Zero-thresholding of KH and KJ sign images for multiple scale segmentation of surfaces

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Abstract

This paper compares the KH and KJ sign images in the context of the zero-thresholding at single and multiple scales. It points out that consistent zero-thresholding of curvatures remains necessary for multiple scale surface segmentation. Even though KJ sign image is a good choice for single scale surface segmentation with a zero-thresholding formula $\epsilon_J = \epsilon_K$*, the KH sign image is a better choice for multiple scale surface segmentation, whose zero-thresholding formula* $\epsilon_K \geq 2|H|\epsilon_H + \epsilon_H^2$ *maintains effectiveness for both KH and KJ sign images.*

Keywords

Zero-thresholding, KH and KJ sign image, Gaussian and mean curvature, multiple scale segmentation.

1 Introduction

Surface segmentation is an important stage in early vision processing. It can produce symbolic descriptions of surfaces for later processing, such as surface reconstruction or recognition.

Apparently, descriptions of segmented surfaces depend on the given shape category, which is determined by some geometrical properties of surfaces. They also depend on the spatial scales at which segmentation is implemented since the significance of surface shapes, like other surface features, may change at different scales. Surface segmentation at multiple scales is therefore necessary.

As the stringent properties of surfaces, curvatures play a fundamental role in surface segmentation. A simple combination of the principal curvature signs can classify surfaces into six shape types: *flat, peak, pit, ridge, valley and saddle* as illustrated in Tab. 1.

A more sophisticated combination of principal curvatures, which makes an implicit comparison of both principal curvatures in the mean curvature H,

Table 1: Surface shapes from the C_1 and C_2 curvature signs.

C_1 \ C_2	$-$	0	$+$
$-$	peak	ridge	saddle
0	ridge	flat	valley
$+$	saddle	valley	pit

Table 2: Surface shapes from the K and H curvature signs.

K \ H	$-$	0	$+$
$-$	saddle ridge	minimal	saddle valley
0	ridge	flat	valley
$+$	peak	(none)	pit

leads to the KH sign image:

$$K = C_1 \cdot C_2 \tag{1}$$

$$H = \frac{1}{2}(C_1 + C_2) \tag{2}$$

It classifies surface shapes into eight types: *flat, peak, pit, ridge, valley, saddle ridge, saddle valley* and *minimal* as illustrated in Tab. 2, where the ninth combination $H = 0,\ K > 0$ is excluded as an impossible case since $H = 0$ implies $C_1 = -C_2$, leading to $K < 0$. Unfortunately, this theoretically impossible case may occur in practical processing as the "phantom shape" if improper zero-thresholdings of K and H are applied. An investigation on this problem[5] resulted in the consistent zero-thresholding inequality of K and H.

Certainly, the "phantom" shape type can be replaced with a new shape type *sphere* by appealing to a non-linear transformation from K-H plane to the K-J plane as shown in Fig. 1 and Fig. 2 in the next section, or by using a series of complex mappings from C_1-C_2 plane to K-J plane as discussed in [4].

Although the adoption of the *sphere* type has little substantial effect in segmentation results in the presence of data noise and discretisation errors, the removal of the "phantom" type could be significant for zero-thresholdings of curvatures. It raises a question: *whether the KJ sign image makes the consistent zero-thresholding redundant?*

This paper gives a negative answer by comparing the KH and KJ sign images in the context of the zero-thresholding at both single and multiple scales.

2 KH sign image vs. KJ sign image

It will be more intuitive if we represent the KH sign classification in the K-H plane[1] as in Fig. 1:

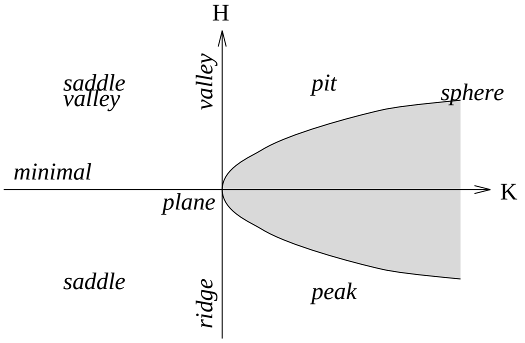

Figure 1: Surface shapes represented in the K-H plane.

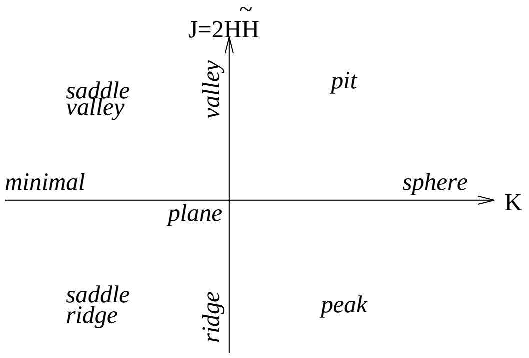

Figure 2: Surface shapes represented in the K-J plane.

From the definitions of K and H, it is easy to see that they subject to $H^2 \geq K$. Therefore, a shaded region which denotes the forbidden area $H^2 < K$ must appear on the K-H plane, This open region has the parabolic curve $H^2 - K = 0$ as its boundary which corresponds to the shape type *sphere*. This region also covers the positive K axis which corresponds to the "phantom shape" $(H = 0,\ K > 0)$.

By defining a functional

$$\Phi(\alpha) = H^2 - \alpha K \qquad \alpha \in [0,\ 1] \tag{3}$$

the shaded region and its boundary can be described as a family of parabolic curves in the form:

$$\Phi(\alpha) = 0 \qquad \alpha \in [0,\ 1],\quad K \in [0,\ \infty) \tag{4}$$

where $\Phi(1) = 0$ is the parabolic boundary curve $H^2 = K$, and $\Phi(0) = 0$ is the positive K axis $H = 0,\ K \geq 0$, *i.e.*, a degenerated parabolic curve.

As for those parabolic curves $\Phi(\alpha) = 0, \alpha \in (0,\ 1]$, note that $\Phi(\alpha)$ is continuous, and monotonic with respect to α for any $K \geq 0$. Thus, if under a transformation from (K, H) to (K, J), the boundary curve $\Phi(1) = 0$ and the positive K axis $\Phi(0) = 0$ $(K \geq 0)$ in the K-H plane are mapped to the same positive axis in the K-J plane: $J = 0,\ K \geq 0$, so will be the whole shaded region $\Phi(\alpha) = 0,\ \alpha \in [0,\ 1)$. Therefore, the whole forbidden region will be transformed to null in the new plane and the "phantom shape" type will be replaced with the *sphere* shape type as illustrated in Fig. 2.

This task can be achieved by a simple (K, H) to (K, J) transformation: $J^2 = 4\ \Phi(0) \cdot \Phi(1)$, that is

$$J^2 = 4\ H^2 \cdot (H^2 - K) \tag{5}$$

or

$$J = 2H\tilde{H} \tag{6}$$

where

$$\tilde{H} = \sqrt{H^2 - K} \tag{7}$$

It is just the same as the transformation obtained by Besl in [4], using a series of complex valued mappings from the C_1-C_2 plane to the K-J plane, where $\tilde{H}$ is presented as ΔH:

$$\Delta H = \tilde{H} = \frac{1}{2}(C_1 - C_2) \geq 0 \tag{8}$$

3 Zero-thresholding at single scale

For the KH sign image, a perturbation analysis in [5] has made explicit the relationship between the zero thresholds of H and K (see Appendix), giving the consistent KH zero-thresholding inequality below:

$$\epsilon_K \geq 2|H|\epsilon_H + \epsilon_H^2 \tag{9}$$

By this formula, a zero-threshold value ϵ_K can be yielded from the zero-threshold value ϵ_H in order to prevent the "phantom shape" $(K > 0,\ H = 0)$ from occurring.

[1] Obviously, such a representation will also involve in the magnitudes of curvatures rather than their signs alone as illustrated by the shaded region in Fig. 1.

Note that the major advantage of the KJ sign image is the removal of the "phantom shape" from the surface shape category. The consistency required between different zero-thresholds is therefore relaxed along with this removal. As J and K have the same dimension, it is hopeful that a single zero-threshold could be shared by both J and K, such as,

$$\epsilon_J = \epsilon_K \tag{10}$$

However, this is only a conjecture since the fact that K and J have the same dimension does not itself indicate more. Whether the conjecture is true should be justified by further analysis.

As shown in [1], an algebraic error analysis of curvature computation based on local surface approximations usually produces some unrealistic results since too many terms are involved in the estimation, thus forcing the upper error bound to be overestimated seriously, giving little help for determining the zero-thresholds for curvatures. So it is preferred to start an analysis straightly from the principal curvatures themselves instead of those variables that are used to calculate the principal curvatures through approximation.

Suppose that small perturbations ξ_1 in principal curvature C_1 and ξ_2 in C_2 have the common bound δ:

$$0 < |\xi_1|,\ |\xi_2| \leq \delta \tag{11}$$

If starting the analysis directly from Besl's definitions of J as in Eq. (6) and $\tilde{H}$ as in Eq. (8), we shall get the following error estimation:

$$\begin{aligned} |E_H| &= \left|\frac{(C_1+\xi_1)+(C_2+\xi_2)}{2} - H\right| \\ &= \frac{1}{2}|\xi_1+\xi_2| \\ &\leq \delta \end{aligned} \tag{12}$$

$$\begin{aligned} |E_{\tilde{H}}| &= \left|\frac{(C_1+\xi_1)-(C_2+\xi_2)}{2} - \tilde{H}\right| \\ &\leq \frac{1}{2}|\xi_1-\xi_2| \\ &\leq \frac{1}{2}(|\xi_1|+|\xi_2|) \\ &\leq \delta \end{aligned} \tag{13}$$

and

$$\begin{aligned} |E_J| &= |2(H+E_H)(\tilde{H}+E_{\tilde{H}}) - J| \\ &= 2|HE_{\tilde{H}} + \tilde{H}E_H + E_H E_{\tilde{H}}| \\ &\leq 2(|H|\delta + |\tilde{H}|\delta + \delta^2) \\ &\leq 2(2(|C_1|+|C_2|)\delta + \delta^2) \end{aligned} \tag{14}$$

It should be noticed that the error bound given by Eq. (14) might still be overestimated since many intermediate terms, such as $H, \tilde{H}, E_H$ and $E_{\tilde{H}}$, other than the primary terms C_1 and C_2 are involved in the expressions (cf.[2]). In fact, by setting $\epsilon_H = \delta$, Eq. (14) leads to the following constraint between the thresholds of J and K:

$$\epsilon_J = 2\epsilon_k \tag{15}$$

because

$$\begin{aligned} \epsilon_J &\geq 2(2(|C_1|+|C_2|)\delta + \delta^2) \\ &\geq 2(2|H|\delta + \delta^2) \\ &= 2(2|H|\epsilon_H + \epsilon_H^2) \end{aligned} \tag{16}$$

Changing the definition of J in Eq. (6) to its equivalent one based only on primary terms C_1 and C_2:

$$J = \frac{C_1^2 - C_2^2}{2} \tag{17}$$

and taking a similar analysis, we will get:

$$\begin{aligned} |E_J| &= \left|\frac{2C_1\xi_1 - 2C_2\xi_2 + \xi_1^2 - \xi_2^2}{2}\right| \\ &\leq 2(|C_1|+|C_2|)\delta + \delta^2 \\ &= 2(|C_1|+|C_2|)\epsilon_H + \epsilon_H^2 \end{aligned} \tag{18}$$

This estimate provides a better error bound than in Eq. (14), resulting in a zero-thresholding formula in terms of H and ϵ_H:

$$\epsilon_J \geq 2|H|\epsilon_H + \epsilon_H^2 \tag{19}$$

which puts the zero-threshold of J at the same position as the zero-threshold of K as given in Eq. (9), thus justifying the conjecture $\epsilon_J = \epsilon_K$. In turn, it shows the consistent zero-thresholding formula of the KH sign images remaintains effective even for the KJ sign images[2].

4 Zero-thresholding at multiple scales

While an empirical imposition of a zero-threshold to both J and K is feasible in a single scale processing, it is improper to use a unique value as the zero curvature threshold for all scales; it is also impractical to set up a sequence of zero thresholds for curvatures at individual scales, where the scale effects change from scale to scale, including a decreasing noisy pollution but an increasing surface distortion when the smoothing scale varies from fine to large.

A zero-thresholding formula working at multiple scales should be able to change the threshold value

[2] The above analysis also shows that to keep Eq. (10) valid J should be calculated directly from principal curvatures C_1 and C_2 as in Eq. (17) instead of from intermediate terms H and $\tilde{H}$.

adaptively. The formula $\epsilon_J = \epsilon_K$ does not have such a mechanism, as it takes no account of the scale effects. But the formula $\epsilon_K \geq 2|H|\epsilon_H + \epsilon_H^2$ can do this job as it can introduce scale effects through the term of H in the inequality. Once a multiple scale processing has been started at a fine scale, the zero-threshold of K will be determined with respect to the whole surface S by:

$$\epsilon_H = max\ [\epsilon_0,\ \underset{\text{S}}{min}\ |H|] \tag{20}$$

where the ϵ_0 is a small positive number, say 0.00005, dependent on the given data and the task. Then according to the following formula as proposed in [5]:

$$\epsilon_K = 2\ \underset{\text{S}}{Average}\ |H|\,\epsilon_H + \epsilon_H^2 \tag{21}$$

the zero-thresholding goes on for larger scales without any supervision, where H changes continuously when the scale increases.

5 Conclusion

The comparison of the zero-thresholdings for the KH and KJ sign images shows that the consistent zero-thresholding is still required in the multiple scale surface segmentation. While the KJ sign image is a good choice for single scale surface segmentation, with a zero-thresholding formula $\epsilon_J = \epsilon_K$, the KH sign image is a better choice for multiple scale surface segmentation, its zero-thresholding formula $\epsilon_K \geq 2|H|\epsilon_H + \epsilon_H^2$, maintains effectiveness for either case. This is the reason why the KH sign image is still widely used in the multiple scale processing, even though KJ sign image is elegant enough to avoid the "phantom shape" at individual scales.

Appendix
Consistent zero-thresholding inequality

Suppose that small perturbations ξ_1 in the principal curvature C_1 and ξ_2 in C_2 have the common bound δ:

$$0 < |\xi_1|,\ |\xi_2| \leq \delta \tag{22}$$

These perturbations introduce errors E_H in H and E_K in K. Both errors can be estimated by

$$\begin{aligned} |E_H| &= \left|\frac{(C_1+\xi_1)+(C_2+\xi_2)}{2} - H\right| \\ &= \frac{1}{2}|\xi_1+\xi_2| \\ &\leq \delta \end{aligned} \tag{23}$$

$$\begin{aligned} |E_K| &= |(C_1+\xi_1)(C_2+\xi_2) - K| \\ &= |C_1\xi_2 + C_2\xi_1) + \xi_1\xi_2| \\ &\leq (|C_1|+|C_2|)\delta + \delta^2 \end{aligned} \tag{24}$$

Since $|C_1|+|C_2| \geq 2|H|$, setting $\epsilon_H = \delta$ as the zero-threshold of H leads to the zero-threshold of K:

$$\begin{aligned} |\epsilon_K| &\geq (|C_1|+|C_2|)\delta + \delta^2 \\ &\geq 2|H|\epsilon_H + \epsilon_H^2 \end{aligned} \tag{25}$$

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