UNIVERSAL NATURAL SHAPES

From the supereggs of Piet Hein to the cosmic egg of Georges Lemaître

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Introduction

From the Introduction and the Epilogue of d'Arcy Thompson's "On Growth and Form" [7], respectively, we quote the following: "The search for differences or fundamental contrasts between the phenomena of organic or inorganic, of animate or inanimate things, has occupied many men's minds, while *the search for community of principles or essential similitudes has been pursued by few;* ... things animate and inanimate, *we dwellers in the world and this world wherein we dwell are bound alike by physical and mathematical law".*

We aim to show that honeycombs and shells, crystals and galaxies, DNA-molecules and flowers, stems, tissues and pollen grains of plants, etc. and the relativistic space-time universe itself, in accordance with similar natural curvature conditions, all do assume shapes with similar geometrical formal descriptions.

1 On the geometry of "what dwells in the world"

Let x and y denote Cartesian co-ordinates in a plane $\mathbb{R}^2$. Then the equation of the circle of radius r which is centered at the origin O is given by $x^2 + y^2 = r^2$ and the equation of the ellipse of axes a and $b \neq a$ which is centered at the origin is given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

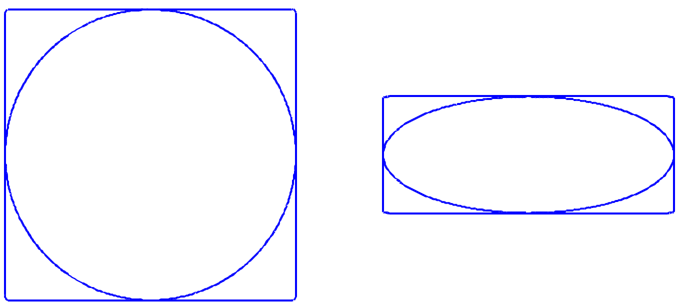

Figure 1: circles and ellipses

Going back to Lamé, the circles and the ellipses, as well as the squares and rectangles which are also shown in Figure 1, are all included in the family of the so-called "superellipses", i.e. the planar curves given by Cartesian equations of the form

$$\left|\frac{x}{A}\right|^p + \left|\frac{y}{B}\right|^p = 1, \tag{1}$$

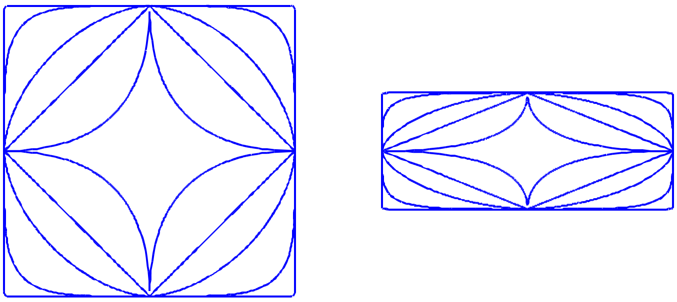

Figure 2: supercircles and superellipses

(whereby $p, A, B \in \mathbb{R}_0^+$; both cases $A = B = r$ and $A = a \neq b = B$ thus being dealt with by the same equation).

Turning to *polar co-ordinates* ρ and θ, such that $x = \rho \cdot \cos\theta$ and $y = \rho \cdot \sin\theta$, when in addition introducing a coefficient $\frac{m}{4}$ (which allows to yield more particular symmetries of rotation around O than those related to the 4 quadrants of the Cartesian co-ordinate system) and when moreover giving "independence" to the exponents occurring in the three terms of equation (1), this equation becomes

$$\rho = \left\{ \left| \frac{\cos \frac{m\theta}{4}}{A} \right|^{n_2} + \left| \frac{\sin \frac{m\theta}{4}}{B} \right|^{n_3} \right\}^{-1/n_1}, \tag{2}$$

(whereby $n_1 \in \mathbb{R}_0, n_2, n_3 \in \mathbb{R}$). Some examples, up to scale, of planar curves given by a polar equation (2), whereby in each case here $A = B$ and for which further, respectively, in case (a) : $m = 3, n_1 = 4.5$ and $n_2 = n_3 = 10$, in case (b) : $m = 4, n_1 = 12$ and $n_2 = n_3 = 15$, in case (c) : $m = 5$ and $n_1 = n_2 = n_3 = 4$ and in case (d) : $m = 7, n_1 = 10$ and $n_2 = n_3 = 6$, are shown in the following figure.

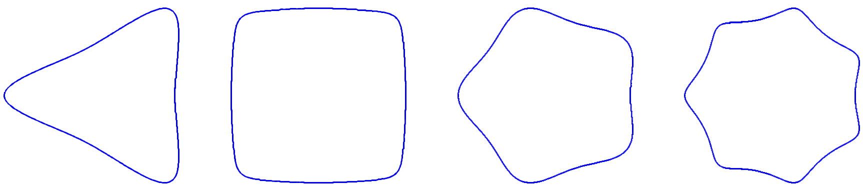

Figure 3: curves (a),(b),(c) and(d) given by equation (2)

These curves pretty accurately describe the shapes of the Nuphar luteum petiole (a) and the stems of Scrophularia nodosa (b), Equisetum (c) and Raspberry (d), respectively. Similarly, for instance different kinds of starfish correspond to curves given by equation (2) for $A = B$ and further, respectively, in case (e) : $m = 5, n_1 = 2$ and $n_2 = n_3 = 7$, and in case (f) : $m = 5, n_1 = 2$ and $n_2 = n_3 = 13$.

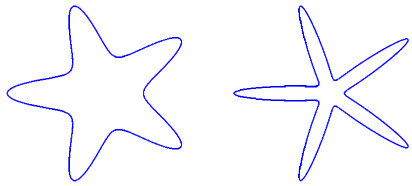

Figure 4: curves (e) and (f) given by equation (2)

The planar curves given by polar equation (2) can in some sense be interpreted as being obtained starting from *the unit circle centered at* 0 ($\rho = 1$) by the *transformation* given by the right hand side of (2), (for any choice of parameters A, B, m, n_1, n_2, n_3). And, similarly, instead of thus transforming this unit circle, all planar curves determined by polar equations $\rho = f(\theta)$, (whereby f basically can be any positive real function), can be thus transformed into the planar curves with polar equations

$$\rho = f(\theta) \cdot \left\{ \left| \frac{\cos \frac{m\theta}{4}}{A} \right|^{n_2} + \left| \frac{\sin \frac{m\theta}{4}}{B} \right|^{n_3} \right\}^{-1/n_1} . \tag{3}$$

The following figure, up to scale, shows in (b) and (c) such transformations with parameters $m = 2.5, n_1 = \frac{1}{1.3}$ and $n_2 = n_3 = 2.7$ and, respectively, $m = 2.5$ and $n_1 = n_2 = n_3 = 5$, of the rose-curve of Grandi (a) with equation $\rho = f(\theta) = \cos(2.5\theta)$ into "superroses".

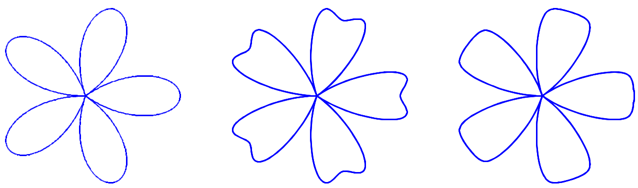

Figure 5: a rose-curve (a) and two corresponding superroses (b) and (c)

And starting for instance from the *logarithmic spiral* (a) $\rho = f(\theta) = e^{0.2\theta}$, by transformation (3) with parameter $m = 4$ and $n_1 = n_2 = n_3 = 100$ in case (b) and $m = 10$ and $n_1 = n_2 = n_3 = 5$ in case (c), respectively, up to scale, the following "*superspirals*" are obtained.

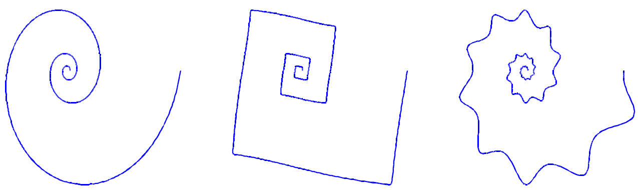

Figure 6: a logarithmic spiral (a) and two corresponding superspirals (b) and (c)

What is briefly stated above concerning *curves in 2D-planes* can readily be extended to *curves and surfaces in 3D-spaces*, (and for that matter to submanifolds of any dimensions and codimensions). Instead of supercircles and superellipses then of course "*superspheres*" and "*superellipsoids*" come into play. And, as in the case of planar curves, also for space-curves and for surfaces in space, the *geometrical transformations* thus obtained in analogy with formulae (2) and (3) provide *unifications of wide ranges of natural and abstract shapes*.

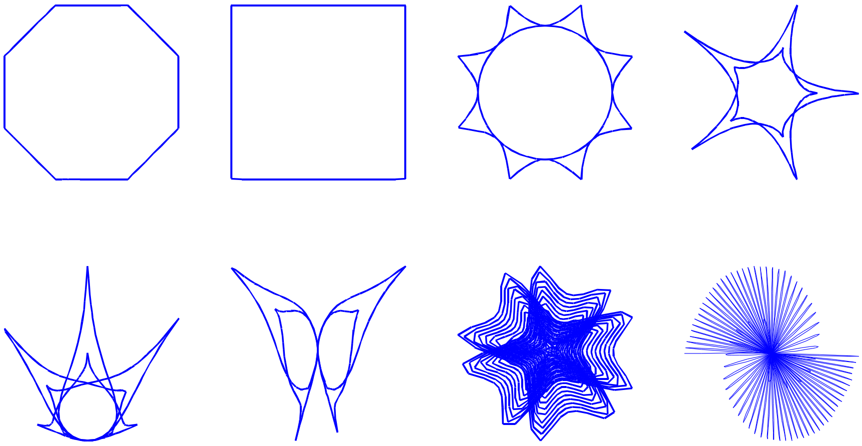

Figure 7: some further curves in 2D with equation (2)

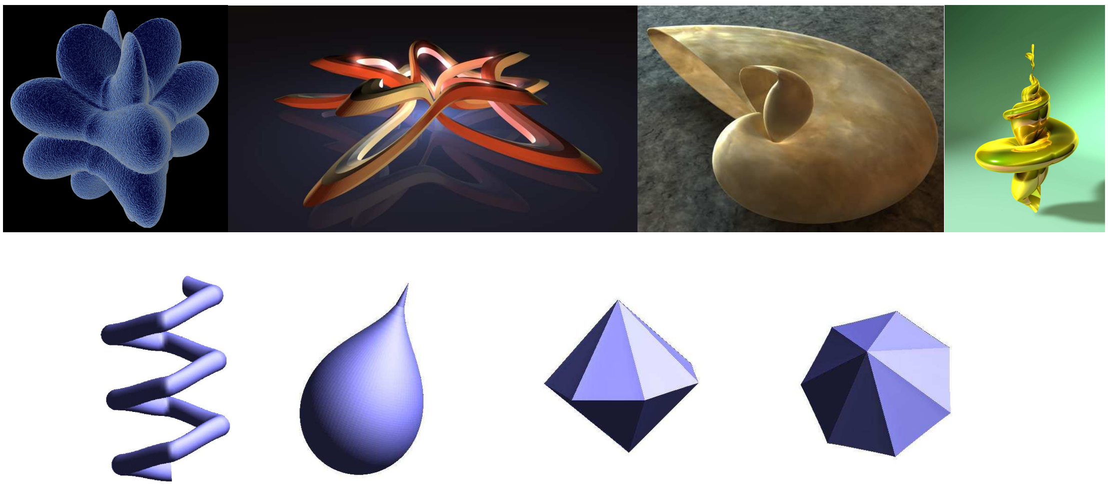

Figure 8: some analogous surfaces in 3D

For more details and for a multitude of examples of natural shapes occurring in animals and in plants, in their cells and tissues, in snowflakes and spider-webs, in crystals and in fluid-currents, etc. etc., and which are so accurately described by such transformations, either directly by one of them individually or indirectly by some natural combination of a few of them, see [2, 6, 5]. Following Piet Hein, in these publications the first named author and his coworker Bert Beirinckx used the term "*superformula*" for the equations and transformations like (2) and (3). In the last named author's Kragujevac-talk "On Geometry and Natural Philosophy", of which this article is a written echo, the transformations and the curves and the surfaces given by (2) and (3) and the like, alternatively were called Gielis-transformations, -curves and -surfaces too (to which the first author did not protest, maybe also due to the fact that he was not there). Anyway in particular in [1], the origins of the above observations were traced back to Piet Hein's "supereggs" ("superellipsoïds" of revolution) as source of inspiration in trying to geometrically describe the shapes of planar sections of bamboo stems.

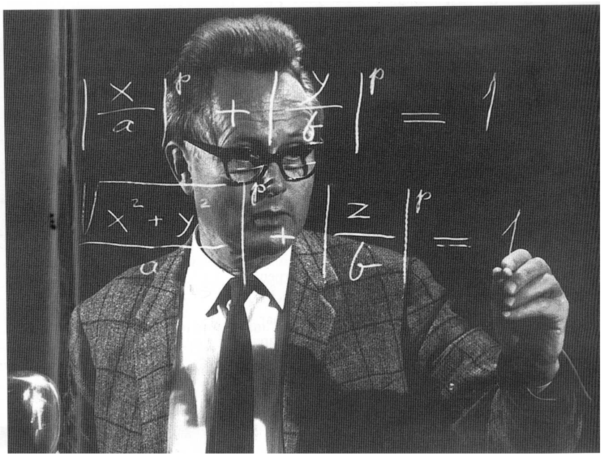

Figure 9: Piet Hein

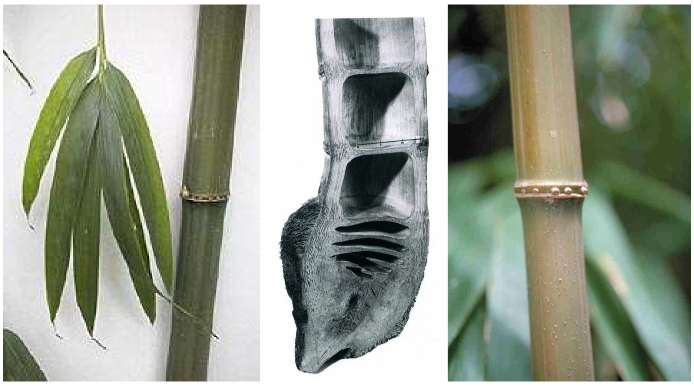

Figure 10: "square" bamboo

Also, e.g. in [7, 6] for such "superforms" occurring in 2D-planes and in 3D-spaces, arguments concerning various principles of natural optimisations are discussed, precisely in relation to their geometrical shapes, sometimes at length and sometimes more briefly. Finally, we remark that the 2D- and 3D-hyperbolic versions of the above superformula's of elliptic type indeed merit seriously to be taken in consideration as well.

2 On the geometry of "the world in which we dwell"

The purpose now is to show that, on top of the *curves* and the *surfaces* given by formulae like the ones before, which from a purely geometrical point of view satisfy natural conditions on their curvatures and which naturally do occur in the worlds of physics, chemistry, geology, biology,... and, for that matter, also in the worlds of art and technology (see e.g. [3, 4]), of all possible "things" indeed also our *relativistic space-time universe* itself is basically determined by similar curvature conditions and described by similar formulae.

The *physical space-times* which are *ideal submanifolds* in the sense of Bang-yen Chen [8, 9, 10] were recently studied by the latter-named two authors, in part also together with Franki Dillen and Mira Petrovic [11, 12]. Thus in particular some *space-time models of Georges Lemaître* (in the literature also referred to as Friedmann-Lemaître-Robertson-Walker space-times), show up like "par excellence" as ideal hypersurfaces in Minkowski spaces. These space-times also are at the origin of the so-called "big bang"-theories. In this respect, and so much more appropriately, Lemaître himself used the terms "primitive atom" and "cosmic egg", the latter one especially giving a most beautiful image of "the beginning" of "the life" of "our cosmos".

Figure 11: Professor Lemaître of the University of Leuven

From the formal point of view, a Robertson-Walker metric on $\mathbb{R}^4$ is given in local co-ordinates (x, y, z, t) by

$$ds^2 = -dt^2 + \left\{c(t)\cdot\left[1+\frac{k}{4}\left(x^2+y^2+z^2\right)\right]\right\}^{-2}\cdot\left(dx^2+dy^2+dz^2\right),\ (k=+1, 0\, or\, -1), \quad (4)$$

whereby the space-slices of the full space-time at any given time t are *Riemannian 3D-spaces of constant curvature* $c(t)$. By way of visualisation in 2D-rather than in 4D-representation, such metrics are carried for instance by surfaces of revolution in 3D-space; (and to go from the geometry of such surfaces to the Riemannian geometry of general warped products, see [13, 14]). The transformations of the flat Euclidean 3D-Pythagorean metric $ds^2 = dx^2 + dy^2 + dz^2$ into 3D-metrics of any constant curvature by multiplication with appropriate functions are essentially due to Riemann and von Helmholtz in their studies on the foundations of geometry via, as they respectively approached this, hypotheses and facts [15, 16]. Bearing in mind that, in the words of Chern [17], "While algebra and analysis provide the foundations of mathematics, geometry is at the core", it seems worthwhile from time to time to reflect somewhat on both these great texts, in particular in the light of the origins of geometry as field of mathematics based on the physical geometrical experiences that humans have by their, visual and other, sensations and perceptions. In space-time metrics like (4) one may observe, on the one hand, the conformal deformations of flat space-metrics to metrics of constant spatial curvature ($=, < or > 0$) which may change in the course of time, and, on the other hand, a hyperbolic twist in the line-direction, which further might include a speed of light-normalisation, which all in all amount to *formal deformations of the theorem of Pythagoras quite alike the above supertransformations relate to the equation of Euclidean circles or spheres.* In this context we additionally remark in connection with the so-called supercircles that already in the very first talk on his geometry, Riemann explicitly mentioned the geometric tangent $2D$-indicatrix $x^4+y^4=1$ as a relevant extension of the Euclidean circle $x^2+y^2=1$, thus conceiving of the special Riemann-Finsler metrics

$$ds = \left\{ \sum_{i=1}^{n} \left(dx^i \right)^p \right\}^{-\frac{1}{p}} ;$$

(for recent work of Chern e.a. on Riemann-Finsler geometry and of Miron e.a. on Lagrange geometry, among others in relation to Hilbert's variational problem 23, see e.g. [18]).

It could be reminded for instance that the *rotational surfaces of Delaunay*, i.e. those of constant mean curvature H, (planes and catenoïds for $H=0$, spheres and circular cylinders and unduloïds and nodoïds for $H \neq 0$), do effectively occur in various fields of the natural sciences. The geometrically natural curvature conditions alluded to before concern on the one hand, in their simplest form, the *constancy of curvatures* like the *mean curvature* H (expressing uniform surface tension) or the *Gauss curvature* K (for surfaces M in say Euclidean 3D-space, or, similarly, the sectional curvature K for general Riemannian spaces, whose constancy as shown by Riemann-von Helmholtz-Lie amounts to satisfying "the axiom of free mobility"), and thus are examples of realizations of basic symmetries or still of classical variational optima. On the other hand, the geometrical natural curvature conditions mentioned before concern the realizations of equalities in the *optimal inequalities* which are known to hold between various scalar-valued curvatures of all submanifolds. In particular, such inequalities give inevitable relations between curvatures which fundamentally relate to the *intrinsic nature* of these submanifolds (like their sectional curvatures, their scalar Riemannian curvature τ which averages the sectional curvatures, and, more generally, all their recently introduced *Chen-curvatures*, which together give like the DNA-structure of the Riemannian manifolds involved) and curvatures which fundamentally relate to the shape which these submanifolds assume in their ambient world, i.e. *extrinsic curvatures* (as being expressed by the second fundamental form h and the normal curvature tensor $R^{\perp}$, the most basic of which is the *squared mean curvature* H^2, expressing the submanifold's surface tension). For all surfaces M in Euclidean 3D-space, already Euler showed that always $K \leq H^2$ everywhere on M, equality holding at every point if and only if, according to a theorem of Meusnier, M is either planar or spherical. The so-called Chen-inequalities generalise this Euler-inequality concerning the dimensions and codimensions of the submanifolds involved, concerning the kind of ambient spaces involved and concerning the kinds of intrinsic and extrinsic curvatures involved. And a submanifold is said to be *an ideal submanifold* in some space when it actually realizes at everyone of its points the equality in such generally holding inequality. Given their intrinsic structures, which normally submanifolds have the tendency rather not to change, ideal submanifolds thus experience the least possible external curvature which might be imposed on them by the forms they assume in given surrounding worlds, much like in some sense experiencing the least possible stress which the living conditions in these ambient spaces may impose on the submanifolds, the creatures, which happen to live in these worlds. It is as such, namely as ideal Lorentz-hypersurfaces in Minkowski-spaces, that in particular also the Lemaître-universes naturally emerged in the recent studies mentioned before. The submanifold's curvature conditions discussed above, like the *constancy* or invariance of certain curvatures and the *Chen-inequalities*, as well as other natural ones like e.g. the *Willmore-conditions* [19], etc., in some sense all express variational principles (see e.g. also [20]) along the lines studied in particular by d'Arcy Thompson.

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