Uncalibrated Euclidean Reconstruction

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1 Introduction

This paper presented a review of recent techniques for Euclidean reconstruction from a single moving camera, with unconstrained motion and unknown *constant parameters*.

In the classical approach, called *autocalibration* or *self-calibration*, camera motion and parameters are recovered first, using rigidity; then structure is easily computed. Recently, new methods based on the idea of *stratification* have been proposed. They upgrade the *projective* structure, achievable from correspondences only, to the *Euclidean* structure, by exploiting all the available constraints.

The contribution of this paper is to give a critical, unified view of some of the most promising techniques. Such a comparative account sheds light on the relations between different methods, presented in different ways and formalisms in the original research articles.

This document have been adapted from [11].

2 Notation and basics

This section introduces the mathematical background on perspective projections necessary for our purposes. Our notation follows [6].

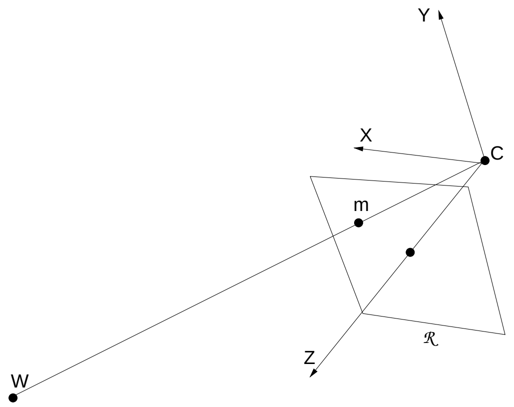

Figure 1: The pinhole camera model, with the camera reference frame (X,Y,Z) depicted. Z is also called the *optical axis* .

A pinhole camera is modeled by its *optical center* C and its *retinal plane* (or *image plane*) $\mathcal{R}$. A 3-D point W is projected into an image point m given by the intersection of $\mathcal{R}$ with the line containing C and W.

Let $\mathbf{w} = (x, y, z)$ be the coordinates of W in the world reference frame (fixed arbitrarily) and $\mathbf{m}$ the pixel coordinates of m. In homogeneous (or projective) coordinates

$$\tilde{\mathbf{m}} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \qquad \tilde{\mathbf{w}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{1}$$

the transformation from $\tilde{\mathbf{w}}$ to $\tilde{\mathbf{m}}$ is given by the matrix $\tilde{\mathbf{P}}$:

$$\kappa \tilde{\mathbf{m}} = \tilde{\mathbf{P}} \tilde{\mathbf{w}}, \tag{2}$$

where κ is a scale factor called *projective depth.* If $\tilde{\mathbf{P}}$ is suitably normalized, κ becomes the true orthogonal distance of the point from the focal plane of the camera.

The camera is therefore modeled by its *perspective projection matrix* (henceforth simply *camera matrix*) $\tilde{\mathbf{P}}$, which can be decomposed, using the QR factorization, into the product

$$\tilde{\mathbf{P}} = \mathbf{A}[\mathbf{R} \mid \mathbf{t}]. \tag{3}$$

The matrix $\mathbf{A}$ depends on the *intrinsic parameters* only, and has the following form:

$$\mathbf{A} = \begin{bmatrix} \alpha_u & \gamma & u_0 \\ 0 & \alpha_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

where $\alpha_u = -f k_u$, $\alpha_v = -f k_v$ are the focal lengths in horizontal and vertical pixels, respectively (f is the focal length in millimeters, k_u and k_v are the effective number of pixels per millimeter along the u and v axes), (u_0, v_0) are the coordinates of the *principal point*, given by the intersection of the optical axis with the retinal plane (Fig. 1), and γ is the *skew* factor.

The camera position and orientation (*extrinsic parameters*), are encoded by the 3×3 rotation matrix $\mathbf{R}$ and the translation $\mathbf{t}$, representing the rigid transformation that aligns the camera reference frame (Fig. 1) and the world reference frame.

2.1 Epipolar geometry

Let us consider the case of two cameras (see Fig. 2).

If we take the first camera reference frame as the world reference frame, we can write the two following general camera matrices:

$$\tilde{\mathbf{P}} = \mathbf{A}[\mathbf{I}|\mathbf{0}] = [\mathbf{A}|\mathbf{0}] \tag{5}$$

$$\tilde{\mathbf{P}}' = \mathbf{A}'[\mathbf{R}|\mathbf{t}]. \tag{6}$$

A three-dimensional point $\mathbf{w}$ is projected onto both image planes, to points $\tilde{\mathbf{m}} = \tilde{\mathbf{P}}\tilde{\mathbf{w}}$ and $\tilde{\mathbf{m}}' = \tilde{\mathbf{P}}'\tilde{\mathbf{w}}$, which constitute a *conjugate pair.* From the left camera we obtain:

$$\kappa' \tilde{\mathbf{m}}' = \mathbf{A}'[\mathbf{R}|\mathbf{t}]\tilde{\mathbf{w}} = \mathbf{A}'[\mathbf{R}|\mathbf{t}] \left(\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbf{A}'\mathbf{R} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \mathbf{A}'\mathbf{t}. \tag{7}$$

From the right camera we obtain: $\kappa \mathbf{A}^{-1} \tilde{\mathbf{m}} = [\mathbf{I}|\mathbf{0}]\ \tilde{\mathbf{w}} = [x\ y\ z]^\top$. Substituting the latter in (7) yields:

$$\kappa' \tilde{\mathbf{m}}' = \kappa \mathbf{A}'\mathbf{R}\mathbf{A}^{-1}\tilde{\mathbf{m}} + \mathbf{A}'\mathbf{t} = \kappa \mathbf{H}_\infty \tilde{\mathbf{m}} + \tilde{\mathbf{e}}' \tag{8}$$

where $\mathbf{H}_\infty = \mathbf{A}'\mathbf{R}\mathbf{A}^{-1}$ and $\tilde{\mathbf{e}}' = \mathbf{A}'\mathbf{t}$ (the reason for this notation will be manifest in the following).

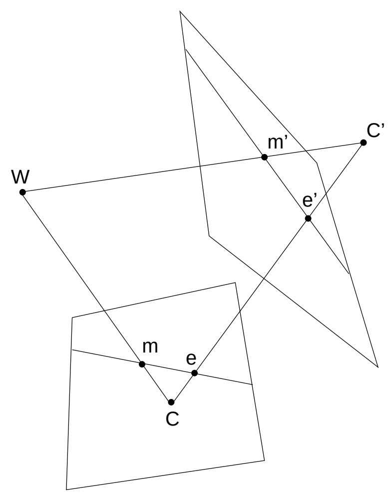

Figure 2: Epipolar geometry. The epipole of the first camera e is the projection of the optical center C' of the second camera (and vice versa) .

Equation (8) means that $\tilde{\mathbf{m}}'$ lies on the line going trough $\tilde{\mathbf{e}}'$ and the point $\mathbf{H}_\infty\tilde{\mathbf{m}}$. In projective coordinates the collinearity of these three points can be expressed with the external product: $\tilde{\mathbf{m}}'^\top(\tilde{\mathbf{e}}' \wedge \mathbf{H}_\infty\tilde{\mathbf{m}}) = 0$, or

$$\tilde{\mathbf{m}}'^\top \mathbf{F}\tilde{\mathbf{m}} = 0, \tag{9}$$

where $\mathbf{F} = [\tilde{\mathbf{e}}']_\wedge \mathbf{H}_\infty$ is the *fundamental matrix*, relating conjugate points, and $[\tilde{\mathbf{e}}']_\wedge$ is a matrix such that $\tilde{\mathbf{e}}' \wedge \mathbf{x} = [\tilde{\mathbf{e}}']_\wedge \mathbf{x}$. From (9) we can see that $\tilde{\mathbf{m}}'$ belongs to the line $\mathbf{F}\tilde{\mathbf{m}}$ in the second image, which is called the *epipolar line* of $\tilde{\mathbf{m}}$. It's easy to see that $\tilde{\mathbf{e}}'^\top\mathbf{F} = \mathbf{0}$, meaning that all the epipolar lines contain the point $\tilde{\mathbf{e}}'$, which is called the *epipole* (Fig. 2).

Since $\mathbf{F}\tilde{\mathbf{e}} = \mathbf{F}^\top\tilde{\mathbf{e}}' = \mathbf{0}$ the rank of $\mathbf{F}$ is in general two and, being defined up to a scale factor, it depends upon seven parameters. In the most general case, the only geometrical information that can be computed from pairs of images is the fundamental matrix. Its computation requires a minimum of eight point correspondences to obtain a unique solution [29, 51].

It can be seen that (9) is equivalent to

$$(\mathbf{A}'^{-1}\tilde{\mathbf{m}}')^\top [\mathbf{t}]_\wedge \mathbf{R}(\mathbf{A}^{-1}\tilde{\mathbf{m}}) = 0. \tag{10}$$

Changing to normalized coordinates, $\tilde{\mathbf{n}} = \mathbf{A}^{-1}\tilde{\mathbf{m}}$, one obtain the original formulation of the Longuet-Higgins [27] equation,

$$\tilde{\mathbf{n}}'^\top \mathbf{E}\tilde{\mathbf{n}} = 0 \tag{11}$$

involving the *essential matrix*

$$\mathbf{E} = [\mathbf{t}]_\wedge \mathbf{R}, \tag{12}$$

which can be obtained when intrinsic parameters are known. $\mathbf{E}$ depends upon five independent parameters (rotation and translation up to a scale factor). From (10) it is easy to see that

$$\mathbf{F} = \mathbf{A}'^{-\top}\mathbf{E}\mathbf{A}^{-1}. \tag{13}$$

Unlike the fundamental matrix, whose only property is being of rank two, the essential matrix is characterized by the two constraints found by Huang and Faugeras [25] which are the nullity of the determinant and the equality of the two non-zero singular values. Indeed, the following Theorem holds:

Theorem 2.1 *A real matrix* $\mathbf{E}$ 3×3 *can be factorized as product of a nonzero skew-symmetric matrix and a rotation matrix if and only if* $\mathbf{E}$ *has two identical singular values and a zero singular value.*

For a proof see [13, 8].

2.2 Homography of a plane

Given two views of a scene, there is a linear projective transformation (an *homography*) relating the projection $\mathbf{m}$ of the point of a plane Π in the first view to its projection in the second view, $\mathbf{m}'$. This application is given by a 3×3 invertible matrix $\mathbf{H}_\Pi$ such that:

$$\tilde{\mathbf{m}}' = \mathbf{H}_\Pi \tilde{\mathbf{m}}. \tag{14}$$

It can be seen that, given the two projection matrices,

$$\tilde{\mathbf{P}} = \mathbf{A}[\mathbf{I} \mid \mathbf{0}], \qquad \tilde{\mathbf{P}}' = \mathbf{A}'[\mathbf{R} \mid \mathbf{t}] \tag{15}$$

(the world reference frame is fixed on the first camera) and a plane Π of equation $\mathbf{n}^\top \mathbf{x} = d$, the following holds [30]:

$$\mathbf{H}_\Pi = \mathbf{A}'(\mathbf{R} + \mathbf{t}\frac{\mathbf{n}^\top}{d})\mathbf{A}^{-1}. \tag{16}$$

$\mathbf{H}_\Pi$ is the homography matrix for the plane Π. If $d \rightarrow \infty$,

$$\mathbf{H}_\infty = \mathbf{A}'\mathbf{R}\mathbf{A}^{-1}. \tag{17}$$

This is the homography matrix for the *infinity plane*, which maps vanishing points to vanishing points and depends only on the rotational component of the rigid displacement. It can be easily seen that:

$$\mathbf{H}_\Pi = \mathbf{H}_\infty + \tilde{\mathbf{e}}'\frac{\mathbf{n}^\top}{d}\mathbf{A}^{-1} \tag{18}$$

where $\tilde{\mathbf{e}}' = \mathbf{A}'\mathbf{t}$.

3 The reconstruction problem

Consider a set of three-dimensional points viewed by N cameras with matrices $\{\tilde{\mathbf{P}}^i\}_{i=1\ldots N}$. Let $\tilde{\mathbf{m}}^i_j \simeq \tilde{\mathbf{P}}^i \tilde{\mathbf{w}}_j$ be the (homogeneous) coordinates of the projection of the j-th point onto the i-th camera. The *reconstruction problem* can be cast in the following way: given the set of pixel coordinates $\{\tilde{\mathbf{m}}^i_j\}$, find the set of camera matrices $\{\tilde{\mathbf{P}}^i\}$ and the scene structure $\{\tilde{\mathbf{w}}_j\}$ such that

$$\tilde{\mathbf{m}}^i_j \simeq \tilde{\mathbf{P}}^i \tilde{\mathbf{w}}_j. \tag{19}$$

Without further restrictions we will, in general, obtain a projective reconstruction [34, 43, 5] defined up to an arbitrary projective transformation. Indeed, if $\{\tilde{\mathbf{P}}^i\}$ and $\{\tilde{\mathbf{w}}_j\}$ satisfy (19), also $\{\tilde{\mathbf{P}}^i\tilde{\mathbf{T}}\}$ and $\{\tilde{\mathbf{T}}^{-1}\tilde{\mathbf{w}}_j\}$ satisfy (19) for any 4×4 nonsingular matrix $\tilde{\mathbf{T}}$.

A projective reconstruction can be computed starting from points correspondences only, without any a-priori knowledge [17, 18, 45, 44, 19, 3, 2, 48]. Despite it conveys some useful in formations [40, 39], we would like to obtain a *Euclidean reconstruction*, a very special one that differs from the true reconstruction by an unknown similarity transformation. This is composed by a rigid displacement (due to the arbitrary choice of the world reference frame) plus a a uniform change of scale (due to the well-known depth-speed ambiguity: it is impossible to determine whether a given image motion is caused by a nearby object with slow relative motion or a distant object with fast relative motion).

Maybank and Faugeras [31, 10] proved that, if intrinsic parameters are constant, Euclidean reconstruction is achievable. The procedure is known as *autocalibration*.

In this approach, the internal unchanging parameters of the camera are computed from at least three views. Once the intrinsic parameters are known, the problem of computing the extrinsic parameters (motion) from point correspondences is the well-known relative orientation problem, for which a variety of methods have been developed [24, 13, 23]. In principle, from the set of correspondences $\{\tilde{\mathbf{m}}_i\}$ one can compute the fundamental matrix, from which the essential matrix is immediately obtained with (13). Motion parameters

$\mathbf{R}$ and the direction of translation $\mathbf{t}$ are obtained directly from the factorization (12) of $\mathbf{E}$. In [28] direct and iterative methods are compared.

Recently, new approaches based on the idea of *stratification* [30, 7] have been introduced. Starting from a projective reconstruction, which can be computed from the set of correspondences $\{\tilde{\mathbf{m}}_j^i\}$ only, the problem is computing the *proper* $\tilde{\mathbf{T}}$ that upgrades it to a Euclidean reconstruction, by exploiting all the available constraints. To this purpose the problem is stratified into different representations: depending on the amount of information and the constraints available, it can be analyzed at a projective, affine[1], or Euclidean level.

4 Autocalibration

In many practical cases, the intrinsic parameters are unknown and the only information that can be be extracted from a sequence are point correspondences, which allow to compute a set of fundamental matrices. *Autocalibration* consist in computing the intrinsic parameters, or – in general – Euclidean information, starting from fundamental matrices (or, equivalently, from point correspondences). In this section we will see which constraints are available for the autocalibration.

4.1 Two-views constraints

In the uncalibrated case, the fundamental matrix can be computed form point correspondences. As we saw in Section 2.1, the seven parameters of the fundamental matrix are available to describe the geometric relationship between the two views (the epipolar geometry); the five parameters of the essential matrix are needed to describe the rigid displacement, thus at most two independent constraint are available for the computation of the intrinsic parameters from the fundamental matrix.

These two constraints must come from the characterization of essential matrices given by Theorem 2.1. Indeed, the condition that the matrix $\mathbf{E}$ has a zero singular value and two non-zero equal singular values is equivalent to the following conditions, found by Huang and Faugeras [25]:

$$\det(\mathbf{E}) = 0 \quad \text{and} \quad \mathbf{trace}((\mathbf{E}\mathbf{E}^\top))^2 - 2\mathbf{trace}((\mathbf{E}\mathbf{E}^\top)^2) = 0. \tag{20}$$

The first condition is automatically satisfied, since $\det(\mathbf{F}) = 0$, but the second condition can be decomposed [29] in two independent polynomial relations that are equivalent to the two equations found by Trivedi [47].

This is an algebraic interpretation of the so-called *rigidity constraint*, namely the fact that for any fundamental matrix $\mathbf{F}$ there exist two intrinsic parameters matrix $\mathbf{A}$ and $\mathbf{A}'$ and a rigid motion represented by $\mathbf{t}$ and $\mathbf{R}$ such that $\mathbf{F} = \mathbf{A}'^{-\top}([\mathbf{t}]_\wedge \mathbf{R})\mathbf{A}^{-1}$.

By exploiting this constraint, Hartley [13] devised an algorithm to factorize the fundamental matrix that yields the five motion parameters and the two different focal lengths. He also pointed out that no more information could be extracted from the fundamental matrix without making additional assumptions (like, for example, that intrinsic parameters are constant).

A geometric interpretation of the rigidity constrain leads to the Kruppa equations, which will be briefly reviewed in Section 4.3.

4.2 N-views constraints

The case of three views is not a straightforward generalization of the two-views case. The epipolar geometry can be described using the *canonical decomposition* introduced by Luong [30] or the *trifocal tensor*, both of which uses the minimal number of parameters, that turns out to be 18. The rigid displacement is described by 11 parameters: 6 for 2 rotations, 4 for two direction of translations and 1 ratio of translation norms[2]. Therefore, three views gives seven constraints on the intrinsic parameters. If they are *constant*, three views are sufficient to recover all the five intrinsic parameters.

[1] An affine reconstruction differs from the true one by an affine transformation.

[2] Between three views there are three displacements, 1-2,2-3,and 1-3. Rotation 1-3 can be obtained by composing the others two, but the third translation cannot be obtained by summing the first two, because their norm is unknown (we only consider direction).

In the general case of n views, Luong demonstrated that $11n - 15$ parameters are need (at least) to describe the epipolar geometry, using his canonical decomposition. The rigid displacement is described by $6n - 7$ parameters: $3(n-1)$ for rotations, $2(n-1)$ for translations, and $n-2$ ratios of translation norms. There are, thus, $5n - 8$ constraints available for computing the intrinsic parameters.

As pointed out in [30], the $n(n-1)/2$ fundamental matrices are not independent, hence the $n(n-1)$ constraints like (20) that can be derived from them are not independent. Nevertheless they can be used for computing the intrinsic parameters, since redundancy improves stability.

4.3 Kruppa equations

With a minimum of three displacements, we can obtain the internal parameters of the camera using a system of polynomial equations due to Kruppa [26], which are derived from a geometric interpretation of the rigidity constraint [31, 8].

The unknown in the Kruppa equations is the matrix $\mathbf{K} = \mathbf{A}\mathbf{A}^\top$, called the *Kruppa coefficients matrix*, that represents the dual of the image of the *absolute conic* (see [6] for details). From $\mathbf{K}$ one can easily obtain the intrinsic parameters by means of Cholesky factorization ($\mathbf{K}$ is symmetric and positive definite), or in closed form:

$$\text{if} \quad \mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_4 & k_5 \\ k_3 & k_5 & 1 \end{bmatrix} \quad \text{then} \quad \mathbf{A} = \begin{bmatrix} \sqrt{k_1 - {k_3}^2 - \frac{(k_2 - k_3 k_5)^2}{k_4 - {k_5}^2}} & \frac{k_2 - k_3 k_5}{\sqrt{k_4 - {k_5}^2}} & k_3 \\ 0 & \sqrt{k_4 - {k_5}^2} & k_5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{21}$$

Kruppa equations were rediscovered and derived by Maybank and Faugeras [31]. Recently Hartley [16] provided a simpler form, based on the Singular Value Decomposition of the fundamental matrix. Let $\mathbf{F}$ be written as $\mathbf{F} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ (with SVD), and

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^\top \\ \mathbf{v}_2^\top \\ \mathbf{v}_3^\top \end{bmatrix} \quad \mathbf{D} = \operatorname{diag}(r, s, 0).$$

Then the Kruppa equations write (the derivation can be found in [16])

$$\frac{\mathbf{v}_2^\top \mathbf{K} \mathbf{v}_2}{r^2 \mathbf{u}_1^\top \mathbf{K} \mathbf{u}_1} = \frac{-\mathbf{v}_2^\top \mathbf{K} \mathbf{v}_1}{rs \mathbf{u}_1^\top \mathbf{K} \mathbf{u}_2} = \frac{\mathbf{v}_1^\top \mathbf{K} \mathbf{v}_1}{s^2 \mathbf{u}_2^\top \mathbf{K} \mathbf{u}_2}. \tag{22}$$

From (22) one obtains two independent quadratic equations in the five parameters of $\mathbf{K}$ for each fundamental matrix (i.e., for each displacement). Moreover, assuming that $\gamma = 0$, which is a good approximation for usual cameras, one has the additional constraint $k_3 k_5 = k_2$ [28]. There are basically two classes of methods for solving the resulting system of equations (assuming that more than three views are available) [28, 50]:

- Partition the equations set in groups of five and solve each group with a global convergent technique for systems of polynomial equations, like homotopy continuation methods [35, 42]. Each system will give a set of solutions and the solution common to all of them is chosen. This method – presented in [28] – has the great advantage of global convergence, but is computationally expensive. Moreover, the number of systems to be solved rapidly increases with the number of displacements.

- The over-constrained system of equation is solved with a non-linear least-squares technique (Levenberg-Marquardt [12], or Iterated Extended Kalman Filter [32]). The problem with non-linear least-squares is that a starting point close to the solution is needed. This can be obtained by applying globally convergent methods to subsets of equations (like in the previous case), or by making the additional assumption that (u_0, v_0) is in the center of the image, thereby obtaining (from just one fundamental matrix) two quadratic equations in two variables k_1, k_4, which can be solved analytically [16]. This technique is used in [50].

In [28] the authors compare three solving methods: the homotopy continuation method, Levenberg-Marquardt and the Iterated Extended Kalman Filter. From the simulations reported, it appears that all the methods give comparable results. However, the homotopy continuation method is suitable for the case of few displacements, as it would be difficult to use all the constraints provided by a long sequence, and its computational cost would be too high. Iterative approaches (Levenberg-Marquardt and Iterated Extended Kalman Filter) are well suited to the case where more displacements are available. The main limitation of all these methods is the sensitivity to the noise in the localization of points.

4.4 Mendonça and Cipolla

Mendonça and Cipolla method for autocalibration [33] is based on the exploitation of the rigidity constraint (via Theorem 2.1). A cost function is designed, which takes the intrinsic parameters as arguments, and the fundamental matrices as parameters, and return a positive value proportional to the difference between the two non-zero singular value of the essential matrix. Let $\mathbf{F}_{ij}$ the fundamental matrix relating views i and j, and let $\mathbf{A}_i$ and $\mathbf{A}_j$ be the respective intrinsic parameters matrices. Let ${}^1\sigma_{ij} > {}^2\sigma_{ij}$ be the non zero singular values of $\mathbf{E}_{ij} = \mathbf{A}_j^\top \mathbf{F}_{ij} \mathbf{A}_j$. The cost function is

$$\mathrm{C}(\mathbf{A}_i\ i = 1 \ldots n) = \sum_{i=1}^{n} \sum_{j>n}^{n} w_{ij} \frac{{}^1\sigma_{ij} - {}^2\sigma_{ij}}{{}^2\sigma_{ij}}, \tag{23}$$

where w_{ij} are normalized weight factors.

The algorithm shows excellent convergence properties. Remarkably,convergence is achieved in the 90% of the cases, even when true values are perturbed with a relative error of 200% . On the other hand intrinsic parameters are recovered with fair, but not excellent, accuracy (5% with 5 views and 1.0 pixels standard deviation image noise).

5 Stratification

Let us assume that a projective reconstruction is available, that is a sequence $\{\tilde{\mathbf{P}}^i_{\text{proj}}\}$ of camera matrices such that:

$$\tilde{\mathbf{P}}^0_{\text{proj}} = [\mathbf{I} \mid \mathbf{0}]; \qquad \tilde{\mathbf{P}}^i_{\text{proj}} = [\mathbf{Q}^i \mid \mathbf{q}^i]. \tag{24}$$

We are looking for a Euclidean reconstruction, that is a 4×4 nonsingular matrix $\tilde{\mathbf{T}}$ that upgrades the projective reconstruction to Euclidean. If $\{\tilde{\mathbf{w}}_j\}$ is the sought Euclidean structure, $\tilde{\mathbf{T}}$ must be such that: $\tilde{\mathbf{m}}^i_j = \tilde{\mathbf{P}}^i_{\text{proj}} \tilde{\mathbf{T}} \tilde{\mathbf{T}}^{-1} \tilde{\mathbf{w}}_j$, hence

$$\tilde{\mathbf{P}}^i_{\text{eucl}} \simeq \tilde{\mathbf{P}}^i_{\text{proj}} \tilde{\mathbf{T}}\ , \tag{25}$$

where the symbol $\simeq$ means "equal up to a scale factor."

5.1 Using additional information

Projective reconstruction differs from Euclidean by an unknown projective transformation in the 3-D projective space, which can be seen as a suitable change of basis. Thanks to the fundamental theorem of projective geometry [41], a collineation in space is determined by five points, hence the knowledge of the true (Euclidean) position of five points allows to compute the unknown 4×4 matrix $\tilde{\mathbf{T}}$ that transform the Euclidean frame into the projective frame. Moreover, if intrinsic parameters $\mathbf{A}$ are known, then $\tilde{\mathbf{T}}$ can be computed by solving a linear system of equations (see (54) in Sec. 5.2.5).

5.2 Euclidean reconstruction from constant intrinsic parameters

The challenging problem is to recover $\tilde{\mathbf{T}}$ without additional information, using only the *hypothesis of constant intrinsic parameters*. The works by Hartley [17], Pollefeys and Van Gool [37], Heyden and Åström [20], Triggs [46] and Bougnoux [4] will be reviewed, but first we will make some remarks that are common to most of the methods.

We can choose the first Euclidean-calibrated camera to be $\tilde{\mathbf{P}}^0_{\text{eucl}} = \mathbf{A}[\mathbf{I} \mid \mathbf{0}]$, thereby fixing arbitrarily the rigid transformation:

$$\tilde{\mathbf{P}}^0_{\text{eucl}} = \mathbf{A}[\mathbf{I} \mid \mathbf{0}] \qquad \tilde{\mathbf{P}}^i_{\text{eucl}} = \mathbf{A}[\mathbf{R}^i \mid \mathbf{t}^i]. \tag{26}$$

With this choice, it is easy to see that $\tilde{\mathbf{P}}^0_{\text{eucl}} = \tilde{\mathbf{P}}^0_{\text{proj}}\tilde{\mathbf{T}}$ implies

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{r}^\top & s \end{bmatrix} \tag{27}$$

where $\mathbf{r}^\top$ is an arbitrary vector of three elements $[r_1\ r_2\ r_3]$. Under this parameterization $\tilde{\mathbf{T}}$ is clearly non singular, and being defined up to a scale factor, it depends on eight parameters (let $s = 1$).

Substituting (24) in (25) one obtains

$$\tilde{\mathbf{P}}^i_{\text{eucl}} \simeq \tilde{\mathbf{P}}^i_{\text{proj}}\tilde{\mathbf{T}} = [\mathbf{Q}^i\mathbf{A} + \mathbf{q}^i\mathbf{r}^\top \mid \mathbf{q}^i], \tag{28}$$

and from (26)

$$\tilde{\mathbf{P}}^i_{\text{eucl}} = \mathbf{A}[\mathbf{R}^i \mid \mathbf{t}^i] = [\mathbf{A}\mathbf{R}^i \mid \mathbf{A}\mathbf{t}^i], \tag{29}$$

hence

$$\mathbf{Q}^i\mathbf{A} + \mathbf{q}^i\mathbf{r}^\top \simeq \mathbf{A}\mathbf{R}^i. \tag{30}$$

This is the basic equation, relating the unknowns $\mathbf{A}$ (five parameters) and $\mathbf{r}$ (three parameters) to the available data $\mathbf{Q}^i$ and $\mathbf{q}^i$. $\mathbf{R}$ is unknown, but must be a rotation matrix.

Affine reconstruction. Equation (30) can be rewritten as

$$\mathbf{Q}^i + \mathbf{q}^i\mathbf{r}^\top\mathbf{A}^{-1} \simeq \mathbf{A}\mathbf{R}^i\mathbf{A}^{-1} = \mathbf{H}^i_\infty, \tag{31}$$

relating the unknown vector $\mathbf{a}^\top = \mathbf{r}^\top\mathbf{A}^{-1}$ to the homography of the infinity plane (compare (31) with (18)). It can be seen that $\tilde{\mathbf{T}}$ factorizes as follows

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{a}^\top & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix}. \tag{32}$$

The right-hand matrix is an *affine transformation*, not moving the infinity plane, whereas the left-hand one is a transformation moving the infinity plane.

Substituting the latter into (25) we obtain:

$$\tilde{\mathbf{P}}^i_{\text{eucl}} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{bmatrix} = \tilde{\mathbf{P}}^i_{\text{affi}} \simeq \tilde{\mathbf{P}}^i_{\text{proj}} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{a}^\top & 1 \end{bmatrix} = [\mathbf{H}^i_\infty | \mathbf{q}^i] \tag{33}$$

Therefore, the knowledge of the homography of the infinity plane (given by $\mathbf{a}$) allows to compute the Euclidean structure up to an affine transformation, that is an *affine reconstruction*.

From affine to Euclidean. Another useful observation is, if $\mathbf{H}_\infty$ is known and the intrinsic parameters are constant, the intrinsic parameters matrix $\mathbf{A}$ can easily be computed [1, 17, 30, 49].

Let us consider the case of two cameras. If $\mathbf{A}' = \mathbf{A}$, then $\mathbf{H}_\infty$ is exactly known (with the right scale), since

$$\det(\mathbf{H}_\infty) = \det(\mathbf{A}\mathbf{R}\mathbf{A}^{-1}) = 1. \tag{34}$$

From (17) we obtain $\mathbf{R} = \mathbf{A}'^{-1}\mathbf{H}_\infty\mathbf{A}$, and, since $\mathbf{R}\mathbf{R}^\top = \mathbf{I}$, it is easy to obtain:

$$\mathbf{H}_\infty \mathbf{K} \mathbf{H}_\infty^\top = \mathbf{K} \tag{35}$$

where $\mathbf{K} = \mathbf{A}\mathbf{A}^\top$ is the Kruppa coefficients matrix. As (35) is an equality between 3×3 symmetric matrices, we obtain a linear system of six equations in the five unknown k_1, k_2, k_3, k_4, k_5 . In fact, only four equations are independent [30, 49], hence at least three views (with constant intrinsic parameters) are required to obtain an over-constrained linear system, which can be easily solved with a linear least-squares technique.

Note that two views would be sufficient under the usual assumption that the image reference frame is orthogonal ($\gamma = 0$), which gives the additional constraint $k_3 k_5 = k_2$.

If points at infinity (in practice, sufficiently far from the camera) are in the scene, $\mathbf{H}_\infty$ can be computed from point correspondences, like any ordinary plane homography [49]. Moreover, with additional knowledge, it can be estimated from vanishing points or parallelism [7, 9], or constrained motion [1].

In the rest of the section, some of the most promising stratification techniques will be reviewed.

5.2.1 Hartley

Hartley [17] pioneered this kind of approach. Starting from (30), we can write

$$(\mathbf{Q}^i + \mathbf{q}^i \mathbf{a}^\top)\mathbf{A} \simeq \mathbf{A}\mathbf{R}^i. \tag{36}$$

By taking the QR decomposition of the left-hand side we obtain an upper triangular matrix $\mathbf{B}^i$ such that $(\mathbf{Q}^i + \mathbf{q}^i \mathbf{a}^\top)\mathbf{A} = \mathbf{B}^i \mathbf{R}^i$, so (36) rewrites $\mathbf{B}^i \mathbf{R}^i = \lambda^i \mathbf{A}\mathbf{R}^i$ or

$$\frac{1}{\lambda^i}\mathbf{A}^{-1}\mathbf{B}^i = \mathbf{I}. \tag{37}$$

The scale factor $1/\lambda^i$ can be chosen so that the sum of the squares of the diagonal entries of $(1/\lambda^i)\mathbf{A}^{-1}\mathbf{B}^i$ equals three. We seek $\mathbf{A}$ and $\mathbf{a}$ that minimizes

$$\sum_{i>0} \left\| \frac{1}{\lambda^i}\mathbf{A}^{-1}\mathbf{B}^i - \mathbf{I} \right\|^2 . \tag{38}$$

Each camera excluding the first, gives six constraints in eight unknowns, so three cameras are sufficient. In practice there are more than three cameras, and the non-linear least squares problem can be solved with Levenberg-Marquardt minimization algorithm [12]. As noticed in the case of Kruppa equations, a good initial guess for the unknowns $\mathbf{A}$ and $\mathbf{a}$ is needed in order for the algorithm to converge to the solution.

Given that from $\mathbf{H}^i_\infty$ the computation of $\mathbf{A}$ is straightforward, a guess for $\mathbf{a}$ (that determines $\mathbf{H}^i_\infty$) is sufficient. The *cheirality constraints* [14] are exploited by Hartley to estimate the infinity plane homography, thereby obtaining an approximate affine (or *quasi-affine*) reconstruction.

5.2.2 Pollefeys and Van Gool

In this approach [37], a projective reconstruction is first updated to affine reconstruction by the use of the *modulus constraint* [30, 38]: since the left-hand part of (31) is conjugated to a (scaled) rotation matrix, all eigenvalues must have equal moduli. Note that this holds if and only if intrinsic parameters are constant. To make the constraint explicit we write the characteristic polynomial:

$$\det(\mathbf{Q}^i + \mathbf{q}^i \mathbf{a}^\top - \lambda \mathbf{I}) = l_3\lambda^3 + l_2\lambda^2 + l_1\lambda + l_0. \tag{39}$$

The equality of the roots of the characteristic polynomial is not easy to impose, but a simple necessary condition holds:

$$l_3 l_1^3 = l_2^3 l_0. \tag{40}$$

This yields a fourth order polynomial equation in the unknown $\mathbf{a}$ for each camera except the first, so a finite number of solutions can be found for four cameras. Some solutions will be discarded using the modulus constraint, that is more stringent than (40).

As discussed previously, autocalibration is achievable with only three views. It is sufficient to note that, given three cameras, for every plane homography, the following holds [30]:

$$\mathbf{H}^{1,3} = \mathbf{H}^{2,3}\mathbf{H}^{1,2}. \tag{41}$$

In particular it holds for the infinity plane homography, so

$$\mathbf{H}^{i,j}_{\infty} = \mathbf{H}^{j}_{\infty}{\mathbf{H}^{i}_{\infty}}^{-1} \simeq (\mathbf{Q}^j + \mathbf{q}^j\mathbf{a}^\top)(\mathbf{Q}^i + \mathbf{q}^i\mathbf{a}^\top)^{-1}. \tag{42}$$

In this way we obtain a constraint on the plane at infinity for each pair of views. Let us write the characteristic polynomial:

$$\det((\mathbf{Q}^j + \mathbf{q}^j\mathbf{a}^\top)(\mathbf{Q}^i + \mathbf{q}^i\mathbf{a}^\top)^{-1} - \lambda\mathbf{I}) = 0 \quad \Longleftrightarrow \tag{43}$$

$$\det((\mathbf{Q}^j + \mathbf{q}^j\mathbf{a}^\top) - \lambda(\mathbf{Q}^i + \mathbf{q}^i\mathbf{a}^\top)) = 0 \tag{44}$$

Writing the constraint (40) for the three views, a system of three polynomial of degree four in three unknowns is obtained. Here, like in the solution of Kruppa equations, homotopy continuation methods could be applied to compute all the $4^3 = 64$ solutions.

In practice more than three views are available, and we must solve a non-linear least-squares problem: Levenberg-Marquardt minimization is used by the author.

5.2.3 Heyden and Åström

The method proposed by Heyden and Åström [20] is again based on (30), which can be rewritten as

$$\tilde{\mathbf{P}}^i_{\mathrm{proj}} \begin{bmatrix} \mathbf{A} \\ \mathbf{r}^\top \end{bmatrix} \simeq \mathbf{A}\mathbf{R}^i. \tag{45}$$

Since $\mathbf{R}^i{\mathbf{R}^i}^\top = \mathbf{I}$ it follows that:

$$\tilde{\mathbf{P}}^i_{\mathrm{proj}} \begin{bmatrix} \mathbf{A} \\ \mathbf{r}^\top \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{r}^\top \end{bmatrix}^\top {\tilde{\mathbf{P}}^i}{}^\top_{\mathrm{proj}} = \tilde{\mathbf{P}}^i_{\mathrm{proj}} \begin{bmatrix} \mathbf{A}\mathbf{A}^\top & \mathbf{A}\mathbf{r} \\ \mathbf{r}^\top\mathbf{A}^\top & \mathbf{r}^\top\mathbf{r} \end{bmatrix} {\tilde{\mathbf{P}}^i}{}^\top_{\mathrm{proj}} \simeq \mathbf{A}\mathbf{R}^i{\mathbf{R}^i}^\top\mathbf{A}^\top = \mathbf{A}\mathbf{A}^\top. \tag{46}$$

The constraints expressed by (46) are called the Kruppa constraints [20]. Note that (46) contains five equations, because the matrices of both members are symmetric, and the homogeneity reduces the number of equations with one. Hence, each camera matrix, apart from the first one, gives five equations in the eight unknowns $\alpha_u, \alpha_v, \gamma, u_0, v_0, r_1, r_2, r_3$. A unique solution is obtained when three cameras are available. If the unknown scale factor is introduced explicitly, (46) rewrites:

$$0 = f_i(\mathbf{A}, \mathbf{r}, \lambda_i) = \lambda_i^2\mathbf{A}\mathbf{A}^\top - \tilde{\mathbf{P}}^i_{\mathrm{proj}} \begin{bmatrix} \mathbf{A}\mathbf{A}^\top & \mathbf{A}\mathbf{r} \\ \mathbf{r}^\top\mathbf{A}^\top & \mathbf{r}^\top\mathbf{r} \end{bmatrix} {\tilde{\mathbf{P}}^i}{}^\top_{\mathrm{proj}}. \tag{47}$$

Therefore, 3 cameras yield 10 equations in 8 unknowns.

5.2.4 Triggs

Triggs [46] proposed a method based on the *absolute quadric* and, independently from Heyden and Åström, he derived an equation closely related to (46). The absolute quadric $\mathbf{\Omega}$ consists of planes tangent to the absolute conic [6], and in a Euclidean frame, is represented by the matrix

$$\mathbf{\Omega}_{\mathrm{euc}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}. \tag{48}$$

If $\tilde{\mathbf{T}}$ is a projective transformation acting as in (25), then it can be verified [46] that it transforms $\mathbf{\Omega}_{\mathrm{euc}}$ into $\mathbf{\Omega} = \tilde{\mathbf{T}}\mathbf{\Omega}_{\mathrm{euc}}\tilde{\mathbf{T}}^\top$. Since the projection of the absolute quadric yields the dual image of the absolute conic [46], one obtain

$$\tilde{\mathbf{P}}^i_{\mathrm{proj}}\mathbf{\Omega}{\tilde{\mathbf{P}}^i}{}^\top_{\mathrm{proj}} \simeq \mathbf{K} \tag{49}$$

from which, using (27), (46) follows immediately. Triggs, however, does not assume any particular form for $\tilde{\mathbf{T}}$, hence the unknown are $\mathbf{K}$ and $\mathbf{\Omega}$. Note that both these matrix are symmetric and defined up to a scale factor.

Let $\mathbf{k}$ be the matrix composed by the the six elements of the lower triangle of $\mathbf{K}$, and $\boldsymbol{\omega}$ be the matrix composed by the six elements of the lower triangle of $\tilde{\mathbf{P}}^i_{\text{proj}}\mathbf{\Omega}\tilde{\mathbf{P}}^{i\top}_{\text{proj}}$, then (49) is tantamount to saying that the two vectors are equal up to a scale, hence

$$\mathbf{k} \wedge \boldsymbol{\omega} = \mathbf{0} \tag{50}$$

in which the unknown scale factor is eliminated. For each camera this amounts to 15 bilinear equations in $9+5$ unknowns, since both $\mathbf{k}$ and $\boldsymbol{\omega}$ are defined up to a scale factor. Since only five of them are linearly independent, at least three images are required for a unique solution.

Triggs uses two methods for solving the non-linear least-squares problem: sequential quadratic programming [12] on $N \geq 3$ cameras, and a quasi-linear method with SVD factorization on $N \geq 4$ cameras. He recommend to use data standardization [15] and to enforce $\det(\mathbf{\Omega}) = 3$. The sought transformation $\tilde{\mathbf{T}}$ is computed by taking the eigen-decomposition of $\mathbf{\Omega}$.

5.2.5 Bougnoux

This methods [4] is different from the previous ones, because it does not require constant intrinsic parameters and because it achieves only an approximate Euclidean reconstruction, without obtaining meaningful camera parameters as a by-product.

Let us write (25) in the following form:

$$\tilde{\mathbf{P}}^i_{\text{eucl}} = \left[\begin{array}{c|c} \mathbf{q}_1^{i\top} & \\ \mathbf{q}_2^{i\top} & \mathbf{q}^i \\ \mathbf{q}_3^{i\top} & \end{array}\right] \simeq \tilde{\mathbf{P}}^i_{\text{proj}}\tilde{\mathbf{T}} \tag{51}$$

where $\mathbf{q}_1^{i\top}, \mathbf{q}_2^{i\top}, \mathbf{q}_3^{i\top}$ are the rows of $\tilde{\mathbf{P}}^i_{\text{eucl}}$. The usual assumptions $\gamma = 0$ and $\alpha_u = \alpha_v$, are used to constraint the Euclidean camera matrices:

$$\gamma = 0 \iff (\mathbf{q}_1^i \wedge \mathbf{q}_3^i)^\top (\mathbf{q}_2^i \wedge \mathbf{q}_3^i) = 0 \tag{52}$$

$$\alpha_u = \alpha_v \iff ||\mathbf{q}_1^i \wedge \mathbf{q}_3^i|| = ||\mathbf{q}_2^i \wedge \mathbf{q}_3^i||. \tag{53}$$

Thus each camera, excluding the first, gives two constraints of degree four. Since we have six unknown, at least four cameras are required to compute $\tilde{\mathbf{T}}$. If the principal point (u_0, v_0) is forced to the image center, the unknowns reduce to four and only three cameras are needed.

The non-linear minimization required to solve the resulting system is rather unstable and must be started close to the solution: an estimate of the focal length and $\mathbf{r}$ is needed. Assuming known principal point, no skew, and unit aspect ratio, the focal length α_u can be computed from the Kruppa equations in closed form [4]. Then, given the intrinsic parameters $\mathbf{A}$, an estimate of $\mathbf{r}$ can be computed by solving a *linear* least-squares problem. From (46) the following is obtained:

$$\mathbf{Q}^i\mathbf{A}\mathbf{A}^\top\mathbf{Q}^{i\top} + \mathbf{Q}^i\mathbf{A}\mathbf{r}\mathbf{q}^{i\top} + (\mathbf{Q}^i\mathbf{A}\mathbf{r}\mathbf{q}^{i\top})^\top + ||\mathbf{r}||^2\mathbf{q}^i\mathbf{q}^{i\top} = \lambda^2\mathbf{A}\mathbf{A}^\top. \tag{54}$$

Since $[\mathbf{A}\mathbf{A}^\top]_{3,3} = \mathbf{K}_{3,3} = 1$, then λ is fixed. After some algebraic manipulation [4], one ends up with four linear equations in $\mathbf{A}\mathbf{r}$. This method works also with varying intrinsic parameters, although, in practice, only the focal length is allowed to vary, since principal point is forced to the image center and no skew and unit aspect ratio are assumed. The estimation of the camera parameters is inaccurate, nevertheless Bougnoux proves that the reconstruction is correct up to an anisotropic homotethy, which he claims to be enough for the reconstructed model to be usable.

5.3 Discussion

Methods based on stratification have appeared only recently, and only preliminary and partial results are available. In many cases they show a graceful degradation as noise increases, but the issue of initialization is not always addressed. Hartley's algorithm leads to a minimization problem that requires a good initial guess; this is computed using a quite complicated method, involving the cheirality constraints. Pollefeys-VanGool's algorithm leads to an easier minimization, and this justify the claim that convergence toward a global minimum is relatively easily obtained. It is unclear, however, how the initial guess has to be chosen. The method proposed by Heyden and Åström was evaluated only on one example, and was initialized close to the ground-truth. Experiments on synthetic data reported by Triggs, suggest that his non-linear algorithm is stable and requires only approximate initialization (the author reports that initial calibration may be wrong up to 50%). Bougnoux's algorithm is quite different form the others, since its goal is not to obtain an accurate Euclidean reconstruction. Assessment of reconstruction quality is only visual.

6 Varying intrinsic parameters

Can the constant intrinsic parameters hypothesis be relaxed, if more than three views are available? Let's count the unknowns, in the case that n_k parameters are *known* and n_c parameters are *constant*. Every view apart from the first one introduces $5-n_k-n_c$ unknowns; the first view introduces $5-n_k$ unknowns, therefore the unknown intrinsic parameters can be computed provided that

$$5n - 8 \geq (n-1)(5 - n_k - n_c) + 5 - n_k, \tag{55}$$

which is equivalent to the following equation reported in [36]:

$$nn_k + (n-1)n_c \geq 8. \tag{56}$$

Despite the problem for constant intrinsic parameters is far from being completely solved, the more general one in which intrinsic parameters are varying is gaining the attention of researchers. In fact, Bougnoux's method already copes with varying parameters. Heyden and Åström [21] proposed a method that works with varying and unknown focal length and principal point. Later, they proved [22] that it is sufficient to know any of the five intrinsic parameters to make Euclidean reconstruction, even if all other parameters are unknown and varying (this can be obtained as a special case of Eq.56). A similar method that can work with different types of of constraints has been recently presented in [36]. Mendonça and Cipolla algorithm [33] copes with varying parameters as well.

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