Geometry Explorer: Combining Dynamic Geometry, Automated Geometry Theorem Proving and Diagrammatic Proofs

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1 Introduction

In this extended abstract, we describe *Geometry Explorer* [2], a prototype system that allows users to create Euclidean geometry constructions using a dynamic geometry interface, specify conjectures about them and then use a full-angle method [1] prover to automatically produce diagram independent, human-readable proofs to theorems. Our system can then automatically generate novel diagrammatic proofs of the forward-chaining and backward-chaining reasoning used by the geometry theorem prover.

2 Dynamic Geometry

Dynamic geometry software allows users to create Euclidean geometry diagrams using ruler and compass construction tools, in a similar way to how they are sketched on paper. Unlike traditional diagrams, constructions can be moved after they have been created, causing the position of dependent constructions to update automatically, thereby generating new diagram instances. This allows exploration of diagram properties and can help users to find new conjectures in ways that are not possible with static diagrams. This concept is best demonstrated with an example.

Example 2.1 (Nine Point Circle Theorem). *Let AD be the altitude on BC and let the midpoints of the sides AB, BC and CA of $\triangle ABC$ be E, F and G respectively. Show that D, E, F and G are cyclic.*

The hypotheses of the example can be constructed as a dynamic diagram within the Geometry Explorer GUI and then explored by manipulating construction positions (see Figure 1). After the user specifies the conjecture for the example theorem, the integrated *full-angle method* theorem prover automatically generates a proof.

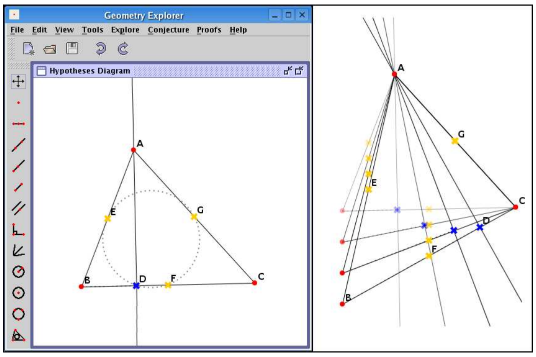

Figure 1: The Geometry Explorer GUI after the hypotheses of the example has been constructed (left) and exploring dynamic diagram instances by changing the position of free point B (right).

3 The Full-Angle Method

The full-angle method [1] has been demonstrated to prove hundreds of geometry theorems automatically whilst producing proofs which are short, human-readable and diagram independent. These proofs include non-trivial examples featuring in American Mathematics Monthly and The International Mathematical Olympiad. The full-angle method relies on a single high-level geometric invariant called the *full-angle* to prove theorems.

Definition 3.1. *The full-angle between the ordered pair of lines u and v is written as $\angle[u,v]$ and can be thought of intuitively as the rotation required to make the line u parallel to line v. Two full-angles $\angle[m,n]$ and $\angle[u,v]$ are equal if there exists a rotation R such that $R(m) \parallel u$ and $R(n) \parallel v$.*

Definition 3.2. *For all $u \parallel v$, $\angle[u,v] = \angle[0]$ is a constant and is said to be a flat full-angle.*

Definition 3.3. *For all $u \perp v$, $\angle[u,v] = \angle[1]$ is a constant and is said to be a right full-angle.*

Definition 3.4. *For any line w, addition of full-angles is defined as $\angle[u,v] = \angle[u,w] + \angle[w,v]$.*

The following steps are used to find a proof:

1. In *predicate form*, the hypotheses is put into a so-called *Geometry Information Basis (GIB)*.

2. *Exhaustive forward-chaining* is applied to the GIB to discover new facts, using rules such as:

   F3 If M and N are the midpoints of AB and AC respectively then $MN \parallel BC$.

   F5 If O is the midpoint of CA and $AB \perp BC$ then O is the circumcenter of $\triangle ABC$.

3. The theorem conjecture is represented as $\angle[0] = \sum f_i$, where f_i is a full-angle.

4. Full-angles are replaced with equal expressions using *conditional rewrite rules* such as:

   R1 $\angle[AB, CD] = \angle[AB, EF]$ if $CD \parallel EF$.

R2 $\angle[AB, CD] = \angle[AB, EF] + \angle[1]$ if $CD \perp EF$.

R6 $\angle[AB, BC] = \angle[AD, CD]$ if A, B, C and D are cyclic.

5. A search algorithm is used to find a sequence of rewrites that transforms the full-angle equation to $\angle[0] = \angle[0]$, giving a *backward-chaining proof.*

4 Diagrammatic Proofs

Geometry Explorer can automatically generate *diagrammatic proofs* using the full-angle method. Forward-chaining reasoning is shown as a graph (see Figure 2), where nodes represent known geometry facts. Each is shown using an image of the hypotheses diagram, which has been marked with appropriate geometry notation. Edges show which facts inferred others, with labels showing the rule used.

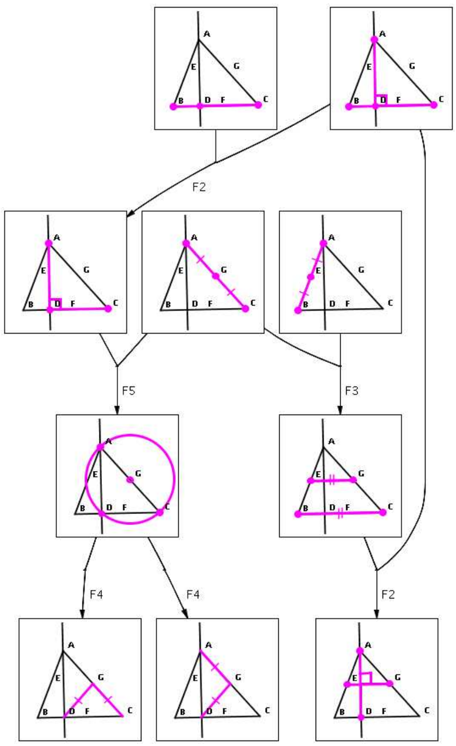

Figure 2: A diagrammatic forward-chaining proof of some interesting discovered facts from the example hypotheses.

Backward-chaining proofs are shown as a graph (see Figure 3), where each node represents a full-angle, using annotated diagrams like before. Full-angles x and y joined to z with a "+" node means that $x + y = z$. Using rule R6, the example theorem conjecture is represented as $\angle[0] = \angle[EF, FD] + \angle[DG, GE]$, as shown at the top of the graph in Figure 3. Edges labelled with rule names show rewrite rule applications and the cancellations at the bottom of the graph complete the proof.

5 Conclusions

Our system takes advantage of the diagram independent reasoning used by the full-angle method to automatically generate diagrammatic proofs, allowing proof steps to be investigated geometrically. The diagrammatic forward-chaining proofs were found to be intuitive diagrammatically, but these cannot be used to prove difficult theorems unaided. The diagrammatic backward-chaining proofs are powerful, but common rewrite rules are unintuitive diagrammatically. It would be desirable to find a geometry theorem proving method that is both powerful and diagrammatically intuitive.

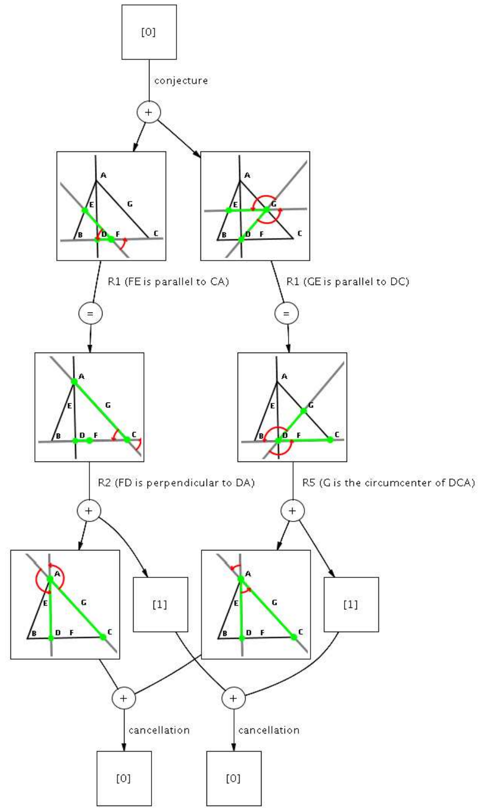

Figure 3: A diagrammatic backward-chaining proof showing that D, E, F and G are cyclic from the example hypotheses.

6 Acknowledgements

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References

[1] Chou, S.-C., X.-S. Gao and J.-Z. Zhang, *Automated generation of readable proofs with geometric invariants, II. Theorem proving with full-angles*, Journal of Automated Reasoning **17** (1996), pp. 349–370.

[2] Wilson, S. and J. D. Fleuriot, *Combining dynamic geometry, automated geometry theorem proving and diagrammatic proofs*, in: *Proceedings of the European Joint Conferences on Theory and Practice of Software (ETAPS) Satellite Workshop on User Interfaces for Theorem Provers (UITP)*, Edinburgh, UK, 2005.