Extra Examples for "Scott Continuity in Generalized Probabilistic Theories"

Robert Furber

May 27, 2020

1 Introduction

The purpose of this note is to draw out some counterexamples using the construction described in [9]. In Section 3 we show that there is an order-unit space A such that there exist $2^{\aleph_0}$ pairwise non-isometric base-norm spaces E such that $A \cong E^*$ as order-unit spaces. This contrasts with the case of C*-algebras, where if there is a predual (and therefore the C*-algebra is a W*-algebra), it is unique.

In Section 4 we show that there is a base-norm space E, whose dual order-unit space $A = E^*$ is therefore bounded directed-complete, such that there is a Scott-continuous unital map $f : A \to A$ that is not the adjoint of any linear map $g : E \to E$. Again, this contrasts with the situation for W*-algebras, where Scott-continuous maps $A \to B$ correspond to adjoints of linear maps between preduals $B_* \to A_*$. In Section 5 we show that there are base-norm and order-unit spaces E such that $E \cong E^{**}$, but the evaluation mapping $E \to E^{**}$ is not an isomorphism, *i.e.* E is not reflexive, using an example of R. C. James from Banach space theory.

In Section 6 we use an example due to J. W. Roberts to show that there are base-norm spaces E admitting Hausdorff vectorial topologies in which the base and unit ball are compact, but are not dual spaces, and similarly that there are order-unit spaces A admitting Hausdorff vectorial topologies in which the unit interval and unit ball are compact, but are not dual spaces. The latter also shows that there are compact effect modules in which the operation of taking convex combinations is continuous that are not "compact effect modules" in the sense of [6, §3.4, §4.4].

We begin with a section containing some general material about the constructions used and then draw out the consequences in the later sections, as described above.

2 Functoriality

We start by proving functoriality of $\mathcal{BN}$ and $\mathcal{OU}$. Because any constraction $f : E \to F$ between normed spaces can be restricted to an affine map between the unit balls, [6, Corollary 2.4.9] implies that for any contraction there is a (functorial) trace-preserving map $\mathcal{BN}(f) : \mathcal{BN}(E) \to \mathcal{BN}(F)$ such that $B(\mathcal{BN}(f)) = f|_{\mathrm{Ball}(E)}$. We can use the pre-existing linear structure of f to get a more direct expression, which can also be applied to make $\mathcal{OU}$ a functor.

Given a contraction $f : E \to F$, we define

$$\mathcal{BN}(f)(x,\lambda) = (f(x),\lambda) = \mathcal{OU}(f)(x,\lambda)$$

Proposition 2.1. *With the above definition on morphisms, $\mathcal{BN}$: $\mathbf{Norm}_1 \to$ $\mathbf{BNS}$ and $\mathcal{OU}$: $\mathbf{Norm}_1 \to \mathbf{OUS}$ are functors, which restrict to functors $\mathcal{BN}$: $\mathbf{Ban}_1 \to \mathbf{BBNS}$ and $\mathcal{OU} : \mathbf{Ban}_1 \to \mathbf{BOUS}$.*

Proof. We already know from [9, Propositions 3.2 and 3.3] that if E is a Banach space, $\mathcal{BN}(E)$ is a Banach base-norm space and $\mathcal{OU}(E)$ is a Banach order-unit space, so as $\mathbf{Ban}_1$, $\mathbf{BBNS}$ and $\mathbf{BOUS}$ are full subcategories of $\mathbf{Norm}_1$, $\mathbf{BNS}$ and $\mathbf{OUS}$, we only need to show that $\mathcal{BN}$ is a functor $\mathbf{Norm}_1 \to \mathbf{BNS}$ and $\mathcal{OU}$ is a functor $\mathbf{Norm}_1 \to \mathbf{OUS}$. So let $f : E \to F$ be a contraction between normed spaces, *i.e.* f is a linear map such that $\|f(x)\|_F \leq \|x\|_E$ for all $x \in E$. We want to show that $\mathcal{BN}(f)$ and $\mathcal{OU}(f)$ are linear and positive, the proof of which is the same in each case, and that $\mathcal{BN}(f)$ is trace-preserving and $\mathcal{OU}(f)$ is unital.

As the proof of linearity and positivity is the same in each case, we show it explicitly for $\mathcal{BN}(f)$ only. For the linearity, observe that if $\alpha \in \mathbb{R}$, and $(x_1, \lambda_1), (x_2, \lambda_2) \in \mathcal{BN}(E)$:

$$\begin{aligned}
\mathcal{BN}(f)(\alpha(x_1,\lambda_1) + (x_2,\lambda_2)) &= \mathcal{BN}(f)(\alpha x_1 + x_2, \alpha\lambda_1 + \lambda_2) \\
&= (f(\alpha x_1 + x_2), \alpha\lambda_1 + \lambda_2) \\
&= (\alpha f(x_1) + f(x_2), \alpha\lambda_1 + \lambda_2) \\
&= \alpha(f(x_1), \lambda_1) + (f(x_2), \lambda_2) \\
&= \alpha\mathcal{BN}(f)(x_1,\lambda_1) + \mathcal{BN}(f)(x_2,\lambda_2),
\end{aligned}$$

which proves the linearity.

By definition, $(x, \lambda) \in \mathcal{BN}(E)_+$ is equivalent to $\|x\|_E \leq \lambda$. So if $(x, \lambda) \in \mathcal{BN}(E)_+$ we have $\mathcal{BN}(f)(x, \lambda) = (f(x), \lambda) \in \mathcal{BN}(F)_+$ because $\|f(x)\|_F \leq \|x\|_E \leq \lambda$. This proves that $\mathcal{BN}(f)$ is positive.

To show that $\mathcal{BN}(f)$ is trace-preserving, we have $\tau(\mathcal{BN}(f)(x,\lambda)) = (\tau(f(x),\lambda)) = \lambda = \tau(x,\lambda)$ for all $(x,\lambda) \in \mathcal{BN}(E)$. To show that $\mathcal{OU}(f)$ is unital, we have $\mathcal{OU}(f)(0,1) = (f(0),1) = (0,1)$, so $\mathcal{OU}(f)$ is unital.

The proof that $\mathcal{BN}$ and $\mathcal{OU}$ are functorial is the same, so for variety we show this only for $\mathcal{OU}$: $\mathcal{OU}(id)(x,\lambda) = (x,\lambda)$, and $\mathcal{OU}(g \circ f)(x,\lambda) = (g(f(x)),\lambda) = \mathcal{OU}(g)(\mathcal{OU}(f)(x,\lambda))$. □

In order to disambiguate which space is involved, we put a superscript π_1^E : $E \times \mathbb{R} \to E$, so $\pi_1^E : \mathcal{OU}(E) \to E$ and $\pi_1^E : \mathcal{BN}(E) \to E$.

Lemma 2.2. *The affine isomorphism $\pi_1 : B(\mathcal{BN}(E)) \to \mathrm{Ball}(E)$ is natural. Therefore $\mathcal{BN}(f)$ is the unique map extending* $\mathrm{Ball}(E)$ *on the bases.*

Proof. Let $f : E \to F$ be a contraction between normed spaces. For all $(x, 1) \in B(\mathcal{BN}(E))$ we have

$$\pi_1^F(B(\mathcal{BN}(f))(x,1)) = \pi_1^F(f(x),1) = f(x) = \mathrm{Ball}(f)(\pi_1^E(x,1)),$$

and so $\pi_1^F \circ B(\mathcal{BN}(f)) = \mathrm{Ball}(f) \circ \pi_1^F$, *i.e.* π_1 is natural. As $B : \mathbf{BNS} \to \mathbf{Conv}$ is fully faithful [6, Proposition 2.4.8], this implies $\mathcal{BN}(f)$ is the unique map extending $\mathrm{Ball}(f)$. □

We have seen how to turn a contraction into a trace-preserving map. Now we show how to go the other way. For each trace-preserving map $f : \mathcal{BN}(E) \to \mathcal{BN}(F)$ we can define a map $\Pi_1(f) : E \to F$ by $\Pi_1(f) = \pi_1^F \circ f \circ \kappa_1^E$, or

$$\Pi_1(f)(x) = \pi_1(f(x, 0))$$

Lemma 2.3. *If E, F are normed spaces, $f : \mathcal{BN}(E) \to \mathcal{BN}(F)$ a trace-preserving map of base-norm spaces, then $\Pi_1(f) : E \to F$ is a contraction, and Π_1 defines a functor from the full subcategory of* **BNS** *on spaces of the form $\mathcal{BN}(E)$ to* $\mathbf{Norm}_1$*. If $f(0, 1) = (0, 1)$, then $\mathcal{BN}(\Pi_1(f)) = f$.*

Proof. The maps κ_1 and π_1 are contractions, and f is a contraction because it is a trace-preserving map of base-norm spaces [6, Proposition 2.2.12]. Therefore $\Pi_1(f) = \pi_1^F \circ f \circ \kappa_1^E$ is a contraction.

Given $(x, \lambda) \in \mathcal{BN}(E)$, we have $f(x, \lambda) - f(x, 0) = f(0, \lambda)$ by linearity, so $f(x, 0) = f(x, \lambda) - f(0, \lambda)$, and $\pi_1(f(x, 0)) = \pi_1(f(x, \lambda)) - \lambda\pi_1(f(0, 1))$. Therefore

$$\begin{aligned}
\mathcal{BN}(\Pi_1(f))(x, \lambda) &= (\Pi_1(f)(x), \lambda) = (\pi_1(f(x, 0)), \lambda) \\
&= (\pi_1(f(x, \lambda)) - \lambda\pi_1(f(0, 1)), \lambda) \\
&= (\pi_1(f(x, \lambda)), \pi_2(f(x, \lambda))) - \lambda(\pi_1(f(0, 1)), 0) \\
&= f(x, \lambda) - \lambda(\pi_1(f(0, 1)), 0)
\end{aligned}$$

So if $\pi_1(f(0, 1)) = 0$, or equivalently, since f is trace-preserving, $f(0, 1) = (0, 1)$, then $\mathcal{BN}(\Pi_1(f))(x, \lambda) = f(x, \lambda)$ for all $(x, \lambda) \in \mathcal{BN}(E)$ so $\mathcal{BN}(\Pi_1(f)) = f$.

It remains to prove the functoriality of Π_1. We have

$$\Pi_1(\mathrm{id})(x) = \pi_1(\mathrm{id}(x, 0)) = \pi_1(x, 0) = x$$

so $\Pi_1(\mathrm{id}) = \mathrm{id}$. Now, if E, F, G are normed spaces and $f : \mathcal{BN}(E) \to \mathcal{BN}(F)$ and $g : \mathcal{BN}(F) \to \mathcal{BN}(G)$ are trace-preserving maps of base-norm spaces, for all $x \in E$ we have

$$\begin{aligned}
(\Pi_1(g) \circ \Pi_1(f))(x) &= \Pi_1(g)(\Pi_1(f)(x)) \\
&= \pi_1(g(\pi_1(f(x, 0)), 0)).
\end{aligned}$$

By the reasoning we gave above to prove that $\mathcal{BN}(\Pi_1(f)) = f$ when $f(0, 1) = (0, 1)$, we have, for all $y \in \mathcal{BN}(F)$, $g(\pi_1(y), 0) = g(y) - g(0, \pi_2(y))$. So

$$(\Pi_1(g) \circ \Pi_1(f))(x) = \pi_1(g(f(x, 0)) - g(0, \pi_2(f(x, 0)))),$$

and $\pi_2(f(x, 0)) = 0$ because f is trace-preserving, so $g(0, \pi_2(f(x, 0))) = 0$ by linearity, meaning

$$(\Pi_1(g) \circ \Pi_1(f))(x) = \pi_1(g(f(x, 0))) = \Pi_1(g \circ f)(x).$$

This shows that $\Pi_1(g) \circ \Pi_1(f) = \Pi_1(g \circ f)$, completing the proof of functoriality. □

Corollary 2.4. *If E, F are normed spaces such that $\mathrm{Ball}(E) \cong \mathrm{Ball}(F)$ as an affine isomorphism, then there is an isometric isomorphism $E \cong F$.*

Proof. Let $f : \mathrm{Ball}(E) \to \mathrm{Ball}(F)$ be an affine isomorphism. We have $(\pi_1^F)^{-1} \circ f \circ \pi_1^E : B(\mathcal{BN}(E)) \to B(\mathcal{BN}(F))$ is an affine isomorphism by Lemma 2.2, so by [6, Proposition 2.4.8] there exists an isomorphism of base-norm spaces $g : \mathcal{BN}(E) \to \mathcal{BN}(F)$ such that $B(g) = (\pi_1^F)^{-1} \circ f \circ \pi_1^E$. Then by the functoriality of Π_1 (Lemma 2.3), $\Pi_1(g) : E \to F$ is an isometric isomorphism. $\square$

The following lemma about pairings is useful.

Lemma 2.5. *For $i \in \{1, 2\}$ let E_i, F_i be normed spaces, $\langle \text{-}, \text{-} \rangle_i : E_i \times F_i \to \mathbb{R}$ a bilinear pairing such that $\mathrm{lev}_i : F_i \to E_i^*$ is an isometric isomorphism. For linear maps $g : E_1 \to E_2$ and $f : F_2 \to F_1$, we have $g^* = \mathrm{lev}_1 \circ f \circ \mathrm{lev}_2^{-1}$ iff for all $x \in E_1$ and $y \in F_2$, $\langle g(x), y \rangle_2 = \langle x, f(y) \rangle_1$.*

Proof. We have $g^* = \mathrm{lev}_1 \circ f \circ \mathrm{lev}_2^{-1}$ iff $g^* \circ \mathrm{lev}_2 = \mathrm{lev}_1 \circ f$. This holds iff for all $x \in E_1$ and $y \in F_2$, $g^*(\mathrm{lev}_2(y))(x) = \mathrm{lev}_1(f(y))(x)$. We see that

$$g^*(\mathrm{lev}_2(y))(x) = \mathrm{lev}_2(y)(g(x)) = \langle g(x), y \rangle_2,$$

and

$$\mathrm{lev}_1(f(y))(x) = \langle x, f(y) \rangle_1,$$

which shows that $g^*(\mathrm{lev}_2(y))(x) = \mathrm{lev}_1(f(y))(x)$ iff $\langle g(x), y \rangle_2 = \langle x, f(y) \rangle_1$, as required. $\square$

We finish this introductory section with a fact about order-unit spaces that characterizes Scott continuity for maps out of spaces of the form $\mathcal{OU}(E^*)$.

Proposition 2.6. *Let (A, A_+, u) be an order unit space and $(a_i)_{i \in I}$ a directed set converging in A to $a \in A$. Then a is the least upper bound of $(a_i)_{i \in I}$.*

Proof. We first show that a is an upper bound for $(a_i)_{i \in I}$. As $a_i \to a$ in norm, for each $\epsilon > 0$ there exists $j \in I$ such that for all $k \geq j$ $\|a_k - a\| \leq \epsilon$. Given $i \in I$, we can find $j \in I$ with the aforementioned property for ϵ, and then by directedness of $(a_i)_{i \in I}$ there exists $k \in I$ such that $a_k \geq a_i, a_j$. Since $\|a_k - a\| \leq \epsilon$, we have $a_k \leq a + \epsilon u$, and therefore $a_i \leq a + \epsilon u$. Since this holds for all $\epsilon > 0$, by the archimedean property $a_i \leq a$. Since this holds for all $i \in I$, a is an upper bound for $(a_i)_{i \in I}$.

To show that a is less than all other upper bounds, let $a' \in A$ be an upper bound for $(a_i)_{i \in I}$, *i.e.* for all $i \in I$, $a_i \leq a'$. For all $\epsilon > 0$, there exists $j \in I$ such that $\|a_j - a\| \leq \epsilon$, so $-\epsilon u \leq a_j - a$, and therefore $a \leq a_j + \epsilon u \leq a' + \epsilon u$. Since this holds for all $\epsilon > 0$, the archimedean property implies $a \leq a'$. $\square$

Corollary 2.7. *Let E be a normed space and (B, B_+, v) an order-unit space. Every map of order-unit spaces $f : \mathcal{OU}(E^*) \to (B, B_+, v)$ is Scott continuous.*

Proof. If $(a_i)_{i \in I}$ is a directed set in $\mathcal{OU}(E^*)$ with supremum $a \in \mathcal{OU}(E^*)$, by [9, Proposition 3.7] $a_i \to a$ in norm. As f is a positive unital map, it is bounded [6, Proposition 1.2.8], and therefore continuous, so $f(a_i) \to f(a)$ in norm in B. By Proposition 2.6, $f(a)$ is the supremum of $(f(a_i))_{i \in I}$. Therefore f is Scott continuous. $\square$

3 An Order-unit Space with Many Isometric Preduals

We will use the following variant of the Banach-Stone theorem.

Proposition 3.1. *Let X, Y be compact Hausdorff spaces. Then $\mathcal{BN}(C(X)) \cong \mathcal{BN}(C(Y))$ as base-norm spaces iff $X \cong Y$ by a homeomorphism.*

Proof. If $f : Y \to X$ is a homeomorphism, then $C(f) : C(X) \to C(Y)$ is an isomorphism of Banach spaces by Gelfand duality, and so $\mathcal{BN}(C(f)) : \mathcal{BN}(C(X)) \to \mathcal{BN}(C(Y))$ is an isomorphism of base-norm spaces by Proposition 2.1.

For the other direction, if $f : \mathcal{BN}(C(X)) \to \mathcal{BN}(C(Y))$ is an isomorphism of base-norm spaces, Corollary 2.4 implies $\Pi_1(f) : C(X) \to C(Y)$ is an isometric isomorphism. By the Banach-Stone theorem [1, p. 170, Théorème 3] [22, Theorem 83] there exists a homeomorphism[1] $h : X \to Y$. $\square$

This gives us a large supply of non-isomorphic base-norm spaces to act as pre-duals. For the other part of the counterexample, we will need non-isometric Banach spaces whose dual spaces are nonetheless isometric. In the following, $\mathcal{G}$ is the underlying functor of the Giry monad, *i.e.* for any measure space (X, Σ) $\mathcal{G}(X, \Sigma)$ is a measurable space whose underlying set is the set of probability measures on (X, Σ). This was originally described in [10], but our notation follows [6, §1.6], where the definition of $\mathcal{G}$ in detail can be found. For a compact Hausdorff space X, $\mathcal{R}(X)$ is the base of the base-norm $C(X)^*$, given the weak-* topology, and is the underlying functor of the Radon monad [6, §1.5, p.178, and §3.6].

We can define convex combinations on $\mathcal{G}(X, \Sigma)$ as follows. Let $\nu_1, \nu_2 \in \mathcal{G}(X, \Sigma)$ and $\alpha \in [0, 1]$. For all $S \in \Sigma$ we define

$$(\alpha\nu_1 + (1 - \alpha)\nu_2)(S) = \alpha\nu_1(S) + (1 - \alpha)\nu_2(S).$$

By the continuity of $+$ and scalar multiplication in $\mathbb{R}$, it is clear that $(\alpha\nu_1 + (1 - \alpha)\nu_2)$ is countably additive, and it is easy to verify that it is in fact a probability measure. We use this structure in the proof of the following statement. This defines a "preconvex structure" in the sense of [11] on $\mathcal{G}(X, \Sigma)$. In the following, for a compact Hausdorff space X, we will use $\mathcal{B}a(X)$ to refer to the Baire σ-algebra (the coarsest σ-algebra such that all real-valued continuous functions on X are measurable) and $\mathcal{B}o(X)$ for the Borel σ-algebra (the σ-algebra generated by the open sets of X). We then define $\mathrm{Ba}(X) = (X, \mathcal{B}a(X))$ and $\mathrm{Bo}(X) = (X, \mathcal{B}o(X))$.

Proposition 3.2. *Let X, Y be uncountable compact metrizable spaces. Then $C(X)^* \cong C(Y)^*$ as base-norm spaces, and therefore isometrically.*

Proof. Compact metrizable spaces are separable and complete [14, Chapter I, 4.2], *i.e.* they are Polish spaces, so by a theorem of Kuratowski [14, Chapter II, 15.6], all uncountable compact metrizable spaces are Borel isomorphic to each other. That is to say, if X, Y are compact metrizable spaces, there exists

[1] It is not necessarily the case that $C(h) = g$ because not every isometric isomorphism is a *-homomorphism.

a measurable isomorphism $f : \mathrm{Bo}(X) \to \mathrm{Bo}(Y)$. As $\mathcal{G}$ is a functor, $\mathcal{G}(f) : \mathcal{G}(\mathrm{Bo}(X)) \to \mathcal{G}(\mathrm{Bo}(Y))$ is a measurable isomorphism. It is easy to verify that it is also an affine isomorphism on the preconvex structures. For each compact Hausdorff space X, the map $\rho_X : \mathcal{G}\mathrm{Ba}(X) \to \mathrm{Ba}\mathcal{R}(X)$ defined by

$$\rho_X(\nu)(a) = \int_X a \, \mathrm{d}\nu$$

where $\nu \in \mathcal{G}(\mathrm{Ba}(X))$ and $a \in C(X)$, is a bijection by a version of the Riesz representation theorem [19, §14.3 Theorem 8] (in fact it is a natural measurable isomorphism [6, Theorem 1.6.8]). Now, $\mathcal{R}(X)$, being the base of $C(X)^*$, is convex, which leads us to ask if ρ_X is affine. We show that $\rho_X(\alpha\nu_1 + (1 - \alpha)\nu_2)(a) = \alpha\rho_X(\nu_1)(a) + (1-\alpha)\rho_X(\nu_2)(a)$ for all real-valued bounded measurable functions a. It follows that this is so for all continuous functions a, and therefore ρ_X is an affine isomorphism.

Observe that if $a = \chi_S$ for $S \in \mathcal{B}a(X)$, then

$$\begin{aligned} \rho_X(\alpha\nu_1 + (1-\alpha)\nu_2)(\chi_S) &= (\alpha\nu_1 + (1-\alpha)\nu_2)(S) = \alpha\nu_1(S) + (1-\alpha)\nu_2(S) \\ &= \alpha\rho_X(\nu_1)(\chi_S) + (1-\alpha)\rho_X(\nu_2)(\chi_S), \end{aligned}$$

establishing what we want for this special case. By the linearity of ρ_X, this extends to the simple functions, *i.e.* the linear span of functions of the form χ_S. Then, as every bounded measurable function a is the pointwise limit of a sequence of simple functions $(a_i)_{i\in\mathbb{N}}$ [21, Theorem 8.8], we have

$$\begin{aligned} \rho_X(\alpha\nu_1 + (1-\alpha)\nu_2)(a) &= \int_X \lim_{i\to\infty} a_i \, \mathrm{d}\alpha\nu_1 + (1-\alpha)\nu_2 \\ &= \lim_{i\to\infty} \int_X a_i \, \mathrm{d}\alpha\nu_1 + (1-\alpha)\nu_2 \\ &= \lim_{i\to\infty} (\alpha\rho_X(\nu_1)(a_i) + (1-\alpha)\rho_X(\nu_2)(a_i)) \\ &= \alpha \left(\lim_{i\to\infty} \int_X a_i \, \mathrm{d}\nu_1 \right) + (1-\alpha) \left(\lim_{i\to\infty} \int_X a_i \, \mathrm{d}\nu_2 \right) \\ &= \alpha\rho_X(\nu_1)(a) + (1-\alpha)\rho_X(\nu_2)(a), \end{aligned}$$

using the dominated convergence theorem [21, Theorem 11.2], the continuity of addition and scalar multiplication, and then the dominated convergence theorem in the opposite direction. As every continuous real-valued function on a compact Hausdorff space is bounded and Baire measurable, we have shown that ρ_X is affine. It follows that its inverse is also affine, and in fact that the convex prestructure we defined on $\mathcal{G}(\mathrm{Ba}(X))$ underlies an $\mathcal{EM}(\mathcal{D})$ structure, although we do not need this.

In the case that X is metrizable, $\mathcal{B}o(X) = \mathcal{B}a(X)$ [5, 4A3K (b)], so combining what we have so far, for any uncountable compact metrizable spaces X, Y, there exists a Borel isomorphism $f : \mathrm{Bo}(X) \to \mathrm{Bo}(Y)$, and therefore $g = \rho_Y \circ \mathcal{G}(f) \circ \rho_X^{-1} : \mathcal{R}(X) \to \mathcal{R}(Y)$ is an affine isomorphism. By [6, Proposition 2.4.8], the base functor $B : \mathbf{BNS} \to \mathcal{EM}(\mathcal{D})$ is fully faithful, so g extends to an isomorphism of base-norm spaces $C(X)^* \to C(Y)^*$. As morphisms of base-norm spaces have operator norm ≤ 1 [6, Proposition 2.2.12], this isomorphism is an isometric isomorphism on the underlying Banach spaces. □

We also need a source of a large number of uncountable compact metrizable spaces.

Proposition 3.3. *There exists a family $(X_i)_{i\in 2^{\mathbb{N}}}$ of compact metrizable spaces such that if $X_i \cong X_j$ as topological spaces then $i = j$.*

Proof. In [15, Chapter 12, Theorem 1.3] a family $(A_i)_{i\in 2^{\mathbb{N}}}$ of $2^{\aleph_0}$ pairwise non-isomorphic countable Boolean algebras is constructed. By Stone duality, their Stone spaces $(Y_i)_{i\in 2^{\mathbb{N}}}$ are pairwise non-homeomorphic, and as each A_i is countable, the corresponding Stone space Y_i is compact metrizable [15, Chapter 3, Proposition 7.23].

Although it possible to prove, by carefully examining the construction, that all but one of the Y_i is uncountable, it is easier to define $X_i = Y_i + [0,1]$ and reason as follows. Let $f : X_i \to X_j$ be a homeomorphism. If $x \in [0,1]$, suppose for a contradiction that $f(x) \in Y_j$. If there were an $x' \in [0,1]$ such that $f(x') \in [0,1]$, this would contradict the connectedness of $[0,1] \subseteq X_i$. But the image of a connected set is connected, and the only connected subsets of Y_j are singletons, as it is a Stone space, so $f([0,1]) = f(x)$, which contradicts the injectivity of f. Therefore $f([0,1]) = [0,1]$. Injectivity implies that $f(Y_i) = f(Y_j)$, so f restricts to a homeomorphism $Y_i \cong Y_j$, and therefore $i = j$. So we have proved that $(X_i)_{i\in 2^{\mathbb{N}}}$ is a pairwise non-homeomorphic family of uncountable compact metrizable spaces. □

We can now obtain our long-awaited counterexample.

Counterexample 3.4. The order-unit space $\mathcal{OU}(C([0,1])^*)$ is bounded directed-complete, and for every uncountable compact metrizable space X, $\mathcal{BN}(C(X))^* \cong \mathcal{OU}(C([0,1]^*))$ as order-unit spaces. Therefore $\mathcal{OU}(C([0,1])^*)$ has $2^{\aleph_0}$ non-isomorphic base-norm spaces as preduals.

Proof. By [9, Proposition 3.6], for any compact metrizable space X, including $[0,1]$, we have $\mathcal{OU}(C(X)^*) \cong \mathcal{BN}(C(X))^*$. We will use this more than once, but for now it shows that $\mathcal{OU}(C([0,1])^*)$ is bounded directed-complete by [9, Lemma 2.1]. If X is an uncountable compact metrizable space, then there is an isometric isomorphism $f : C(X)^* \to C([0,1])^*$ by Proposition 3.2, so $\mathcal{OU}(f)$ is an isomorphism of order-unit spaces $\mathcal{OU}(C(X)^*) \to \mathcal{OU}(C([0,1])^*)$ by Proposition 2.1, so composing this with the isomorphism $\mathcal{BN}(C(X))^* \cong \mathcal{OU}(C(X)^*)$ produces an isomorphism $\mathcal{BN}(C(X))^* \cong \mathcal{OU}(C([0,1])^*)$, as required, *i.e.* for every uncountable metric space X, $\mathcal{BN}(C(X))$ is a predual for $\mathcal{OU}(C([0,1])^*)$.

By Proposition 3.3, there is a continuum-sized family of non-homeomorphic compact metrizable spaces $(X_i)_{i\in 2^{\mathbb{N}}}$. By what we have shown so far, $\mathcal{OU}(C([0,1])^*) \cong \mathcal{BN}(C(X_i))^*$ for all $i \in 2^{\mathbb{N}}$, but if $\mathcal{BN}(C(X_i)) \cong \mathcal{BN}(C(X_j))$, then by Proposition 3.1, $X_i \cong X_j$ and so $i = j$. Therefore $(\mathcal{BN}(C(X_i)))_{i\in 2^{\mathbb{N}}}$ is a pairwise non-isomorphic family of base-norm spaces of size continuum, each of which is a predual for $\mathcal{OU}(C([0,1])^*)$. □

4 An Normal Endomorphism With No Pre-adjoint

What I mean by this a base-norm space (E, E_+, τ), with dual order-unit space (A, A_+, u) (of course $u = \tau$), and a Scott-continuous unital map $f : A \to A$ such that for all trace-preserving maps $g : E \to E$ it is not the case that $f = g^*$.

We start with a criterion for the non-existence of a pre-adjoint, expressed in the contrapositive.

Lemma 4.1. *Let E be a normed space. If $f : E^* \to E^*$ has a pre-adjoint, then for all $x \in E$, there exists $x' \in E$ such that $f^*(\mathrm{ev}(x)) = \mathrm{ev}(x')$.*

Proof. Let $g : E \to E$ be the pre-adjoint of f, so $g^* = f$. We show that for all $x \in E$, $f^*(\mathrm{ev}(x)) = \mathrm{ev}(g(x))$ as follows. Let $y \in E^*$, and observe that

$$\begin{aligned} f^*(\mathrm{ev}(x))(y) &= (\mathrm{ev}(x) \circ f)(y) = \mathrm{ev}(x)(f(y)) = f(y)(x) = g^*(y)(x) \\ &= (y \circ g)(x) = y(g(x)) = \mathrm{ev}(g(x))(y), \end{aligned}$$

as required. □

Lemma 4.2. *Let E be an irreflexive Banach space. Then there exists a contraction $f : E^* \to E^*$ with no pre-adjoint.*

Proof. As E is irreflexive, there exists $\phi \in \mathrm{Ball}(E^{**})$ such that $\phi \neq \mathrm{ev}(x)$ for any $x \in E$. Pick[2] $y_0 \in E^*$ with $\|y_0\| = 1$, and define $f : E^* \to E^*$ by

$$f(y) = \phi(y)y_0.$$

For all $y \in E^*$ we have

$$\|f(y)\| = \|\phi(y)y_0\| = |\phi(y)|\|y_0\| = |\phi(y)| \leq \|y\|,$$

because $\phi \in \mathrm{Ball}(E^{**})$. Therefore f is a contraction. To prove that f has no pre-adjoint, we use Lemma 4.1. As $\|y_0\| = 1$, there exists an element $x \in E$ such that $y_0(x) \neq 0$. Define $\alpha = y_0(x)$. Then for each $y \in E^*$ we have

$$f^*(\mathrm{ev}(x))(y) = (\mathrm{ev}(x) \circ f)(y) = \mathrm{ev}(x)(f(y)) = f(y)(x) = \phi(y)y_0(x) = \alpha\phi(y).$$

So $f^*(\mathrm{ev}(x)) = \alpha\phi$. If there were an x' such that $\mathrm{ev}(x') = \alpha\phi$, then $\mathrm{ev}(\alpha^{-1}x') = \phi$, which contradicts the defining property of ϕ. Therefore $f^*(\mathrm{ev}(x)) \neq \mathrm{ev}(x')$ for any $x' \in E$. By Lemma 4.1 this proves f has no pre-adjoint. □

We can now upgrade this example to an example for order-unit spaces.

Counterexample 4.3. There is a base-norm space (E, E_+, τ) with dual order unit space (A, A_+, u) and a Scott-continuous positive unital map $f : A \to A$ with no pre-adjoint, *i.e.* for all trace-preserving maps $g : E \to E$, $f \neq g^*$.

Proof. Let F be an irreflexive Banach space, and $f : F^* \to F^*$ a contraction with no pre-adjoint (Lemma 4.2). Define $A = \mathcal{OU}(F^*)$ and $E = \mathcal{BN}(F)$ and we know that $A \cong E^*$ by [9, Proposition 3.6]. Then $\mathcal{OU}(f) : A \to A$ and it is Scott continuous by Corollary 2.7. Now suppose for a contradiction that $g : E \to E$ is a trace-preserving pre-adjoint to f, by which we mean $\mathrm{lev} \circ g^* \circ \mathrm{lev}^{-1} = f$, or equivalently, by Lemma 2.5 $\langle g(x, \lambda), (y, \mu)\rangle = \langle (x, \lambda), \mathcal{OU}(f)(y, \mu)\rangle$ for all $(x, \lambda) \in BN(F)$ and $(y, \mu) \in OU(F^*)$.

[2]The space $\{0\}$ is reflexive, so is not equal to E^*, so there must exist an element in E^* different from 0, and hence an element of norm 1.

Then by specializing to the case that $\lambda = \mu = 0$ we see that

$$\begin{aligned}
\Pi_1(g)^*(y)(x) &= (y \circ \Pi_1(g))(x) \\
&= y(\Pi_1(g)(x)) \\
&= y(\pi_1(g(x,0))) \\
&= y(\pi_1(g(x,0))) + \pi_2(g(x,0)) \cdot 0 \\
&= \langle g(x,0),(y,0)\rangle \\
&= \langle (x,0), \mathcal{OU}(f)(y,0)\rangle \\
&= \langle (x,0),(f(y),0)\rangle \\
&= f(y)(x).
\end{aligned}$$

Therefore $\Pi_1(g)^* = f$, contradicting the fact that f has no pre-adjoint. □

5 Irreflexive Base-Norm and Order-Unit Spaces Isomorphic to Their Double Duals

What we have described allows us to adapt various counterexamples from Banach space theory to base-norm and order-unit spaces. Recall that a Banach space E (and also a base-norm or order-unit space) is called *reflexive* if the evaluation mapping $\mathrm{ev} : E \to E^{**}$ is an isometric isomorphism (it suffices that it be surjective for this).

Since a reflexive space has $E \cong E^{**}$, it is natural to ask if this is sufficient. The answer is no, there are separable infinite-dimensional Banach spaces E that are not reflexive, but nonetheless $E \cong E^{**}$. R. C. James constructed a Banach space that is not reflexive but is isomorphic to its double dual [12], and the construction was later modified to produce a space that is not reflexive, but is isometrically isomorphic to its double dual [16, Theorem 6.16].

Proposition 5.1.

(i) There is a base-norm space (E, E_+, τ) that is not reflexive, but $(E, E_+, \tau) \cong (E^{**}, E^{**}_+, \mathrm{ev}(\tau))$ as base-norm spaces.

(ii) There is an order-unit space (A, A_+, u) that is not reflexive, but $(A, A_+, u) \cong (A^{**}, A^{**}_+, \mathrm{ev}(u))$ as order-unit spaces.

Proof. The proof of part (ii) is similar to that of part (i) but using $\mathcal{OU}$ instead of $\mathcal{BN}$, so we only give the proof of part (i) explicitly. Let E be an irreflexive Banach space that is isometrically isomorphic to its double dual by a map $f : E \to E^{**}$. Then, by [9, Counterexample 4.1], $\mathcal{BN}(E)$ is not reflexive. However, by Proposition 2.1 $\mathcal{BN}(f) : \mathcal{BN}(E) \to \mathcal{BN}(E^{**})$, and by [9, Propositions 3.4 and 3.6] the maps $\mathrm{lev} : \mathcal{BN}(E^{**}) \to \mathcal{OU}(E^*)^*$ and $(\mathrm{lev}^{-1})^* : \mathcal{OU}(E^*)^* \to \mathcal{BN}(E)^{**}$ are isomorphisms of base-norm spaces, so $\mathcal{BN}(E) \cong \mathcal{BN}(E)^{**}$. □

6 Compact Convex Sets that are not Locally Convex

Roberts [17] (see also [13] or [18, §5.6]) constructed a non-empty compact convex subset of a topological vector space with no extreme points. As every compact convex subset of a *locally convex* topological vector space is the closed convex hull of its extreme points (the Krein-Mil'man theorem [20, II.10.4]), this implies that Roberts's set has no affine homeomorphism to a subset of a locally convex space. In the rest of this section, we will simply refer to the Krein-Mil'man theorem by name, without giving a reference.

In [6, Proposition 3.3.2, Theorems 3.3.6 and 3.3.7] the author showed that if a pre-base-norm space (E, E_+, τ) has a locally convex topology $\mathcal{T}$ in which its base is compact, then the associated Smith base-norm space $(E, E_+, \tau, \mathcal{T}_b)$ is isomorphic to the bounded weak-* dual of a Banach order-unit space, which can be taken to be $\mathcal{CE}(B(E))$. This is a categorical version of [3, Theorem 4]. Dually, in [6, Proposition 3.4.1, Theorem 3.4.5] the author showed that if an order-unit space (A, A_+, u) admits a locally convex topology $\mathcal{T}$ in which $[0, 1]_A$ is compact, then the associated Smith order-unit space $(A, A_+, u, \mathcal{T}_b)$ is isomorphic to the bounded weak-* dual of a Banach base-norm space. This is a categorical version of [4, Theorem 6].

The aim of this section is to show how to use the existence of compact convex sets such as Roberts's example to show that there exist base-norm spaces with vector space topologies in which the base is compact that do not arise as dual spaces of order-unit spaces, and there exist order-unit spaces with vector space topologies in which the unit interval is compact that do not arise as dual spaces of base-norm spaces. A consequence of the latter is that we cannot omit the local convexity assumption from the definition of compact effect module in [6, p. 183].

In the original version of [6], the author made certain unnecessary local convexity assumptions regarding bounded convex sets and their embeddings in pre-base-norm spaces. Therefore we need to refer to the later version [8] which has these assumptions removed where possible.

We begin with a lemma about the absolutely convex hull of the base of a base-norm space.

Lemma 6.1. *Let (E, E_+, τ) be a base-norm space, define $X = B(E)$ and $U =$ $\mathrm{absco}(X)$. If $x \in U$ such that $\tau(x) = 1$, then $x \in X$. Similarly, if $x \in U$ and $\tau(x) = -1$, then $x \in -X$.*

Proof. If $X = \emptyset$, then $\tau(x) = \pm 1$ is impossible because the only element of $\mathrm{absco}(X)$ is 0. So we assume for the rest of the proof that $X \neq \emptyset$, and therefore $U = \mathrm{absco}(X) = \mathrm{co}(X \cup -X)$ [6, Lemma 0.1]. If $x \in U$, then there exist $x_+, x_- \in X$ and $\alpha \in [0, 1]$ such that $x = \alpha x_+ + (1 - \alpha)(-x_-)$. If $\tau(x) = 1$, then

$$1 = \tau(\alpha x_+ + (1 - \alpha)(-x_-)) = \alpha\tau(x_+) - (1 - \alpha)\tau(x_-) = 2\alpha - 1,$$

so $2\alpha = 2$ and therefore $\alpha = 1$. From this it follows that $x = x_+ \in X$. The proof that $\tau(x) = -1$ implies $x \in -X$ is similar. □

Throughout, if X is a convex set, we use $\partial(X)$ to mean the set of extreme points of X, *i.e.* the set of elements $x \in X$ such that $\alpha y + (1 - \alpha)z = x$ for $0 < \alpha < 1$ and $y, z \in X$ implies $y = z = x$.

Lemma 6.2. *Let (E, E_+, τ) be a pre-base-norm space, and let $X = B(E)$ and $U = \mathrm{absco}(X)$. Then if $X = \emptyset$, $\partial(U) = \{0\}$, and if $X \neq \emptyset$,*

$$\partial(U) = \partial(X) \cup -\partial(X).$$

In particular if X is non-empty and $\partial(X) = \emptyset$, then $\partial(U) = \emptyset$.

Proof. If $X = \emptyset$ then $U = \text{absco}(X) = \{0\}$, so $\partial(U) = \{0\}$ because U consists of only one point. So we now assume that $X \neq \emptyset$. We first show that $\partial(X) \subseteq \partial(U)$.

Let $x \in \partial(X)$, and suppose we have $y, z \in U$ and $0 < \alpha < 1$ such that $x = \alpha y + (1-\alpha)z$. As $\tau(x) = 1$, we have

$$1 = \tau(\alpha y + (1-\alpha)z) = \alpha\tau(y) + (1-\alpha)\tau(z).$$

Since $y, z \in U$, we have $\tau(y), \tau(z) \in [-1, 1]$, and as 1 is an extreme point of $[-1, 1]$, this implies that $\tau(y) = \tau(z) = 1$, which by Lemma 6.1 implies $y, z \in X$. Then the fact that $x \in \partial(X)$ implies $y = z = x$. All together, this shows that $x \in \partial(U)$, and therefore that $\partial(X) \subseteq \partial(U)$. The proof that $-\partial(X) \subseteq \partial(U)$ is similar, using the negative case of Lemma 6.1, and from this we deduce that $\partial(X) \cup -\partial(X) \subseteq \partial(U)$.

For the opposite inclusion, suppose that $x \in \partial(U)$. As $x \in U$, there exist $x_+, x_- \in X$ and $\alpha \in [0, 1]$ such that $x = \alpha x_+ + (1-\alpha)(-x_-)$, and as it is an extreme point, either $\alpha \in \{0, 1\}$ or $x_+ = -x_-$. The latter cannot occur because $\tau(x_+) = 1$ while $\tau(-x_-) = -1$, so we are left with the conclusion that $\alpha = 0$ or $\alpha = 1$, *i.e.* either $x \in X$ or $x \in -X$. We conclude the argument under the assumption that $x \in X$, as the proof with $x \in -X$ is similar. We show that $x \in \partial(X)$ as follows. If $y, z \in X$ and $0 < \alpha < 1$ such that $x = \alpha y + (1-\alpha)z$, then by the fact that $x \in \partial(U)$, $y = z = x$, so $x \in \partial(X)$ *a fortiori.* □

Counterexample 6.3. There exists a Banach base-norm space (E, E_+, τ) with a vector space topology $\mathcal{T}$ on E such that $B(E)$ is compact in $\mathcal{T}$ but has no extreme points. Therefore (E, E_+, τ) is not isomorphic to the dual space of any order-unit space (in any topology). The unit ball of (E, E_+, τ) is $\text{absco}(B(E))$ and is an absolutely convex set with no extreme points that is compact in $\mathcal{T}$.

Proof. By Roberts's counterexample [17, 13] there exists a Hausdorff topological vector space F and a compact convex subset $X \subseteq F$ with no extreme points. By [8, Proposition 2.13] there is a base-norm space (E, E_+, τ) and a Hausdorff linear topology $\mathcal{T}$, and an affine homeomorphism $X \cong B(E)$. It follows that $B(E)$ is compact, and therefore complete in its unique uniformity [2, II.4.1 Theorem 1], so by [8, Proposition 2.18], (E, E_+, τ) is a Banach base-norm space.

If (E, E_+, τ) were the dual space of an order-unit space (or even the signed state space of an effect algebra), then $B(E)$ would be compact in the (locally convex) weak-* topology [7, Lemma 4.1], and would therefore have extreme points by the Krein-Mil'man theorem. Therefore (E, E_+, τ) is not the dual space of an order-unit space, nor even the signed state space of an effect algebra.

As $B(E)$ is compact, $\text{absco}(B(E))$ is compact [8, Lemma 0.15 (i)], and therefore radially compact, so $\text{absco}(B(E))$ is the unit ball of E [8, Lemma 0.7]. By Lemma 6.2 it has no extreme points. □

This completes the base-norm space example. For the order-unit space example, we will be using $\mathcal{OU}$. We need some lemmas about the unit ball of $\mathcal{OU}(E)$ and its extreme points first. We write u for $(0, 1)$, the order unit of $\mathcal{OU}(E)$, and recall the fact that $\text{Ball}(\mathcal{OU}(E)) = [-u, u]$, as in any order-unit space [6, Lemma A.5.3]. We will require the characterization $(x, y) \in [-u, u]$ in $\mathcal{OU}(E)$ iff $\|x\|_E + |y| \leq 1$ from [9, Proposition 3.3].

Lemma 6.4. *Let E be a normed space, and let $(x, y) \in [-u, u]$ of $\mathcal{OU}(E)$. Then* *(i) If* $0 < y < 1$*, then there exist* $0 < \alpha < 1$ *and* $x' \in \mathrm{Ball}(E)$ *such that* $(x, y) = \alpha(0, 1) + (1 - \alpha)(x', 0)$.

(ii) If $-1 < y < 0$*, then there exist* $0 < \alpha < 1$ *and* $x' \in \mathrm{Ball}(E)$ *such that* $(x, y) = \alpha(0, -1) + (1 - \alpha)(x', 0)$.

Proof. We give the proof of (i) only, as the proof of (ii) is similar[3]. Define $\alpha = y$, so $0 < \alpha < 1$. We can therefore divide by $1 - \alpha$, so

$$\|x\| + |\alpha| \leq 1 \Rightarrow \|x\| \leq 1 - \alpha \Rightarrow \frac{\|x\|}{1 - \alpha} \leq 1.$$

So we define $x' = \frac{x}{1-\alpha}$, and $\|x'\| = \frac{\|x\|}{1-\alpha} \leq 1$, so $x' \in \mathrm{Ball}(E)$. Then

$$\alpha(0, 1) + (1 - \alpha)(x', 0) = \left((1 - \alpha)\frac{x}{1 - \alpha}, \alpha\right) = (x, y),$$

as required. $\square$

Lemma 6.5. *Let* E *be a normed space. Then*

$$\partial(\mathrm{Ball}(\mathcal{OU}(E))) = \{(0, 1)\} \cup (\partial(\mathrm{Ball}(E)) \times \{0\}) \cup \{(0, -1)\},$$

i.e. *an extreme point of the closed unit ball of* $\mathcal{OU}(E)$ *is either* $(0, 1)$*,* $(0, -1)$ *or of the form* $(x, 0)$ *where* x *is an extreme point of the closed unit ball of* E*. In particular, if* $\mathrm{Ball}(E)$ *has no extreme points, then* $\partial(\mathrm{Ball}(\mathcal{OU}(E))) = \{(0, 1), (0, -1)\}$.

Proof. We proceed in five steps.

1. $(0, 1)$ and $(0, -1)$ are extreme points:

   Suppose $(0, 1) = \alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)$, where $0 < \alpha < 1$ and $(x_i, y_i) \in [-u, u]$ for $i \in \{1, 2\}$. So $\alpha y_1 + (1 - \alpha)y_2 = 1$, and as $y_1, y_2 \in [-1, 1]$ and 1 is an extreme point of $[-1, 1]$, we have $y_1 = y_2 = 1$. As $\|x_i\| + |y_i| \leq 1$, $\|x_i\| \leq 0$ and so $x_1 = x_2 = 0$. This proves that $(0, 1)$ is an extreme point. The proof that $(0, -1)$ is an extreme point is similar.

2. If $x \in \partial(\mathrm{Ball}(E))$, then $(x, 0) \in \partial([-u, u])$:

   Let $x \in \partial(\mathrm{Ball}(E))$, and take $0 < \alpha < 1$ and $(x_1, y_1), (x_2, y_2) \in [-u, u]$. So $\alpha x_1 + (1 - \alpha)x_2 = x$, and therefore $x_1 = x_2 = x$. Since $\|x_i\| + |y_i| \leq 1$, we have $|y_i| \leq 0$, so $y_i = 0$ for each $i \in \{1, 2\}$. Therefore $(x_i, y_i) = (x, 0)$, so $(x, 0)$ is an extreme point.

3. If $y \notin \{-1, 0, 1\}$, $(x, y) \notin \partial([-u, u])$:

   Let $(x, y) \in [-u, u]$ and $y \notin \{-1, 0, 1\}$. By Lemma 6.4, if $y > 0$ then there exists $0 < \alpha < 1$ such that $x' \in \mathrm{Ball}(E)$ such that $(x, y) = \alpha(0, 1) + (1 - \alpha)(x', 0)$, and if $y < 0$ there similarly exists α and x' such that $(x, y) = \alpha(0, -1) + (1 - \alpha)(x', 0)$. In either case, this shows that $(x, y) \notin \partial([-u, u])$.

4. If $y \in \{-1, 1\}$ then $x = 0$:

   We have $\|x\| + |y| \leq 1$, and therefore $|y| = 1$, so $\|x\| \leq 0$, which implies $x = 0$.

[3] Using $\alpha = -y$.

5. If $y = 0$ and $x \notin \partial(\mathrm{Ball}(E))$ then $(x, y) \notin \partial([-u, u])$:

   As $x \notin \partial(\mathrm{Ball}(E))$, there exist $x_1, x_2 \in \mathrm{Ball}(E)$ and $0 < \alpha < 1$ such that $\alpha x_1 + (1-\alpha)x_2 = x$. Then $(x, 0) = \alpha(x_1, 0) + (1-\alpha)(x_2, 0)$. As $x_i \in \mathrm{Ball}(E)$, we have $\|x_i\| + |0| = \|x_i\| \leq 1$, so $(x_i, 0) \in [-u, u]$, and therefore $(x, 0) \notin \partial([-u, u])$.

In total, we have shown that if $(x, y) \in \{(0, 1), (0, -1)\} \cup (\partial(\mathrm{Ball}(E)) \times \{0\})$, then it is an extreme point of $[-u, u]$ in $\mathcal{OU}(E)$, and that if (x, y) is not in that set, then it is not an extreme point. This characterizes the extreme points of $[-u, u]$ in $\mathcal{OU}(E)$. If $\partial(\mathrm{Ball}(E))$ is empty, then $\partial(\mathrm{Ball}(E)) \times \{0\}$ is empty, so the only extreme points are $(0, 1)$ and $(0, -1)$, *i.e.* u and $-u$. □

We now have enough to complete the following counterexample.

Counterexample 6.6. For any normed space E, such that $\mathrm{Ball}(E)$ has no extreme points and $\mathcal{T}$ is a Hausdorff vector space topology on E in which $\mathrm{Ball}(E)$ is compact[4], if $\mathcal{OU}(E)$ is equipped with the product topology, then $\mathrm{Ball}(\mathcal{OU}(E))$ is compact but its extreme points are only $\{(0, 1), (0, -1)\}$, so $\mathcal{OU}(E)$ is not a dual space. The unit interval $[0, 1]_{\mathcal{OU}(E)}$ is a compact effect module in which the operation of taking convex combinations is continuous, but that is not a compact effect module in the sense of [6, §3.4, §4.4].

Proof. Let E be a normed space, such that $\mathrm{Ball}(E)$ has no extreme points, $\mathcal{T}$ a Hausdorff vectorial topology on E in which $\mathrm{Ball}(E)$ is compact, such as one of the space constructed from Roberts's example in Counterexample 6.3. As the underlying vector space of $\mathcal{OU}(E)$ is $E \times \mathbb{R}$, we can tive it the product topology from $\mathcal{T}$ and the usual topology of $\mathbb{R}$. As $\mathrm{Ball}(E)$ has no extreme points, the extreme points of $\mathrm{Ball}(\mathcal{OU}(E))$ are just $\{(0, 1), (0, -1)\}$, by Lemma 6.5.

We show that $\mathrm{Ball}(\mathcal{OU}(E))$ is compact as follows. As $\mathcal{OU}(E)$ and E are topological vector spaces, the map $f : \mathbb{R}^3 \times E \to \mathcal{OU}(E)$ defined by

$$f(\alpha, \beta, \gamma, x) = \alpha(0, 1) + \beta(x, 0) + \gamma(0, -1)$$

is continuous. The set $C \subseteq \mathbb{R}^3$ defined as

$$C = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid \alpha, \beta, \gamma \geq 0 \text{ and } \alpha + \beta + \gamma\},$$

is compact, so $C \times \mathrm{Ball}(E) \subseteq \mathbb{R}^3 \times E$ is compact, and therefore the image $f(C \times \mathrm{Ball}(E)) \subseteq \mathcal{OU}(E)$ is compact.

By the convexity of $\mathrm{Ball}(\mathcal{OU}(E))$, $f(C \times \mathrm{Ball}(E)) \subseteq [-u, u]$. We have $(0, 1) = f(1, 0, 0, 0)$, $(0, -1) = f(0, 0, 1, 0)$, $(x, 0) = f(0, 1, 0, x)$, if $0 < y < 1$ there exist $x' \in \mathrm{Ball}(E)$ and $0 < \alpha < 1$ such that $(x, y) = f(\alpha, (1-\alpha), 0, x')$, and if $-1 < y < 0$ there exist $x' \in \mathrm{Ball}(E)$ and $0 < \alpha < 1$ such that $(x, y) = f(0, (1-\alpha), \alpha, x')$. All together, this means that $\mathcal{OU}(E) \subseteq f(C \times \mathrm{Ball}(E))$, and therefore $f(C \times \mathrm{Ball}(E)) = \mathcal{OU}(E)$, and so $\mathcal{OU}(E)$ is compact.

If $\mathcal{OU}(E)$ were a dual space, $[-u, u]$ would be the closed convex hull of its extreme points in the weak-* topology, but by Lemma 6.5, the only extreme points of $[-u, u]$ are $(0, 1)$ and $(0, -1)$, their convex hull is $\{(0, \alpha) \mid \alpha \in [-1, 1]\}$, which is compact and therefore closed. Then we can take any $x \in \mathrm{Ball}(E)$, $x \neq 0$, and obtain $(x, 0) \in [-u, u]$ that is not in the closed convex hull of the extreme points of $\mathrm{Ball}(\mathcal{OU}(E))$, which contradicts $\mathcal{OU}(E)$ being a dual space.

[4] Such spaces exist by Counterexample 6.3.

Finally, as $[0,1]_{\mathcal{OU}(E)} = \frac{1}{2}(u + [-u,u])$ (see [6, Lemma 0.2.2]), and this is the image of $[-u,u]$ under a continuous map, $[0,1]_{\mathcal{OU}(E)}$ is a compact effect module. The operation of forming a convex combination is continuous because $\mathcal{OU}(E)$ is a topological vector space. If it were a compact effect module in the sense of [6, §3.4, §4.4], it would be embeddable as a compact subset of a locally convex vector space (by definition), and this would contradict the Krein-Mil'man theorem. $\square$

Acknowledgements

Robert Furber has been financially supported by the Danish Council for Independent Research, Project 4181-00360.

References

[1] Stefan Banach. *Théorie des Opérations Linéaires*, volume 1 of *Monografie Matematyczne*. Instytut Matematyczny Polskiej Akademii Nauk, 1932. 5

[2] Nicolas Bourbaki. *General Topology*. Ettore Majorana International Science. Springer, 1998. 11

[3] David A. Edwards. On the Homeomorphic Affine Embedding of a Locally Compact Cone into a Banach Dual Space Endowed with the Vague Topology. *Proceedings of the London Mathematical Society*, 14:399–414, 1964. 10

[4] Alan J. Ellis. The Duality of Partially Ordered Normed Linear Spaces. *Journal of the London Mathematical Society*, s1-39(1):730–744, 1964. 10

[5] David H. Fremlin. Measure Theory, Volume 4. `https://www.essex.ac.uk/maths/people/fremlin/mt.htm`, 2003. 6

[6] Robert Furber. *Categorical Duality in Probability and Quantum Foundations*. PhD thesis, Radboud Universiteit Nijmegen, 2017. Link available at `http://www.robertfurber.com`. 1, 2, 3, 4, 5, 6, 10, 11, 13, 14

[7] Robert Furber. Categorical Equivalences from State-Effect Adjunctions. In Peter Selinger and Giulio Chiribella, editors, Proceedings of the 15th International Conference on *Quantum Physics and Logic,* Halifax, Canada, 3-7th June 2018, volume 287 of *Electronic Proceedings in Theoretical Computer Science*, pages 107–126. Open Publishing Association, 2019. 11

[8] Robert Furber. Categorical Duality in Probability and Quantum Foundations (Second Version). `www.robertfurber.com`, 2020. 10, 11

[9] Robert Furber. Scott Continuity in Generalized Probabilistic Theories. In Bob Coecke and Matthew Leifer, editors, Proceedings 16th International Conference on *Quantum Physics and Logic,* Chapman University, Orange, CA, USA., 10-14 June 2019, volume 318 of *Electronic Proceedings in Theoretical Computer Science*, pages 66–84. Open Publishing Association, 2020. 1, 2, 4, 7, 8, 9, 11

[10] Michèle Giry. A Categorical Approach to Probability Theory. In B. Banaschewski, editor, *Categorical Aspects of Topology and Analysis*, pages 68–85. Springer, 1980. 5

[11] Stanley Gudder. Convex Structures and Operational Quantum Mechanics. *Communications in Mathematical Physics*, 29(3):249–264, 1973. 5

[12] Robert C. James. Bases and Reflexivity of Banach Spaces. *Annals of Mathematics*, 52(3):518–527, 1950. 9

[13] Nigel J. Kalton and N. Tenney Peck. A Re-examination of the Roberts Example of a Compact Convex Set without Extreme Points. *Mathematische Annalen*, 253(2):89–101, 1980. 9, 11

[14] Alexander S. Kechris. *Classical Descriptive Set Theory*. Number 156 in Graduate Texts in Mathematics. Springer, 1995. 5

[15] J. Donald Monk and Robert Bonnet, editors. *Handbook of Boolean Algebras*. North-Holland, 1989. 7

[16] Terry J. Morrison. *Functional Analysis: An Introduction to Banach Space Theory*. Pure and Applied Mathematics. John Wiley and Sons, 2001. 9

[17] James W. Roberts. A Compact Convex Set with no Extreme Points. *Studia Mathematica*, 60(3):255–266, 1977. 9, 11

[18] Stefan Rolewicz. *Metric Linear Spaces*. Mathematics and its Applications. D. Reidel Publishing Company, 1984. 9

[19] Halsey L. Royden. *Real Analysis*. The Macmillan Company, 1963. 6

[20] Helmut H. Schaefer. *Topological Vector Spaces*, volume 3 of *Graduate Texts in Mathematics*. Springer Verlag, 1966. 10

[21] René L. Schilling. *Measures, Integrals and Martingales*. Cambridge University Press, 2005. 6

[22] Marshall H. Stone. Applications of the Theory of Boolean Rings to General Topology. *Transactions of the American Mathematical Society*, 41(3):375–481, 1937. 5