Critical Configurations of Lines to Geometry Determination of Three Cameras

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Abstract

We address the question of, what structure of a set of lines in space constitutes a critical configuration to three generally positioned cameras. By critical configuration, it is meant a set of lines in space whose image projections to the cameras do not allow unique determination of the cameras' relative positions in space. We approach the question by looking into the trifocal tensor of the cameras, a quantity tightly related to the camera geometry. We focus on structures of lines in space that are common in reality – the linear structures – and examine which of them would not allow the tensor to be uniquely determined. Linear structures of lines include linear ruled surface, linear line congruence, and linear line complex, and more specifically line pencil, point star, and ruled plane. We offer a summary of by how much such families of line set are shy of the general enough structure for a unique determination of the tensor. We also point out how many lines need be visible minimally in each of them, for the full information of the associated structure to be revealed in the image data. We present synthetic and real image results to confirm the findings. The findings are important to the validity and stability of algorithms related to structure from motion and projective reconstruction using line correspondences.

1. Introduction

Determining the relative geometry of a number of camera positions in space is an indispensable problem in computer vision and robotics. Should the camera positions be about different cameras fixated in space, the information is a pre-requisite of relating and in turn fusing the visual data captured by the respective cameras. Should they be about the same camera moving in space, the information is then about the motion of the camera and in turn of the person or vehicle that is holding it. Both cases represent important applications of vision and robotics.

One approach of solving the problem is to let the cameras capture image data of the surroundings, and seek to determine the geometry from such image data. On the approach, much work [4, 5] has been devoted to the case that the image data are considered as image projections of a set of points in space. In particular, what point set in space can or cannot allow the geometry to be uniquely determined, how many points are minimally needed, and how the geometry can be uniquely determined etc. have been extensively studied.

In comparison, less work has been devoted to the case that the image data are about image projections of a set of lines. It is known that the geometry can indeed be recovered from image observations of a line set, as long as the line set is of a structure general enough. However, only limited works are present in the literature on what line structures are *critical configurations* to the determination of camera geometry. Here by critical configuration, it is meant line structures that do not allow the relative geometry of the cameras to be uniquely determined from the image data.

Lines are in many circumstances as amply available as points, especially in the urban environment. There is even evidence [8] that 3-D determination from lines could be more accurate than from points.

This work aims at offering a comprehensive analysis

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Figure 1. Examples of scenes consisting of line structures that are of the class point-star and ruled plane respectively.

of what line sets are critical structures to three generally positioned cameras. We focus on linear line structures, which encompass linear ruled surface, linear line congruence, and linear line complex, and more specifically, line pencil, point star, and ruled plane. Examples of such line structures in real life are shown in Fig. 1. These critical line structures are not uncommon in reality, and thus the work is important to the validity and stability of algorithms related to structure from motion and projective reconstruction using line correspondences.

In all cases, we assume that the three camera positions are general with respect to the set of lines, meaning that the images are not degenerate images like those having multiple lines in space projected to the same image line, a result of accidental alignment.

We approach the problem from the perspective of the trifocal tensor of the cameras, a $3\times3\times3$ quantity closely related to the cameras' relative geometry. Given image projections of a set of say N lines, a $2N \times 27$ matrix we refer to as the *tensor estimation matrix* can be constructed, which constrains the values of the trifocal tensor through a homogeneous system of linear equations. Whether the trifocal tensor can be uniquely determined then depends upon whether the estimation matrix is of enough rank or not. The facts that the trifocal tensor consists of altogether 27 scalar values, and that there are always 9 nonlinear constraints (called the admissability constraints in the literature) relating the entries of the tensor (the constraints include that the trifocal tensor is defined up to arbitrary nonzero scale), together imply that should the rank of the estimation matrix be less than $27-9=18$ the trifocal tensor cannot possibly have a unique solution.

Our findings include that the rank of the estimation matrix is reduced to $7, 11, 15$ if the observed lines come from a line pencil, a point-star, and a ruled plane respectively, which are line families belonging to linear line space; and $12, 19, 23$ if the lines come from a general linear ruled surface, a general linear line congruence, and a general linear line complex, which are subclasses of linear line structures.

We also offer a summary of how many lines need be visible minimally in each of the above line families for the full information of the associated line structure to be revealed in the image data. We present synthetic and real image results to confirm the findings.

2. Previous Work

As discussed, in camera motion determination the ambiguity issue possibly involved in the use of line correspondences has not been given as much attention as to the use of point correspondences. Buchanan [2, 3] pointed out that $(3, 6, 5)$-congruence is a critical line set, and suggested a few line sets that defeat the Liu-Huang algorithm [9]. Maybank [11] used Semple's representation to describe lines so as to let line congruence be parameterized by surfaces of lower degree. However, Semple's representation does not have a direct way of expressing the projection of 3-D lines to image lines. Navab and Faugeras [12] addressed the analysis of the critical sets of lines in the Euclidean domain using two equations from Liu and Huang [9] and related the degeneracy analysis to the Liu-Huang algorithm. The treatment was however in terms of geometric primitives directly related to individual camera positions and orientations, not the collective quantity – the trifocal tensor. We believe there is advantage in treating the degeneracy issue in terms of trifocal tensor. In particular, the rank of the estimation matrix involved in the trifocal tensor's determination could be a measure of the degeneracy level of the observed line set, and the parallel measure in direct geometric primitives is not as accessible. [10] enabled global geometric analysis for multiple images. It made it possible to utilize all incidence conditions involving point and line features of all images simultaneously for consistent recovery of structure and motion from multiple views.

The work of Stein and Shashua [14] was among the very few that looked at the degeneracy of line structures from the viewpoint of trifocal tensor. However, the work focused only on the structure of linear line complex (LLC). It pointed out that the tensor estimation matrix has the rank reduced to 23 if the observed line set belong to an LLC.

This work aims at enumerating all linear line structures that are critical, and pinpointing how critical or how under-determining they are in terms of constraining the trifocal tensor.

3. Preliminaries and Notations

Plücker line coordinates. Given two 3-D points $\mathbf{M^T} \sim (\mathbf{\bar{M}^T}|m)$ and $\mathbf{N^T} \sim (\mathbf{\bar{N}^T}|n)$ of a 3-D line, the Plücker line coordinate is the homogeneous 6-vector $\mathbf{L} \sim (\mathbf{a^T}|\mathbf{b^T})^T$, where $\mathbf{a} = \mathbf{\bar{M}} \times \mathbf{\bar{N}}$, $\mathbf{b} = m\mathbf{\bar{N}} - n\mathbf{\bar{M}}$, and $\mathbf{a}, \mathbf{b}$ satisfy the *Plücker equality*: $\mathbf{\Omega_{L,L}} = \mathbf{a}^T\mathbf{b} = 0$. With this, a line in 3-D can be represented as a point that lies on a non-singular quadric $\mathbf{\Omega}$ in $\mathcal{P}^5$, a representation called the *Klein model* [7].

Camera line projection matrix. The 3×6 line projection matrix [1] that projects 3-D line (in Plücker coordinates) to image line is $\mathbf{\tilde{P}} \sim [det(\mathbf{\bar{P}})\mathbf{\bar{P}^{-T}}|[\mathbf{p}]_\mathbf{x}\mathbf{\bar{P}}]$, where $\mathbf{P} \sim (\mathbf{\bar{P}}|\mathbf{p})$ is the camera matrix.

4. Linear Line Structures

General 3-D lines form a quadruply infinite system P^4 [13]. Specific family of lines that possess 1 degree of freedom (DoF), 2 DoFs, and 3 DoFs are respectively called ruled surface, line congruence, and line complex. In particular, if the constraints that reduce the DoFs of the line set are all linear, the above line structures are called linear. In this work, we restrict our discussion to **linear line structures**, i.e., we focus on *linear ruled surface*, *linear line congruence*, and *linear line complex*.

Linear line congruence is defined by two independent equations $\mathbf{a}_1^T\mathbf{L} = \mathbf{a}_2^T\mathbf{L} = 0$. Linear line complex (LLC) is the 3-D linear manifold of lines defined by one linear equation $\mathbf{a}_1^T\mathbf{L} = 0$. If the 6-vector $\mathbf{a}_1$ satisfies the Plücker equality ($\mathbf{\Omega}_{\mathbf{a}_1,\mathbf{a}_1} = 0$), the LLC is called special; otherwise, the LLC is called general. The degeneracy study of Stein and Shashua [14] was about special LLC.

On top of the above, if the Plücker equality happens to be linear, the linear line structures are reduced to the **linear line space**. In such a case, linear ruled surface becomes *line pencil*, and linear line congruence becomes *point-star* or *ruled plane*.

Sketches of the line structures are shown in Fig. 2 for illustration.

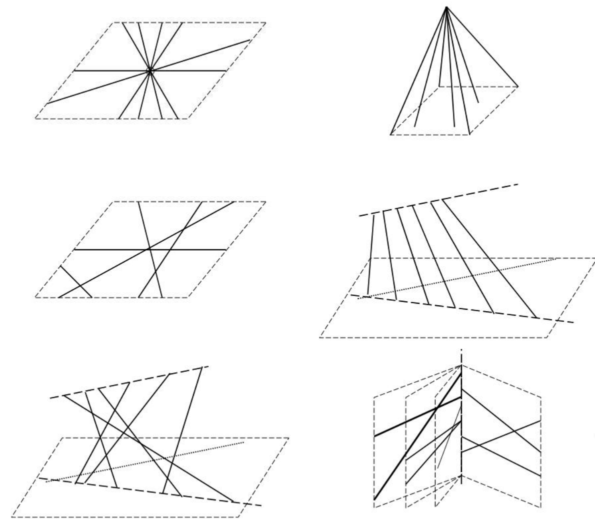

Figure 2. Illustration of the various linear line structures. Top to bottom, left to right: linear line spaces including line pencil, point-star, and ruled plane; linear line structures including linear ruled surface, linear line congruence, and linear line complex.

5. Critical Configurations of Line Structure in 3-View Analysis

Image Line Matrix $\mathbf{\bar{K}}$. For any 3-D line $\mathbf{L}$ visible in three given views, its image projections $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ in the three views are related to $\mathbf{L}$ in Plücker coordinates by:

$$\mathbf{\bar{K}} = \begin{bmatrix} \mathbf{l} \\ \mathbf{l}' \\ \mathbf{l}'' \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{P}} \\ \mathbf{\tilde{P}}' \\ \mathbf{\tilde{P}}'' \end{bmatrix} \mathbf{L} \tag{1}$$

where $\mathbf{\tilde{P}}, \mathbf{\tilde{P}}', \mathbf{\tilde{P}}''$ are the three cameras' line projection matrices, $\mathbf{L}$ is a 6×1 matrix, $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ will be each a 3×1 matrix. Here the matrix $\mathbf{\bar{K}}$ represents an encapsulation of all image data in the three views about the 3-D line, and can be termed the *image line matrix*. If data about N lines are included into the above equation (which we hereafter assume), $\mathbf{L}$ will become a $6 \times N$ matrix, and $\mathbf{l}, \mathbf{l}', \mathbf{l}''$ will be each a $3 \times N$ matrix, and the image line matrix $\mathbf{\bar{K}}$ a $9 \times N$ matrix.

Linear Constraints for Trifocal Tensor. Each line correspondence over three views gives two independent linear equations for the trifocal tensor [6] whose entries can be listed out in a column fashion as a 27×1 vector $\mathbf{t}$. Given N line correspondences, we have $2N$ linear equations for $\mathbf{t}$, which can be collectively captured by a $2N \times 27$ matrix $\mathbf{A}$. This matrix $\mathbf{A}$ is the aforementioned tensor estimation matrix. The tensor-equivalent vector $\mathbf{t}$ and the estimation matrix $\mathbf{A}$ are related by $\mathbf{At} = \mathbf{0}$.

Notice that the tensor estimation matrix $\mathbf{A}$ is totally dependent on the image line matrix $\mathbf{\bar{K}}$ in the following way:

$$\mathbf{l}'^T[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]\mathbf{l}''[\mathbf{l}]_\times = \mathbf{0}^T \tag{2}$$

where $\mathbf{T}_i$, $i = 1, 2, 3$ are the matrix decompositions of the trifocal tensor. In other words, the determination of the trifocal tensor depends upon $\mathbf{A}$, which in turns depends upon $\mathbf{\bar{K}}$.

Through an extensive analysis which involves intermediate results expressible as a number of lemmas and theorems, we have come to the findings that are summarized in Table 1. In brief, the analysis proceeds in this manner. We show that if the $\bar{\mathbf{K}}$ matrix defined by the line correspondences owns respectively certain rank properties, the rank of the estimation matrix $\mathbf{A}$ will be capped accordingly at certain values. We then show that the respectively required properties of the $\bar{\mathbf{K}}$ matrix are indeed there should the line correspondences come from the various line structures mentioned above.

Retrieving trifocal tensor in critical configurations. Notice that the tensor estimation matrix from the observations of linear line congruence and linear line complex is under-ranked but the rank is no less than 18. It means that though the estimation matrix $\mathbf{A}$ itself does not allow a unique determination of the trifocal tensor, but if the 9 admissibility constraints of the tensor are brought in, the tensor can still be determined uniquely.

Line transfer between Views. Empirical experimentation also indicates the following. While the trifocal tensor determined from any degenerate line structure $\mathbf{L}$ are not unique and cannot be used to transfer general lines (more specifically, lines beyond the degenerate line structure $\mathbf{L}$ itself) from any two views of the input triplet of views to the third view, the tensor can be used to transfer lines that are within the line structure $\mathbf{L}$.

Due to space limit, here we cannot possibly include the derivations here; interested readers can refer the details to [15]. In this article, we choose to present empirical experimentation on both synthetic and real image data to prove the findings.

Table 1. A classification of linear line structures according to the rank of the tensor estimation matrix.

Line structure	Rank	Min. No. lines
Line pencil	7	4
Point star	11	6
Ruled plane	15	8
Linear ruled surface	12	6
Linear line congruence	19	10
Linear line complex	23	12

6. Experimental Results

Massive experimentation on synthetic and real image data have been conducted to verify the findings summarized in Table 1. Here we present two sets of results for illustration.

Ruled Plane. A ruled plane structure was generated in 3-D, and three views of 50 lines of it were created and they are shown in Fig. 3(a) respectively. Fig. 3(b) shows the logarithmic values of all singular values for the line structure. Plain visual check already shows that the rank of the estimation matrix was no more than 15.

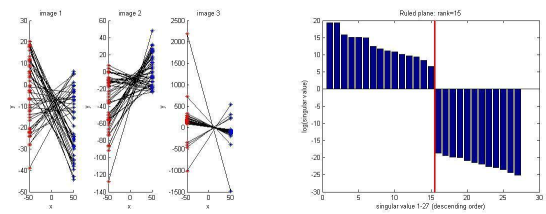

Figure 3. Three views of 50 lines that belong to a ruled plane: rank$(\mathbf{A}) \leq 15$; (a) the three views; (b) magnitudes of all singular values of the tensor estimation matrix.

Point Star. In a real image experiment we constructed a scene that was composed of a cube and square board, with square grids covering every surface of the scene. The images were captured by a consumer-grade camera (Fuji FinePix S602 Zoom camera) at 2048×1536 resolution. Image lines were precisely located by having two widely separated grid-corner points identified on each of them, and line correspondences were manually established carefully. The grid-corner points came from four different planes: three faces of the cube and one square board behind the cube. Lines and subsequently line correspondences could be established from points within a plane as well as across different planes. Three views of 15 lines of a point star were captured as displayed in Fig. 4(a),(b),(c). Fig. 4(d) shows the singular values in logarithmic form. The rank of the estimation matrix $\mathbf{A}$ was indeed no more than 11, as predicted by the findings.

7. Conclusion and Future Work

We have offered a rank classification of the structure of a set of lines in space in regard to the determination of the trifocal tensor from three views of the lines. An unprecedented study, it lays down what linear line structures in space represent critical configurations to

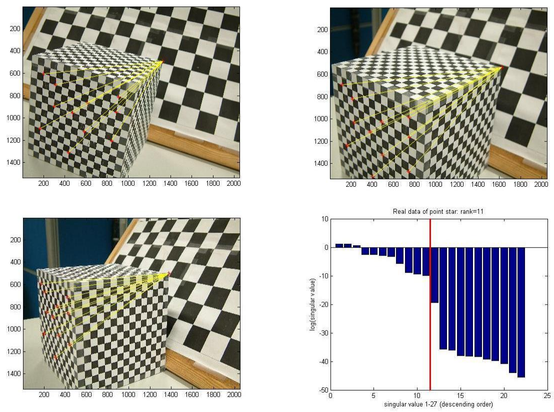

Figure 4. Three views of 11 lines of point star: rank(A) should be no more than 11: (a),(b),(c): the three views, (d) the singular values of the tensor estimation matrix.

the task of determining the camera geometry, how critical they are, and how many lines minimally need be visible for the full information of each of the structures to be revealed in the image data. Experimental results on synthetic and real image data prove empirically that the findings are valid. The analysis on linear line structures lays down a framework for the analysis on structures beyond linear. Possible future work includes extension of the study for nonlinear line structures.

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