-1}$$
$$= \begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{vmatrix} \begin{vmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{vmatrix}$$

Back to Image Eigenspaces

- Entire set of training images treated as one set of points.
- The sample covariance matrix is computed.
- You may think of covariance in terms of a Gaussian r.v.
  - The theory of principle components is more general, but the Gaussian model helps understanding.

More Image Eigenspaces

- The Eigensystem associated with Covariance tells us:
  - The rotation of the principle components (Eigenvectors)
  - Variance along each component (Eigenvalues)
- The maximum number of eigenvectors with non-zero variance is Max(samples-1, orig. dimensions)
  - Low variance means little change - disregard.
- Eigenspace is cheap way to find highest correlation!

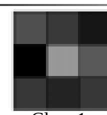
Assumptions

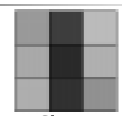
- Each image contains one object.
- Objects imaged by a fixed camera under weak perspective.
- Images are normalized with respect to size.
- Images are zero-mean
- The energy of the pixel values is normalized to 1.
  $\sum_{i=1}^{N} \sum_{j=1}^{N} I_1(i,j)^2 = 1$
- The object is completely visible.
- Each image is represented as a vector.

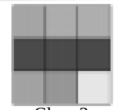
Visualization

- An aside to test your visualization skills
  - Images are points on an NxN dimensional hyper sphere.
  - Cross correlation is the cosine of the angle between the vector from the origin to that point.

An Example


Class 1


Class 2


Class 3

- Class 1 is dark around the edges and bright in the middle.
- Class 2 is light with dark vertical bars.
- Class 3 is light with dark horizontal bars.
- All classes initially use 2 for low value, 7 for high value.
- Each instance is corrupted by sigma=1 Guassian Noise.

The Image Matrices

- Consider 9 training images, 3 from each class.

$$\left| \begin{vmatrix} 1.65 & 3.11 & 2.25 \\ 3.22 & 5.79 & 3.09 \\ 1.10 & 2.47 & 2.96 \end{vmatrix}, \begin{vmatrix} 1.55 & 3.29 & 1.62 \\ 2.91 & 3.88 & .71 \\ 2.35 & 3.60 & 2.46 \end{vmatrix}, \begin{vmatrix} .80 & 2.43 & 2.04 \\ 1.59 & 8.17 & .79 \\ .69 & 1.96 & 4.34 \end{vmatrix}, \right.$$

$$\begin{vmatrix} 6.36 & 2.39 & 9.36 \\ 6.05 & .55 & 6.60 \\ 5.97 & 3.49 & 7.33 \end{vmatrix}, \begin{vmatrix} 6.43 & 1.43 & 7.01 \\ 7.66 & 3.20 & 6.66 \\ 6.96 & 1.82 & 7.52 \end{vmatrix}, \begin{vmatrix} 6.52 & .89 & 7.74 \\ 4.80 & 1.97 & 7.58 \\ 5.75 & 1.06 & 7.24 \end{vmatrix},$$

$$\left. \begin{vmatrix} 8.11 & 8.94 & 5.85 \\ 2.63 & 2.60 & 5.16 \\ 7.20 & 6.09 & 6.12 \end{vmatrix}, \begin{vmatrix} 6.94 & 6.68 & 5.99 \\ 3.63 & 3.15 & 1.37 \\ 8.50 & 6.89 & 6.49 \end{vmatrix}, \begin{vmatrix} 7.02 & 7.73 & 7.08 \\ 2.75 & 2.10 & 1.91 \\ 5.92 & 6.85 & 7.16 \end{vmatrix} \right|$$

Normalized Image Vectors

- Each element is a 9x1 vector representing an image.
- Each vector is normalized to have magnitude (length) one.
  - Mistake: not zero-meaned
- Each vector has the centroid for the set subtracted from it.

$$X = \left| \begin{vmatrix} -.133 \\ .0500 \\ -.102 \\ .0750 \\ .324 \\ .0910 \\ -.188 \\ .00100 \\ -.0630 \end{vmatrix}, \begin{vmatrix} -.117 \\ .127 \\ -.141 \\ .0930 \\ .186 \\ -.152 \\ -.0140 \\ .185 \\ -.0740 \end{vmatrix}, \begin{vmatrix} -.231 \\ -.0450 \\ -.143 \\ -.114 \\ .505 \\ -.163 \\ -.239 \\ -.0710 \\ .0450 \end{vmatrix}, \begin{vmatrix} .0480 \\ -.149 \\ .184 \\ .0710 \\ -.266 \\ .131 \\ .0300 \\ -.0670 \\ .0320 \end{vmatrix}, \begin{vmatrix} .0530 \\ -.202 \\ .0520 \\ .162 \\ -.116 \\ .135 \\ .0860 \\ -.161 \\ .0440 \end{vmatrix}, \begin{vmatrix} .0840 \\ -.229 \\ .124 \\ .0200 \\ -.178 \\ .217 \\ .0410 \\ -.200 \\ .0570 \end{vmatrix}, \begin{vmatrix} .126 \\ .197 \\ -.0290 \\ -.129 \\ -.157 \\ .0370 \\ .0800 \\ .0630 \\ -.0520 \end{vmatrix}, \begin{vmatrix} .0810 \\ .0930 \\ -.00600 \\ -.0660 \\ -.120 \\ -.163 \\ .172 \\ .124 \\ -.0150 \end{vmatrix}, \begin{vmatrix} .0900 \\ .157 \\ .0600 \\ -.113 \\ -.177 \\ -.132 \\ .0310 \\ .126 \\ .0270 \end{vmatrix} \right|$$

Brute Force Correlation

- Correlation is now the dot product of elements in X

$$CC = \begin{vmatrix} 1. & .925 & .914 & .718 & .781 & .723 & .777 & .757 & .752 \\ .925 & 1. & .841 & .715 & .751 & .660 & .848 & .890 & .862 \\ .914 & .841 & 1. & .509 & .596 & .536 & .596 & .631 & .618 \\ .718 & .715 & .509 & 1. & .968 & .976 & .871 & .859 & .871 \\ .781 & .751 & .596 & .968 & 1. & .979 & .836 & .838 & .816 \\ .723 & .660 & .536 & .976 & .979 & 1. & .828 & .798 & .800 \\ .777 & .848 & .596 & .871 & .836 & .828 & 1. & .964 & .974 \\ .757 & .890 & .631 & .859 & .838 & .798 & .964 & 1. & .982 \\ .752 & .862 & .618 & .871 & .816 & .800 & .974 & .982 & 1. \end{vmatrix}$$

Eigenspace Theorem

Let $x_1, \ldots x_n$ be vectors in $\Re^{NxN}$, and $\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$ their average.

Given the $N^2 \times n$ matrix

$X = \left[(x_1 - \bar{x}) \middle| \ldots (x_n - \bar{x}) \right]$

we can write each x_j as

$$x_j = \bar{x} + \sum_{i=1}^{n} g_{j,i} e_i$$

where $e_1, \ldots, e_n$ are the eigen vectors of $Q = XX^T$, and $g_j = \left| g_1, \ldots, g_n \right|$ are the vector components of x_j in eigenspace

First k Principle Axes

- Now consider using only the first k eigenvectors.

$$x_j \approx \sum_{i=1}^{k} g_{j,i} e_i + \bar{x}$$

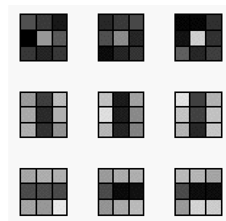
- This assumes an ordering such that:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \text{ and } \lambda_i \approx 0 \text{ for } i > k$$

- Consider the following case
  - 100 training images.
  - 100x100 pixels for each image.
  - What is the maximum possible value for k?
  - A typical value is even smaller.

Eigenspace Example 1

- Consider three examples from the three classes.



Class 1

Class 2

Class 3

Singular Value Decomposition

- The general form for SVD: $\Omega = U D U^T$
- Below are the actual values for this example.
- The eigenvalues are 1.2, 0.55, 0.19 etc.
- The eigenvectors are the columns of the ***U*** matrix.

$$\underbrace{\begin{vmatrix} -.29 & -.080 & -.31 & -.51 & .40 & .30 & -.17 & -.15 & -.51 \\ -.14 & .46 & .10 & -.48 & -.27 & -.17 & -.46 & -.30 & .35 \\ -.23 & -.060 & .34 & .050 & -.49 & -.29 & -.18 & .16 & -.66 \\ -.23 & -.52 & .64 & -.060 & .12 & .16 & .050 & -.44 & .17 \\ .84 & .040 & .16 & -.10 & -.090 & .21 & -.12 & -.32 & -.30 \\ .080 & -.47 & -.42 & -.36 & -.45 & -.29 & .35 & -.22 & .080 \\ -.14 & -.10 & -.38 & .60 & -.080 & -.020 & -.40 & -.54 & -.050 \\ -.14 & .50 & .080 & .080 & .13 & -.21 & .62 & -.47 & -.23 \\ -.20 & .16 & -.050 & .050 & -.53 & .77 & .21 & 0 & .040 \end{vmatrix}}_{U} \underbrace{\begin{vmatrix} 1.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .030 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .010 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}}_{D} \underbrace{\begin{vmatrix} -.29 & -.14 & -.23 & -.23 & .84 & .080 & -.14 & -.14 & -.20 \\ -.080 & .46 & -.060 & -.52 & .040 & -.47 & -.10 & .50 & .16 \\ -.31 & .10 & .34 & .64 & .16 & -.42 & -.38 & .080 & -.050 \\ -.51 & -.48 & .050 & -.060 & -.10 & -.36 & .60 & .080 & .050 \\ .40 & -.27 & -.49 & .12 & -.090 & -.45 & -.080 & .13 & -.53 \\ .30 & -.17 & -.29 & .16 & .21 & -.29 & -.020 & -.21 & .77 \\ -.17 & -.46 & -.18 & .050 & -.12 & .35 & -.40 & .62 & .21 \\ -.15 & -.30 & .16 & -.44 & -.32 & -.22 & -.54 & -.47 & 0 \\ -.51 & .35 & -.66 & .17 & -.30 & .080 & -.050 & -.23 & .040 \end{vmatrix}}_{U^T}$$

Subspace Projection

- Project the images onto the first three Eigenvectors

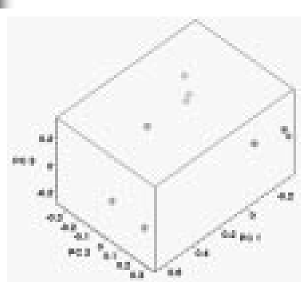
$$P = \begin{vmatrix} .0310 & .0240 & -.227 & -.315 & .384 & .164 & -.0790 & -.0540 & -.105 \\ -.139 & .0490 & .0230 & .145 & .380 & -.0540 & -.269 & -.0390 & -.0980 \\ -.312 & -.208 & -.142 & -.171 & .525 & -.0390 & .0550 & -.0660 & -.105 \\ .0330 & -.0840 & .110 & .155 & -.201 & .179 & .0860 & -.102 & -.0130 \\ .118 & -.169 & .0440 & .271 & -.234 & .0810 & .0970 & -.137 & -.0560 \\ .129 & -.101 & .0300 & .152 & -.168 & .156 & .0410 & -.140 & .0490 \\ .0310 & .148 & .0510 & -.0840 & -.137 & -.0790 & .0330 & .113 & .151 \\ .0380 & .178 & .0850 & -.0690 & -.276 & -.198 & .0260 & .186 & .0880 \\ .0720 & .163 & .0270 & -.0830 & -.274 & -.211 & .0100 & .240 & .0890 \end{vmatrix} \begin{vmatrix} -.285 & -.0840 & -.314 \\ -.138 & .463 & .102 \\ -.234 & -.0570 & .342 \\ -.229 & -.516 & .642 \\ .842 & .0380 & .158 \\ .0810 & -.469 & -.424 \\ -.142 & -.103 & -.384 \\ -.136 & .500 & .0820 \\ -.196 & .163 & -.0510 \end{vmatrix}$$

$$P = \begin{vmatrix} .489 & .373 & .651 & -.209 & -.257 & -.190 & -.193 & -.334 & -.329 \\ .0850 & -.0100 & .00900 & -.281 & -.365 & -.284 & .216 & .297 & .332 \\ -.265 & .337 & -.00400 & -.0300 & .0180 & -.0650 & -.0300 & .0320 & .00700 \end{vmatrix}$$

Class 1 Class 2 Class 3

$$\boxed{P = X^T S}$$

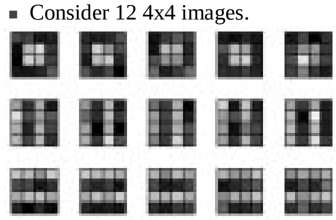
Subspace Projection Pictures



Legend
Class 1
Class 2
Class 3

Eigenspace Example 2

- Consider 12 4x4 images.



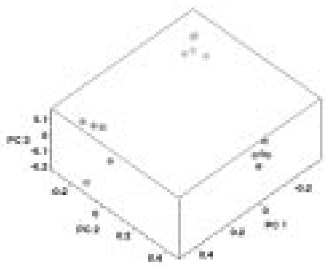
Class 1

Class 2

Class 3

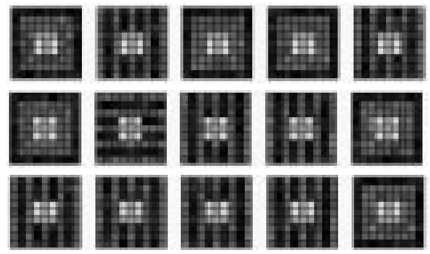
- Again, low value is 2, high value is 7, noise sigma 1.0

Example 2 Subspace 3D



Legend
Class 1
Class 2
Class 3

Eigenspace Example 3

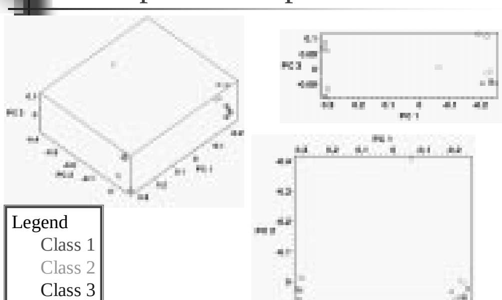


Class 1

Class 2

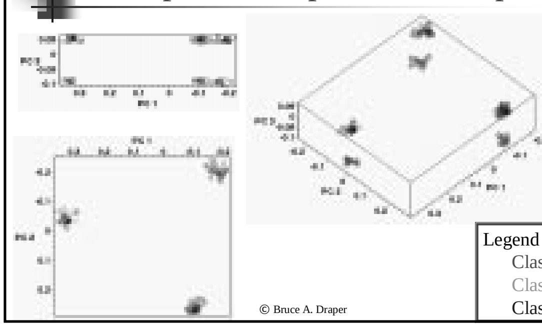
Class 3

Example 3 Subspace



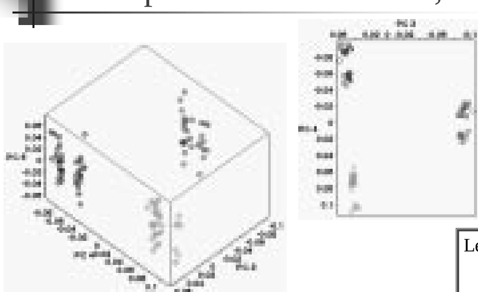
Legend
Class 1
Class 2
Class 3

Example 3 Subspace: 99 Samples



Legend
Class 1
Class 2
Class 3

Example 3: Consider PC 3, 4 & 5



Legend
Class 1
Class 2
Class 3

Example 3 Observations

- The first two Principle Components carry no information with respect to image class.
- However, Principle Components 3 and 4 carry all the information necessary for a nearest neighbors classifier.
- The Eigenvalues, which record variance along each axis, show higher PC's have more variance:

  [4.775, 3.846, .4245, .3970, .07430]
  1 2 3 4 5
- Even when the first principle components are irrelevant to classification task, this does not mean lessor components will be irrelevant as well.

Looking at Eigenvectors…

The eight Eigenvectors with the largest Eigenvalues