β reduction without rule ξ

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Overview

- A concrete representation of lambda terms.
- Locally nameless:
  - indexes for bound positions,
  - names for free variables.
  - Canonical: α conversion is syntactic identity.
- Abstraction, $\mathsf{lam}_x M$, is a defined function.
- Using the defined abstraction, the language looks like conventional notation.
- We can define various reduction relations without rule ξ.
- Only works for some relations.
  - Apparently fails for η.

Preterms and well formedness

- Let i, j, m, n, p, q, range over natural numbers.
- Fix a countable set of *names*, ranged over by x, y, z.
- The raw syntax of *preterms* (ranged over by M, N, P, Q) is

$$\text{pt} ::= \mathrm{X}_n\, x \;\mid\; \mathrm{J}_n\, j \;\mid\; \lceil M, N \rceil_n$$

In preterm syntax, n is the *height* of the preterm, written $hgt\, M$.

Well formedness (written $\mathcal{W}M$) is defined inductively by

$$\frac{}{\mathcal{W}\mathrm{X}_n\, x} \qquad \frac{j < n}{\mathcal{W}\mathrm{J}_n\, j} \qquad \frac{\mathcal{W}P \quad \mathcal{W}Q \quad n \leq hgt\, P \quad n \leq hgt\, Q}{\mathcal{W}\lceil P, Q \rceil_n}$$

- Well formed preterms are called *terms*.

Intended meaning of well formed terms

$$\frac{}{\mathcal{W}\mathrm{X}_n\, x} \qquad \frac{j < n}{\mathcal{W}\mathrm{J}_n\, j} \qquad \frac{\mathcal{W}P \quad \mathcal{W}Q \quad n \leq hgt\ P \quad n \leq hgt\ Q}{\mathcal{W}\lceil P, Q \rceil_n}$$

- $\mathrm{X}_n\, x$ represents $\lambda_1 \ldots \lambda_n\, x$ (so $\mathrm{X}_0\, x$ represents x).
- $\mathrm{J}_n\, j$ represents $\lambda_1 \ldots \lambda_n\, j$.
  - Require $j < n$ for well formedness; otherwise j would be unbound.
- If M_1 represents t_1 and M_2 represents t_2 then $\lceil M_1, M_2 \rceil_0$ represents $(t_1\ t_2)$.
- Terms are de Bruijn closed using only the black text.
- What are the red premises for?

Abstraction defined as a function on preterms

$$\begin{aligned}
\mathrm{lam}_x(\mathsf{X}_n\, y) &:= \text{if } x = y \text{ then } \mathsf{J}_{n+1}\, 0 \text{ else } \mathsf{X}_{n+1}\, y \\
\mathrm{lam}_x(\mathsf{J}_n\, j) &:= \mathsf{J}_{n+1}\, (j+1) \\
\mathrm{lam}_x \lceil M, N\rceil_n &:= \lceil \mathrm{lam}_x M, \mathrm{lam}_x N\rceil_{n+1}
\end{aligned}$$

- Abstraction preserves well formedness and raises height by one.

$$\mathcal{W} M \implies \mathcal{W}(\mathrm{lam}_x M) \qquad hgt\,(\mathrm{lam}_x M) = hgt\, M + 1$$

- Conversely, every term with height a successor is an abstraction.

$$\mathcal{W} M \wedge hgt\, M = n + 1 \implies \exists\, P, x \,.\, M = \mathrm{lam}_x P$$

  The red premises of well-formedness are needed for this lemma.
- We use A, B as metavariables over abstractions.

Examples

- Using $\mathrm{lam}_x M$ we can write lambda terms as usual
- Notations: write
  - $\mathrm{lam}_{xy} M$ for $\mathrm{lam}_x \mathrm{lam}_y M$.
  - $\overline{x}$ for $\mathrm{X}_0\, x$.

Some combinators: (assuming $x \neq y,\ x \neq z,\ y \neq z$)

$$\begin{array}{rclrcl} I & = & \lambda x\,.\,x & \mathrm{lam}_x \overline{x} & = & \mathrm{J}_1\, 0 \\ K & = & \lambda x\, y\,.\,x & \mathrm{lam}_{xy} \overline{x} & = & \mathrm{J}_2\, 0 \\ \mathit{false} & = & \lambda x\, y\,.\,y & \mathrm{lam}_{xy} \overline{y} & = & \mathrm{J}_2\, 1 \end{array}$$

$$\begin{array}{rcl} S & = & \lambda x\, y\, z\,.\,(x\, z)\,(y\, z) \\ \mathrm{lam}_{xyz} \lceil \lceil \overline{x}, \overline{z} \rceil_0, \lceil \overline{y}, \overline{z} \rceil_0 \rceil_0 & = & \lceil \lceil \mathrm{J}_3\, 0, \mathrm{J}_3\, 2 \rceil_3, \lceil \mathrm{J}_3\, 1, \mathrm{J}_3\, 2 \rceil_3 \rceil_3 \end{array}$$

Adequacy

- Let t range over lambda terms (e.g. Nominal Isabelle lambda terms).
- As usual, M ranges over our terms.
- the relation between lambda terms and our terms is given by:

$$x \sim \mathrm{X}_0\, x \qquad \frac{t_1 \sim M_1 \quad t_2 \sim M_2}{(t_1\ t_2) \sim \lceil M_1, M_2 \rceil_0} \qquad \frac{t \sim M}{\lambda x.t \sim \mathrm{lam}_x M}$$

- $\sim$ respects $\mathcal{W}$: $\quad t \sim M \implies \mathcal{W} M$
- $\sim$ is total, single-valued, and injective.
- We must define substitution and check that $\sim$ respects substitution

Lifting

To define instantiation we first introduce a lifting function

$$
\begin{aligned}
(\mathrm{X}_n\, y)^{\uparrow} &:= \mathrm{X}_{n+1}\, y \\
(\mathrm{J}_n\, j)^{\uparrow} &:= \mathrm{J}_{n+1}\, (j{+}1) \\
(\lceil M, N \rceil_n)^{\uparrow} &:= \lceil (M)^{\uparrow}, (N)^{\uparrow} \rceil_{n+1}
\end{aligned}
$$

which we iterate as:

$$
\begin{aligned}
(M)^{\uparrow 0} &:= M \\
(M)^{\uparrow m+1} &:= ((M)^{\uparrow m})^{\uparrow}
\end{aligned}
$$

- Lifting preserves well formedness and raises height by one.

$$
\mathcal{W} M \implies \mathcal{W} (M)^{\uparrow} \qquad\qquad hgt\,(M)^{\uparrow} = hgt\, M + 1
$$

Instantiation

Instantiation is a binary function, $M[N]$.

- If $hgt\, M = 0$ (M is under no binders), $M[N] = M$.
- Otherwise $M[N]$ fills any holes $\mathsf{J}_{n+1}\, 0$ in M and adjusts the rest of the term:

$$\begin{aligned} \mathsf{X}_{n+1}\, y[N] &:= \mathsf{X}_n\, y \\ \mathsf{J}_{n+1}\, 0[N] &:= (N)^{\uparrow n} \\ \mathsf{J}_{n+1}\, (j{+}1)[N] &:= \mathsf{J}_n\, j \\ \lceil M, P \rceil_{n+1}[N] &:= \lceil M[N], P[N] \rceil_n \end{aligned}$$

- Instantiation is not substitution.
- Instantiation preserves well formedness:

$$\mathcal{W}M \wedge \mathcal{W}N \implies \mathcal{W}(M[N]) \;\wedge\; (hgt\, M) - 1 \leq hgt\, M[N]$$

Substitution

Substitution is defined in terms of instantiation:

$$M[x \leftarrow P] := (\text{lam}_x M)[P]$$

- All the expected properties hold.
- Usual substitution lemma:

$$x \neq y \wedge x \notin \text{FV(P)} \wedge \mathcal{W}(\text{M}, \text{P}, \text{N}) \implies M[x \leftarrow N][y \leftarrow P] = M[y \leftarrow P][x \leftarrow N[y \leftarrow P]]$$

Now we can finish adequacy: $\sim$ respects substitution:

$$s \sim M \wedge t \sim N \implies t[x \leftarrow s] \sim N[x \leftarrow M]$$

β reduction as usual

Using abstraction we have a natural definition of β reduction:

$$\frac{\mathcal{W}M \quad \mathcal{W}N}{\lceil \mathrm{lam}_x M, N \rceil_0 \overset{\beta}{\to} M[x \leftarrow N]} \ (\beta)$$

$$\frac{M \overset{\beta}{\to} M' \quad \mathcal{W}N}{\lceil M, N \rceil_0 \overset{\beta}{\to} \lceil M', N \rceil_0} \qquad \frac{\mathcal{W}M \quad N \overset{\beta}{\to} N'}{\lceil M, N \rceil_0 \overset{\beta}{\to} \lceil M, N' \rceil_0}$$

$$\frac{M \overset{\beta}{\to} N}{\mathrm{lam}_x M \overset{\beta}{\to} \mathrm{lam}_x N} \ (\xi)$$

- Any preterm that participates in this relation is well-formed.
- Correct β reduction w.r.t. the meaning of terms given above,
- Still contains rule ξ

Properties of usual β reduction

- As usual, rule ξ is invertible:

$$\mathrm{lam}_x M \xrightarrow{\beta} \mathrm{lam}_x N \implies M \xrightarrow{\beta} N$$

- β reduction does not lower height:

$$M \xrightarrow{\beta} N \implies hgt\, M \leq hgt\, N$$

Generalized lifting

To eliminate rule ξ from our presentation of β reduction, we define *generalized lifting*.

$$(\mathrm{X}_n\, y)^{i\Uparrow} := \mathrm{X}_{n+1}\, y$$

$$(\mathrm{J}_n\, j)^{i\Uparrow} := \begin{cases} \mathrm{J}_{n+1}\, j & (j < i) \\ \mathrm{J}_{n+1}\, (j+1) & (j \geq i) \end{cases}$$

$$(\lceil M, N \rceil_n)^{i\Uparrow} := \lceil (M)^{i\Uparrow}, (N)^{i\Uparrow} \rceil_{n+1}$$

- Preserves well formedness and raises height by one.
- Many useful properties of generalized lifting are used, e.g.
  - Injectivity: $\mathcal{W}(M, N) \wedge (M)^{i\Uparrow} = (N)^{i\Uparrow} \implies M = N$.

We iterate generalized lifting:

$$(M)^{i\Uparrow 0} := M$$

$$(M)^{i\Uparrow m+1} := ((M)^{i\Uparrow m})^{i\Uparrow}$$

Generalized instantiation

Generalized instantiation, $M[N]^i$, leaves terms M of height 0 unchanged, and updates abstractions:

$$\begin{aligned}
\mathsf{X}_{n+1}\, y[M]^i &:= \mathsf{X}_n\, y \\
\mathsf{J}_{n+1}\, i[M]^i &:= (M)^{i \Uparrow n-i} \\
\mathsf{J}_{n+1}\, j[M]^i &:= \begin{cases} \mathsf{J}_n\, j & (j < i) \\ \mathsf{J}_n\, (j-1) & (j > i) \end{cases} \\
\lceil P, Q \rceil_{n+1}[M]^i &:= \lceil P[M]^i, Q[M]^i \rceil_n
\end{aligned}$$

- $A[P]^0 = A[P]$
- $n < hgt\, A \wedge n \leq hgt\, P \implies n \leq hgt\,(A[P]^n)$
- $n < hgt\, A \wedge n \leq hgt\, P \wedge \mathcal{W} A \wedge \mathcal{W} P \implies \mathcal{W}(A[P]^n)$

β without rule ξ

Claim the relation $\bullet > \bullet$ defined without a ξ rule:

$$\frac{\mathcal{W}A \quad n < hgt\ A \quad \mathcal{W}N \quad n \leq hgt\ N}{\lceil A, N \rceil_n > A[N]^n}\ (\beta)$$

$$\frac{M > M' \quad n \leq hgt\ M \quad \mathcal{W}N \quad n \leq hgt\ N}{\lceil M, N \rceil_n > \lceil M', N \rceil_n}$$

$$\frac{N > N' \quad n \leq hgt\ N \quad \mathcal{W}M \quad n \leq hgt\ M}{\lceil M, N \rceil_n > \lceil M, N' \rceil_n}$$

is equivalent to the relation $\bullet \xrightarrow{\beta} \bullet$ given above (and thus to the usual notion of β reduction).

Proof that $M > N \Longrightarrow M \xrightarrow{\beta} N$: by induction on the relation $M > N$.

Both congruence rule cases use invertibility of rule ξ for relation $\xrightarrow{\beta}$.

The converse direction is straightforward. □

Tait–Martin-Löf parallel reduction: Usual presentation

Parallel reduction (non-deterministic):

$$\frac{}{\mathsf{X}_0\, x \overset{p}{\to} \mathsf{X}_0\, x} \qquad \frac{M \overset{p}{\to} M' \quad N \overset{p}{\to} N'}{\lceil \mathsf{lam}_x M, N \rceil_0 \overset{p}{\to} M'[x \leftarrow N']}\ (\beta)$$

$$\frac{M \overset{p}{\to} N}{\mathsf{lam}_x M \overset{p}{\to} \mathsf{lam}_x N}\ (\xi) \qquad \frac{M \overset{p}{\to} M' \quad N \overset{p}{\to} N'}{\lceil M, N \rceil_0 \overset{p}{\to} \lceil M', N' \rceil_0}$$

- Correct w.r.t. usual presentation.
- Overlap between rule (β) and application congruence.

Complete development (deterministic, à la Takahashi):

- Remove overlap, forcing every β step to be taken:

$$\frac{M \overset{cd}{\to} M' \quad N \overset{cd}{\to} N' \quad \color{red}{M \text{ not an abstraction}}}{\lceil M, N \rceil_0 \overset{cd}{\to} \lceil M', N' \rceil_0}$$

Parallel reduction without rule ξ

Parallel reduction:

$$\frac{}{\mathsf{X}_n\, y \gg \mathsf{X}_n\, y} \qquad \frac{j < n}{\mathsf{J}_n\, j \gg \mathsf{J}_n\, j}$$

$$\frac{n \leq hgt\, M \quad M \gg M' \quad n \leq hgt\, N \quad N \gg N'}{\lceil M, N \rceil_n \gg \lceil M', N' \rceil_n}$$

$$\frac{n < hgt\, A \quad A \gg B \quad n \leq hgt\, M \quad M \gg N}{\lceil A, M \rceil_n \gg B[N]^n}$$

Complete development (remove overlap):

$$\frac{n = hgt\, M \quad M \ggg M' \quad n \leq hgt\, N \quad N \ggg N'}{\lceil M, N \rceil_n \ggg \lceil M', N' \rceil_n}$$

Church–Rosser theorem

- With parallel reduction and complete development, we can carry out Takahashi's proof of Church–Rosser.
- Although there is no rule ξ, this proof is no easier than usual.

η?

Consider a standard representation of pure η reduction:

$$\frac{\mathcal{W}M \quad x \notin \mathrm{FV(M)}}{\mathrm{lam}_x \lceil M, (\mathrm{X}_0\, x)\rceil_0 \overset{\eta}{\to} M} \; (\eta)$$

$$\frac{M \overset{\eta}{\to} M' \quad \mathcal{W}N}{\lceil M, N\rceil_0 \overset{\eta}{\to} \lceil M', N\rceil_0} \qquad \frac{\mathcal{W}M \quad N \overset{\eta}{\to} N'}{\lceil M, N\rceil_0 \overset{\eta}{\to} \lceil M, N'\rceil_0} \qquad \frac{M \overset{\eta}{\to} N}{\mathrm{lam}_x M \overset{\eta}{\to} \mathrm{lam}_x N}$$

Rule ξ is not invertible for this relation:

- $\mathrm{lam}_x \lceil \mathrm{lam}_x \lceil \overline{x}, \overline{x}\rceil_0, \overline{x}\rceil_0 \overset{\eta}{\to} \mathrm{lam}_x \lceil \overline{x}, \overline{x}\rceil_0$, but not $\lceil \mathrm{lam}_x \lceil \overline{x}, \overline{x}\rceil_0, \overline{x}\rceil_0 \overset{\eta}{\to} \lceil \overline{x}, \overline{x}\rceil_0$
- We might conjecture a ξ-free system for η, but our proof of correctness (using invertibility of ξ) will fail.
- $\overset{\eta}{\to}$ can reduce height, which the previous relations cannot do.