Mean Shift on Lie Groups

The mean shift algorithm is a nonparametric clustering technique which does not require prior knowledge of the number of clusters, and does not constrain the shape of the clusters. The major limitation of the original mean shift algorithm is that it can only be applied to points in Euclidean spaces. Some of the common parameter spaces that are widely used in computer vision are not Euclidean spaces, but either elements of matrix Lie groups, eg. rigid motion, affine motion, planer homography, or can be represented as product of matrix Lie groups, eg. fundamental matrix. Clustering on these parameter spaces is performed by extending mean shift to Lie groups.

A Lie group is a group G with the structure of an analytic manifold such that the group operations (multiplication and inverse) are analytic. Let T_X be the tangent space to the manifold at point X. The local neighborhood of X can be described by its tangent space T_X. The tangent space to the identity element of the group T_e forms a Lie algebra which is denoted by g. The exponential map, $\exp : g \to G$ maps the vectors in the Lie algebra to the Lie group.

Matrix Lie groups are all the subgroups of the general linear group $\mathbf{GL}(n, R)$ which is the group of nonsingular square matrices. The group operation, matrix multiplication, is associative and every nonsingular matrix has an inverse.

Matrix groups are probably the most well known examples of Lie groups. The exponential map of a matrix is defined by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k. \tag{1}$$

The inverse map

$$\log(X) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (X - e)^k \tag{2}$$

can be defined only on a neighborhood of identity e. When X is distant from the identity element of the group, the series fails to converge.

The distances on manifolds are defined in terms of minimum length curves between points on the manifold. The curve with the minimum length is called the *geodesic* and the length of the curve is the *intrinsic distance*. The intrinsic distance between any two group elements is computed by

$$d(X, Y) = \|\log(X^{-1}Y)\|. \tag{3}$$

Similar to kernel density estimation on Euclidean spaces, the density estimator at a point on the Lie group, X, is

$$f(X) = \frac{c_{k,d}}{nh^d} \sum_{i=1}^{n} k\left(\left\| \frac{\log(X^{-1}X_i)}{h} \right\|^2 \right) \tag{4}$$

where $k(s)$ is the kernel profile.

Let $\{X_i\}_{i=1..n}$ be the data points on the group. The mapping

$$x_i = \log(X^{-1}X_i) \tag{5}$$

transforms the data points to the Lie algebra and X to 0. Using (4) and (5), the mean shift vector at location X is defined as

$$m_h(x) = \frac{\sum_{i=1}^{n} x_i g\left(\left\|\frac{x_i}{h}\right\|^2\right)}{\sum_{i=1}^{n} g\left(\left\|\frac{x_i}{h}\right\|^2\right)} \tag{6}$$

where $g(s) = -k'(s)$. The mean shift vector is on the Lie algebra. The vector is transformed to the Lie group and the location of X is updated as

$$X = X\exp(m_h(x)). \tag{7}$$

Starting at a data point and iteratively updating the location with the mean shift vector, point converges to a local mode of the distribution. The algorithm is illustrated in Figure 1.

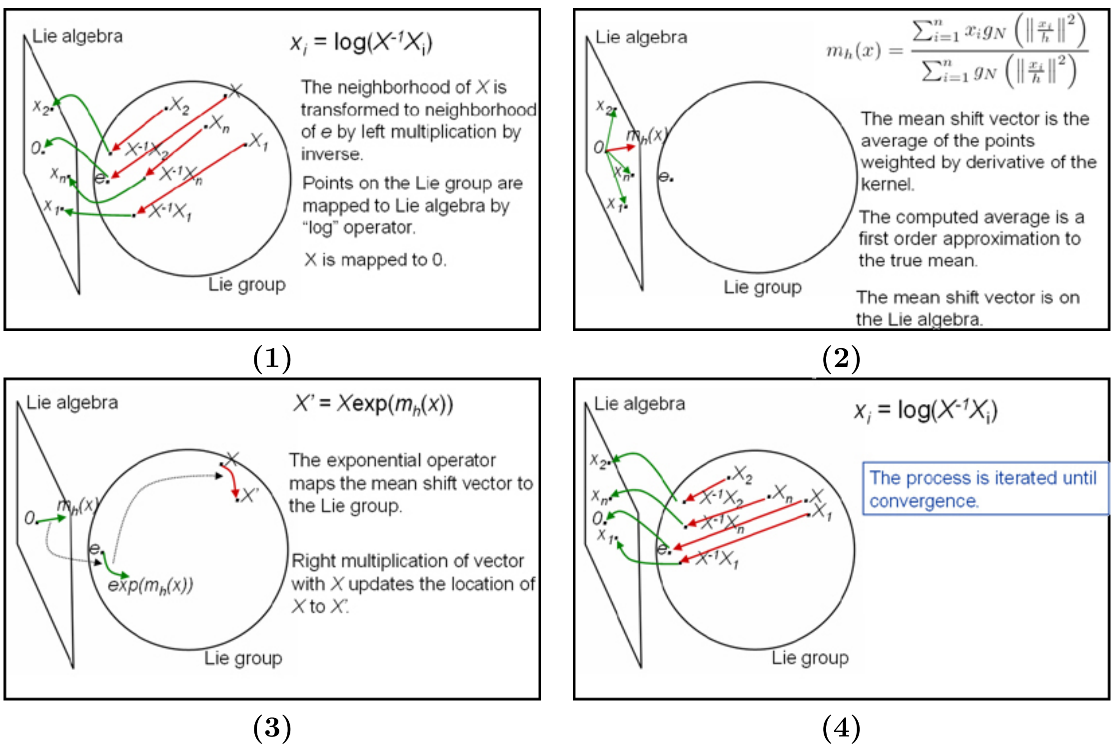
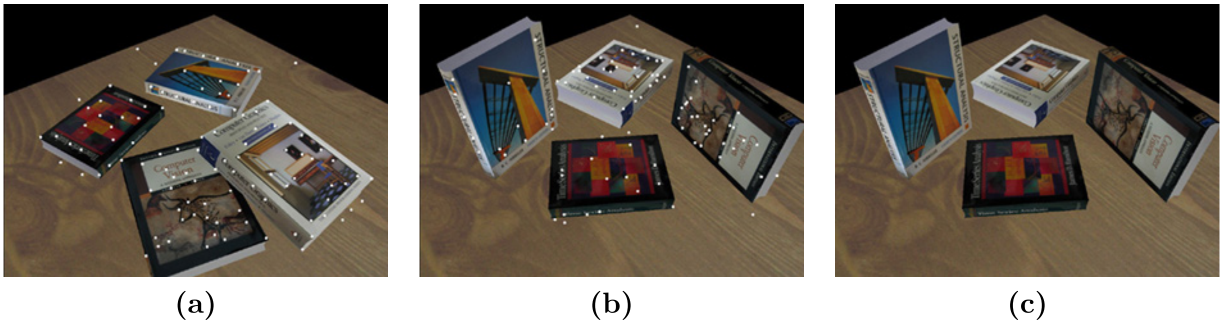


Figure 1: **Mean shift on Lie groups.**

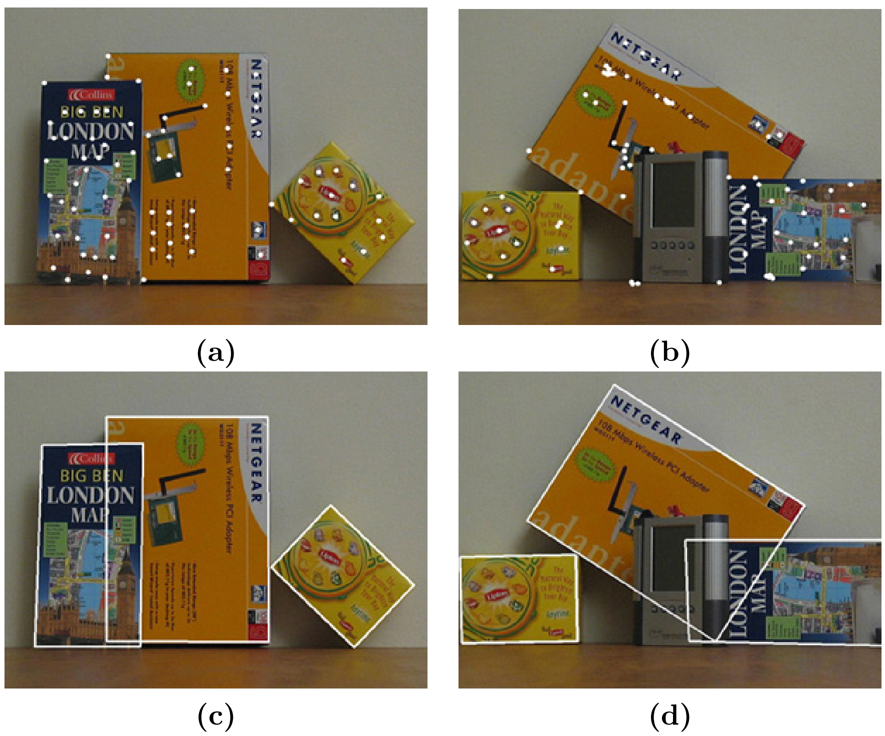
Figure 2 and 3 show two examples of mean shift clustering on Lie groups. Motion hypothesis are generated using elemental subsets of point correspondences and mean shift clustering is performed on the generated hypothesis. The modes of the distribution correspond to the motion parameters.

More details on mean shift clustering on Lie Groups can be found in [1].



(a) Original scene. (b) Transformed scene. (c) Reconstructed scene from (a) with the estimated motion parameters.

Figure 2: **Clustering on 3D rigid motion distribution.**



(a) Original scene. (b) Transformed scene. (c) The boundaries of the bodies. (d) Transformed boundaries with the estimated motion parameters.

Figure 3: **Clustering on 2D rigid motion distribution.**

References

[1] O. Tuzel and P. Subbarao, R. Meer. Simultaneous multiple 3d motion estimation via mode finding on lie groups. In *Proc. 10th IEEE Intl. Conf. on Computer Vision,* Beijing, China, volume 1, pages 18–25, 2005. Available at `http://www.caip.rutgers.edu/riul/research/papers/pdf/liems.pdf`.