Heteroscedastic Errors-in-Variables Regression

Many computer vision problems can be viewed as regression problems which try to estimate a set of parameters from noisy measurements. The measurements and parameters are related by a *constraint equation* and a *noise model* characterizes the noise affecting the measurement. Examples of this include applications such as fundamental matrix estimation, trifocal tensor estimation, triangulation, motion analysis and camera pose estimation. For the purposes of this discussion we shall assume that the constraint is univariate, although the method discussed below can be extended to a multivariate constraints.

The ideal data points $\mathbf{z}_{io}, i = 1, \ldots, n$ and ideal parameters $\boldsymbol{\theta}_o$ satisfy

$$\mathbf{f}(\mathbf{z}_{io}, \boldsymbol{\theta}_o) = 0 \;\; i = 1, \ldots, n \qquad \mathbf{f}(\cdot) \in \mathbf{R}^m \quad \mathbf{z}_{io} \in \mathbf{R}^s \quad \boldsymbol{\theta}_o \in \mathbf{R}^p \tag{1}$$

In most problems, the noise affecting the measurements can be assumed to be additive. However, the errors affecting the different measurements need *not* be identically distributed. Therefore, we are given the measurements $\mathbf{z}_i$ such that

$$\mathbf{z}_i = \mathbf{z}_{io} + \delta\mathbf{z}_i \qquad \delta\mathbf{z}_i \sim GI(0, \sigma^2 \mathbf{C}_{z_i}) \qquad i = 1, \ldots, n \tag{2}$$

where, $GI(.)$ stands for a general, symmetric noise distribution whose first two central moments are available. Such data where each measurement is affected by noise of different parameters is said to be *heteroscedastic* as opposed to *homoscedastic* data for which all the $\mathbf{C}_{z_i}$ are the same. Due to the heteroscedastic noise, simple linear regression methods cannot be applied to such problems since linear regression assumes the data is homoscedastic.

In practice, heteroscedastic data occurs due to one of two reasons.

- The data vectors are supplied by different sources and are inherently heteroscedastic. An example of this is the pose estimation problem in the structure-from-motion problem of [2]. During triangulation, each 3D point is localized with a different covariance. Therefore, for pose computation the data vectors are always heteroscedastic.
- The data vectors $\mathbf{z}_i$ are nonlinear functions of a set of underlying homoscedastic measurements $\mathbf{m}_i$. Although, the $\mathbf{m}_i$ are corrupted by homoscedastic noise, since $\mathbf{z}_i$ are nonlinear functions of $\mathbf{m}_i$, the noise affecting $\mathbf{z}_i$ is heteroscedastic. This occurs more frequently and it is well known that not accounting for the heteroscedasticity of the data leads to unsatisfactory results [1].

Heteroscedastic Errors-in-Variables (HEIV) regression is a regression method which identifies the reasons why linear regression fails for such problems and handles the nonlinearities while still requiring only linear computations.

The HEIV algorithm, handles constraints with *separable parameters*. For such constraints, the constraint (1) can be factorized into two parts. The first part is a nonlinear function of the measurement vector and the second part is the parameter vector which needs to be estimated.

$$f(\mathbf{z}_{io}, \boldsymbol{\theta}_o) = \boldsymbol{\Phi}(\mathbf{z}_{io})\boldsymbol{\theta}_o = 0 \quad i = 1, \ldots, n \qquad\qquad \boldsymbol{\Phi}(\cdot) \in \mathbf{R}^{m \times p} \tag{3}$$

A wide array of computer vision problems fall into this category [1].

The HEIV algorithm is an iterative procedure. At each iteration it is assumed that estimates for the correct data points, $\check{\mathbf{z}}_i$, are available from the previous iteration. At the first iteration these values are initialized to the given measurement vectors.

Given the values of $\check{\mathbf{z}}_i$ and $\check{\boldsymbol{\theta}}$ from the previous iteration, compute the covariance of $\mathbf{f}(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}})$ by error propagation as

$$\mathbf{C}_f(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}}) = \mathbf{J}_{f|\hat{z}_i}(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}})^T \mathbf{C}_{z_i} \mathbf{J}_{f|\hat{z}_i}(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}}) \tag{4}$$

where $\mathbf{J}_{f|\hat{z}_i}(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}})$ is the Jacobian of the constraint (3) at $\check{\mathbf{z}}_i$. Now define the *weighted scatter matrix*, $\mathbf{S}(\check{\boldsymbol{\theta}})$ as

$$\mathbf{S}(\check{\boldsymbol{\theta}}) = \textstyle\sum_{i=1}^{n} \boldsymbol{\Phi}(\mathbf{z}_i)^T \mathbf{C}_f^+(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}}) \boldsymbol{\Phi}(\mathbf{z}_i). \tag{5}$$

In the above equation, $\mathbf{C}_f^+$ denotes the pseudo-inverse of the matrix $\mathbf{C}_f$. Compute the *weighted covariance matrix*, $\mathbf{C}(\check{\boldsymbol{\theta}})$ as

$$\mathbf{C}(\check{\boldsymbol{\theta}}) = \textstyle\sum_{i=1}^{n} (\boldsymbol{\eta}_i \otimes \mathbf{I}_p)^T \mathbf{C}_\varphi(\check{\mathbf{z}}_i)(\boldsymbol{\eta}_i \otimes \mathbf{I}_p) \tag{6}$$

$$\mathbf{C}_\varphi(\check{\mathbf{z}}_i) = \mathbf{J}_{\varphi|\hat{z}_i}(\check{\mathbf{z}}_i)^T \mathbf{C}_{z_i} \mathbf{J}_{\varphi|\hat{z}_i}(\check{\mathbf{z}}_i) \tag{7}$$

where, $\otimes$ denotes the Kronecker product, $\boldsymbol{\varphi} = \mathbf{vec}(\boldsymbol{\Phi}^T)$ and the Lagrange multipliers are given by

$$\boldsymbol{\eta}_i = \mathbf{C}_f^+(\check{\mathbf{z}}_i, \check{\boldsymbol{\theta}}) \mathbf{f}(\mathbf{z}_i, \check{\boldsymbol{\theta}}). \tag{8}$$

Now, solve the generalized eigenproblem,

$$\mathbf{S}(\check{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} = \lambda \mathbf{C}(\check{\boldsymbol{\theta}})\hat{\boldsymbol{\theta}} \tag{9}$$

to obtain $\hat{\boldsymbol{\theta}}$, which is an improved estimate of $\boldsymbol{\theta}$. The corrected measurement estimates after the current iteration are,

$$\hat{\mathbf{z}}_i = \mathbf{z}_i - \mathbf{C}_{z_i} \mathbf{J}_{f|\hat{z}_i}(\check{\mathbf{z}}_i, \hat{\boldsymbol{\theta}}) \mathbf{C}_f(\check{\mathbf{z}}_i, \hat{\boldsymbol{\theta}})^+ \mathbf{f}(\mathbf{z}_i, \hat{\boldsymbol{\theta}}) \tag{10}$$

The smallest eigenvalue of the generalized eigenproblem of 9 converges to one from below. The iteration is stopped when this value is close enough to one. Convergence usually occurs within three or four iterations.

The HEIV algorithm has been applied to numerous problems and has been shown to give results which are at least as good as various nonlinear methods such as Levenberg-Marquardt (LM). Here we show the result of HEIV regression applied to ellipse fitting.

Ellipse fitting is the problem of finding the best ellipse to fit a given set of points. The results of using various estimators to fit an ellipse to a set of image points is shown in Figure 1. Note that

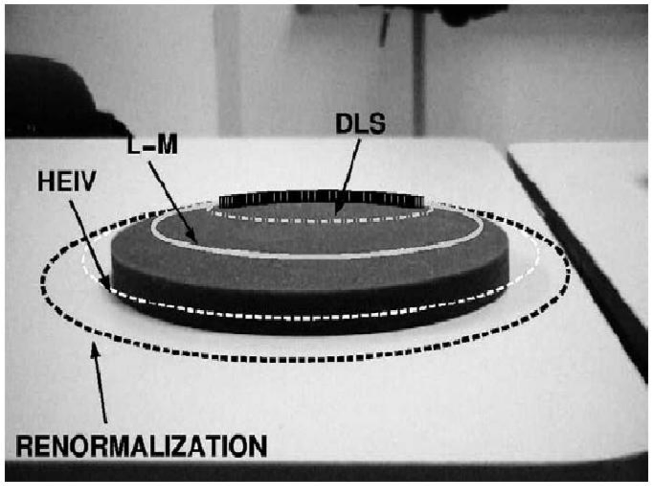

Figure 1: Results of ellipse fitting

all the points lie on a small segment of the ellipse with low curvature. This makes the estimation much harder than if the points were distributed all along the circumference of the ellipse. Both the Direct Least Squares (DLS) and Levenberg-Marquardt (LM) estimators are biased towards smaller ellipses. Renormalization tries to account for this bias but uses small noise assumptions and breaks down in practice. The estimate returned by the HEIV algorithm is closest to the true ellipse.

A thorough analysis of the HEIV estimator, its propoerties and its applications can be found in [1].

References

[1] B. Matei, *Heteroscedastic Errors-In-Variables Models in Computer Vision.* PhD thesis, Department of Electrical and Computer Engineering, Rutgers University, 2001. Available at `http://www.caip.rutgers.edu/riul/research/theses.html`.

[2] R. Subbarao, Y. Genc, and P. Meer, "A balanced approach to 3d tracking from image streams," in *Proc. IEEE and ACM International Symposium on Mixed and Augmented Reality*, October 2005, pp. 70–78. Available at `http://www.caip.rutgers.edu/riul/research/hetero.html`.