Parameterized Regular Expressions and Their Languages

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- $\Sigma$: a finite alphabet
- $\mathcal{V}$: a countably infinite set of variables $x, y, z, \ldots$

A PRE over $\Sigma$ is a regular expression over alphabet $\Sigma \cup \mathcal{V}$. 
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A PRE over $\Sigma$ is a regular expression over alphabet $\Sigma \cup \mathcal{V}$.

$(0x)^*1(xy)^*$ and $(0|1)^*xy(0|1)^*$ are PREs over $\{0, 1\}$. 
Language of PREs?

\[(0x)^* 1(xy)^* \quad (0|1)^* xy(0|1)^*.\]
Each PRE defines a regular language over $(\Sigma \cup \mathcal{V})^*$. 

\[(0x)^*1(xy)^* \quad (0|1)^*xy(0|1)^*\]
Language of PREs?

$$(0x)^*1(xy)^* \quad (0|1)^*xy(0|1)^*.$$ Each PRE defines a regular language over $(\Sigma \cup \mathcal{V})^*$. We want PREs to define languages over $\Sigma$. 
For now, variables are interpreted as **symbols** from $\Sigma$.

Given a PRE $e$ over $\Sigma$ that uses variables $\mathcal{W} \subset \mathcal{V}$:

- A **valuation** for $e$ is a mapping $\nu : \mathcal{W} \rightarrow \Sigma$. 
How to interpret variables in PREs

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$\nu : x \mapsto 0, \ y \mapsto 1$
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Example:

$$e = (0x)^*1(xy)^* \quad \nu : x \mapsto 0, \quad y \mapsto 1$$

$$\nu(e) = (00)^*1(01)^*$$
Semantics for PREs over $\Sigma$: Two alternatives

Let $e$ be a PRE over $\Sigma$. Then

$$\mathcal{L}_{\Diamond}(e) := \bigcup \{ \mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e \}$$

(possibility)
Semantics for PREs over $\Sigma$: Two alternatives

Let $e$ be a PRE over $\Sigma$. Then

$$\mathcal{L}_\diamond(e) := \bigcup \{ \mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e \}$$

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$$\mathcal{L}_\diamond(e) =$$
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$$\Rightarrow 00101 \text{ is in } \mathcal{L}_{\Diamond}(e).$$
Semantics for PREs over $\Sigma$: Two alternatives

Let $e$ be a PRE over $\Sigma$. Then

$\mathcal{L}_\square(e) := \bigcap \{ \mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e \}$  (certainty)
Semantics for PREs over $\Sigma$: Two alternatives

Let $e$ be a PRE over $\Sigma$. Then

$$\blacktriangleleft L_2(e) := \bigcap \{L(\nu(e)) \mid \nu \text{ is a valuation for } e\} \quad \text{(certainty)}$$

Example:

$$e = (0|1)^*xy(0|1)^*$$
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Semantics for PREs over $\Sigma$: Two alternatives

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(0|1)^*10(0|1)^* \cap (0|1)^*11(0|1)^*$$

$10011$ is in $\L\Box(e)$.
Semantics for PREs over $\Sigma$: Two alternatives

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- $10011$ is in $L_{\square}(e)$.
- No word of length $\leq 4$ is in $L_{\square}(e)$. 
Semantics for PREs over $\Sigma$: Two alternatives

Let $e$ be a PRE over $\Sigma$. Then

- $\mathcal{L}_{\Box}(e) := \bigcup \{ \mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e \}$ (possibility)
- $\mathcal{L}_{\Diamond}(e) := \bigcap \{ \mathcal{L}(\nu(e)) \mid \nu \text{ is a valuation for } e \}$ (certainty)
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Finite unions or intersections of regular languages:

$\mathcal{L}_\Diamond(e)$ and $\mathcal{L}_\Box(e)$ are regular languages
Applications of PREs: Graph databases

Graph DBs:

- **Applications**: RDF, SNs, Scientific data, etc.
- **Model**: Edge-labeled directed graphs (that is: NFAs).
Applications of PREs: Graph databases

As it is usual, some data in the graph DB may be missing [Barceló et al. 2011, Calvanese et al. 2011].
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Example: Biological DB

- Proteins $p_1, q_1, p_2, q_2$

![Graph representation of proteins and interactions](image)
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![Graph representation of proteins and relationships](image-url)
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**Example**: Biological DB

- Proteins $p_1, q_1, p_2, q_2$
- We do not know the actual relationship

Incomplete graph DBs are graph DBs with edges labeled in $\mathcal{V}$.

- They can be represented as NFAs over $\Sigma \cup \mathcal{V}$.
- Equivalently, as PREs over $\Sigma$. 
Applications of PREs: Graph Databases

Standard semantics for incomplete DBs: Certain answers.

- Answers that hold regardless of the interpretation of the variables.
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Standard semantics for incomplete DBs: Certain answers.

- Answers that hold regardless of the interpretation of the variables.

How to use PREs to compute certain answers over graph DBs?
PRE’s for querying incomplete graph DB’s
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Paths from $n_3$ to $n_5$. 
PRE’s for querying incomplete graph DB’s

◮ Paths from $n_3$ to $n_5$.
◮ $e = b^*ya| b^*x| b^*xy$. 
PRE’s for querying incomplete graph DB’s

Paths from $n_3$ to $n_5$.

$e = b^*ya | b^*x | b^*xy$.

We can be certain about a word $w \in \Sigma^*$ labeling a path from $n_3$ to $n_5$ in $G$ iff $w \in L_{\square}(e)$. 
The certainty semantics is essential for computing certain answers over incomplete graph DBs.
Applications of PREs: Program analysis

PREs naturally arise in program analysis [Liu & Stoller 2004, de Moor at al. 2003].

- **Alphabet**: Operations on variables; e.g. `def`, `use`, `open`, etc.
- **Variables**: Program variables, pointers, files, etc.

PREs are used in this setting to specify undesired behavior.

**Example**: The undesired property “A variable is used without being defined” can be expressed as follows:

\[(\neg \text{def}(x)) \ast \text{use}(x).\]
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\[(\neg \text{def}(x)) \ast \text{use}(x).\]

These expressions are evaluated over graphs that serve as an abstraction of the program behavior.
Applications of PREs: Program analysis

PREs specify undesired behavior: Assignments of the variables that “satisfy” the PRE represent bugs of the program.

In the program analysis context the possibility semantics is essential for finding where the program fails a specification.
We study basic computational problems of PREs

Despite its importance, basic computational problems associated with PREs have not been addressed.

In this paper: Study standard language-theoretical problems for PREs divided as follows:

- **Decision problems:** Emptiness, universality, containment and membership.
- **Computational problems:** Minimal-size NFAs representing $L_{\Box}(e)$ and $L_{\Diamond}(e)$. 
- Upper bound techniques
- Decision problems
- Computational problems
- Extending the semantics
- Future work
- Upper bound techniques
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NFAs for $\mathcal{L}_\Diamond(e)$ and $\mathcal{L}_\Box(e)$

- Exponentially many valuations: $|\Sigma|^\text{(\# of variables)}$.
- Taking the union gives an exponential NFA for $\mathcal{L}_\Diamond(e)$.
- Taking the intersection gives a doubly-exponential NFA for $\mathcal{L}_\Box(e)$.

We shall see that these are tight bounds...
- Upper bound techniques
- **Decision problems**
- Computational problems
- Extending the semantics
- Future work
In order to do a finer analysis we study two restrictions of PREs:

- **Simple**: No repetition of variables; e.g. \( e = (0|1)^*xy(0|1)^* \).

- **Star-height 0**: No Kleene-star: i.e. finite languages.
Decision problems: Nonemptiness

- **NONEMPTINESS**\(\bigtriangledown\): \(\mathcal{L}\bigtriangledown(e) \neq \emptyset\)?

- **NONEMPTINESS**\(\square\): \(\mathcal{L}\square(e) \neq \emptyset\)?
Decision problems: Nonemptiness

- **NONEMPTINESS**⪨: $\mathcal{L}(e) \neq \emptyset$?

Not different from the case without variables:

$$(0x)^*1^*(xy)^*$$

- **NONEMPTINESS**⪨: $\mathcal{L}(e) \neq \emptyset$?
Decision problems: Nonemptiness

- **Nonemptiness\(\vartriangle\):** \(L(\vartriangle(e)) \neq \emptyset?\)

  Not different from the case without variables:

  \[(0x)^*1^*(xy)^*\]

- **Nonemptiness\(\Box\):** \(L(\Box(e)) \neq \emptyset?\)

**Theorem**

**Nonemptiness\(\Box\) is EXPSPACE-complete.**
Decision problems: Nonemptiness

- \textbf{NONEMPTINESS} \(\Diamond\): \(\mathcal{L}(e) \neq \emptyset\) ?

Not different from the case without variables:

\[(0x)^*1^*(xy)^*\]

- \textbf{NONEMPTINESS} \(\Box\): \(\mathcal{L}(e) \neq \emptyset\) ?

\textbf{Theorem}

\textbf{NONEMPTINESS} \(\Box\) is \textbf{EXPSPACE}-complete.

1. Remains \textbf{EXPSPACE}-hard even over the class of \textit{simple expressions}.
2. For PREs of star-height 0: \textbf{NONEMPTINESS} \(\Box\) is \(\Sigma^P_2\)-complete.
PREs and succinct intersection

Main tool for EXPSPACE-hardness:

Given PRE’s $e_1, \ldots, e_n$ we can construct in polynomial time a PRE $e'$ such that

$$\mathcal{L}_\square(e') \text{ is empty iff } \mathcal{L}_\square(e_1) \cap \cdots \cap \mathcal{L}_\square(e_n) \text{ is empty.}$$
PREs and succinct intersection

Main tool for \textsc{ExpSpace}-hardness:

Given PRE’s \( e_1, \ldots, e_n \) we can construct in polynomial time a PRE \( e' \) such that

\[
\mathcal{L}_{\square}(e') \text{ is empty iff } \mathcal{L}_{\square}(e_1) \cap \cdots \cap \mathcal{L}_{\square}(e_n) \text{ is empty.}
\]

Gives us \textsc{PSPACE}-hardness for \textsc{Nonemptiness} \( \square \), since regular expressions are PRE’s
Corollary (NONEMPMTINESS\(\square\))

There exists a sequence of parameterized regular expressions \(\{e_n\}_{n \in \mathbb{N}}\) such that:

1. Each \(e_n\) is of size polynomial in \(n\).

2. Every word in the language \(L_{\square}(e_n)\) has size at least \(2^{2^n}\).
Minimal size of words in $\mathcal{L}_\Box(e)$: exponential bound

Consider PREs of the form:

$$(0 | 1)^* x_1 \cdot x_2 \cdots x_n (0 | 1)^* \quad (n \geq 1).$$
Minimal size of words in $\mathcal{L}(e)$: exponential bound

Consider PREs of the form:

$$(0 \mid 1)^* x_1 \cdot x_2 \cdots x_n (0 \mid 1)^* \quad (n \geq 1).$$

- If $w \in \mathcal{L}(e)$, then it contains as a subword each $w' \in \{0, 1\}^n$. 
Minimal size of words in $\mathcal{L}_\square(e)$: exponential bound

Consider PREs of the form:

$$(0 \mid 1)^* x_1 \cdot x_2 \cdots x_n (0 \mid 1)^* \quad (n \geq 1).$$

- If $w \in \mathcal{L}_\square(e)$, then it contains as a subword each $w' \in \{0, 1\}^n$.

**de Bruijn sequences of order $n$, which are of size $\geq 2^n$.**
Decision problems: Universality

**Universality** ◊: Is $L_\diamond (e) = \Sigma^*$?

As opposed to nonemptiness, universality is more difficult for the ◊-semantics than for the □-semantics:

- **Universality** □ is $\text{PSPACE}$-complete.
- **Universality** ◊ is $\text{EXPSPACE}$-complete.
  - It remains $\text{EXPSPACE}$-complete even over the class of simple expressions.
Decision problems: Containment

Containment: Is $L_1(e_1) \subseteq L_2(e_2)$?

We can reduce from other problems, since:

- $L$ is empty iff $L \subseteq \emptyset$
Decision problems: Containment

**Containment**: $L_1(e_1) \subseteq L_1(e_2)$?

We can reduce from other problems, since:
- $L$ is empty iff $L \subseteq \emptyset$
- $L$ is $\Sigma^*$ iff $\Sigma^* \subseteq L$.

Thus,

**Containment**: $\square$ and **Containment**: $\Diamond$ are $\text{ExpSpace}$-complete.
- Even if restricted to simple expressions.
Decision problems: Membership

Is \( w \) in \( \mathcal{L}_\Diamond(e) \) or \( \mathcal{L}_\Box(e) \)?

Guess a valuation \( \nu \):

- \( w \in \mathcal{L}(\nu(e)) \) (possibility)
- \( w \notin \mathcal{L}(\nu(e)) \) (certainty)

Gives us \( \text{NP} \) and \( \text{coNP} \) bounds
Decision problems: Membership

Is $w$ in $\mathcal{L}(e)$ or $\mathcal{L}(e)$?

Guess a valuation $\nu$:
- $w \in \mathcal{L}(\nu(e))$ (possibility)
- $w \notin \mathcal{L}(\nu(e))$ (certainty)

Gives us $\text{NP}$ and $\text{coNP}$ bounds (tight).

**Theorem**
- $\textbf{MEMBERSHIP}_\Diamond$ is $\text{NP}$-complete.
- $\textbf{MEMBERSHIP}_\Box$ is $\text{coNP}$-complete.
Decision problems: Membership

We can do a finer analysis:

Proposition

- The complexity of $\text{Membership}_\diamond$ is as follows:
  1. Simple expressions: $\text{NP}$-complete.
  2. Star-height 0 expressions: $\text{NP}$-complete.
  3. Simple and star-height 0 expressions: $\text{Ptime}$.

- The complexity of $\text{Membership}_\square$ is as follows:
  1. Simple expressions: $\text{coNP}$-complete.
  2. Star-height 0 expressions: $\text{coNP}$-complete.
  3. Simple and star-height 0 expressions: $\text{Ptime}$.
Also in the paper:

- Containment when one expression is fixed.
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- **Containment** when one expression is fixed.
- **Membership** when the word is fixed.
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- **Containment** when one expression is fixed.
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- Emptiness of the intersection with a regular language
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- **Containment** when one expression is fixed.
- **Membership** when the word is fixed.
- Emptiness of the intersection with a regular language

These problems are motivated by the application of PREs
- Upper bound techniques

- Decision problems

- Computational problems

- Extending the semantics

- Future work
Computational problems

What is the size of the minimal NFA $A$ such that $L(A) = L_\diamond(e)$ or $L(A) = L_\Box(e)$?
What is the size of the minimal NFA $A$ such that $\mathcal{L}(A) = \mathcal{L}_\Diamond(e)$
or $\mathcal{L}(A) = \mathcal{L}_\square(e)$?

**Theorem**
The sizes of minimal NFAs are:
- necessarily double-exponential for $\mathcal{L}_\square$
- necessarily exponential for $\mathcal{L}_\Diamond$. 

Computational problems
Proof sketch of minimal size NFA for \( L \)

We use the following result by Glaister and Shallit:

If \( L \) is a regular language, and there exists a set of pairs

\[
P = \{(u_i, v_i) \mid 1 \leq i \leq m\} \subseteq \Sigma^* \times \Sigma^*,
\]

such that

1. \( u_iv_i \in L \),
2. \( u_jv_i \notin L \) for \( i \neq j \),

then every NFA accepting \( L \) has at least \( m \) states.
Proof sketch of minimal size NFA for $L_\square$

Consider the following family of PREs:

$$e_n = ((0 | 1)^{n+1})^* \cdot x_1 \cdots x_n \cdot x_{n+1} \cdot ((0 | 1)^{n+1})^* \quad (n \geq 1)$$

- Each $e_n$ is of linear size on $n$.

We shall construct a Fooling Set for $L_\square(e_n)$. 
Proof sketch of minimal size NFA for $\mathcal{L}$

Given a set $S \subset \{0, 1\}^{n+1}$ of size $2^n$:

- $w_S$ is the concatenation in lexicographical order of all words in $S$; and
- $w_{\bar{S},n}$ is the concatenation in lexicographical order of all words in $\{0, 1\}^{n+1}$ that are not in $S$. 

We define:

\[ P_n := \{(w_S, w_{\bar{S}, n}) \mid S \subset \{0, 1\}^{n+1} \text{ and } |S| = 2^n\}, \]
Proof sketch of minimal size NFA for $\mathcal{L}_\square$

We define:

$$P_n := \{(w_S, w_{\bar{S}}, n) \mid S \subset \{0, 1\}^{n+1} \text{ and } |S| = 2^n\},$$

1. There are $\binom{2^{n+1}}{2^n} \geq 2^{2n}$ different subsets of $\{0, 1\}^{n+1}$ of size $2^n$, and thus $|P_n| \geq 2^{2n}$.
2. $(w_S, w_{\bar{S}}, n)$ belongs to $\mathcal{L}_\square(e_n)$, but
3. $(w_{S_1}, w_{\bar{S}_2}, n)$ are not in $\mathcal{L}_\square(e_n)$, for distinct $S_1$ and $S_2$. 
- Upper bound techniques
- Decision problems
- Computational problems
- Extending the semantics
- Future work
Extending Semantics

We can extend semantics and allow replacement of variables by words that belong to some regular language.

- ◊-semantics: Easily becomes non-regular (e.g. \( xx = \) squared words). Regular for finite languages.
We can extend semantics and allow replacement of variables by words that belong to some regular language.

- ◊-semantics: Easily becomes non-regular (e.g. $xx = \text{squared words}$). Regular for finite languages.

- □-semantics: Keeps being regular. Same complexity bounds apply.
- Upper bound techniques
- Decision problems
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Future work

Closure properties:

- The minimal NFA $A$ that accepts $L_{\square}(e_1) \cap L_{\square}(e_2)$ is necessarily of double-exponential size.
Future work

Closure properties:

- The minimal NFA $A$ that accepts $\mathcal{L}_\square(e_1) \cap \mathcal{L}_\square(e_2)$ is necessarily of double-exponential size.

- Perhaps it is possible to construct in polynomial time a PRE $e$ such that $\mathcal{L}_\square(e) = \mathcal{L}_\square(e_1) \cap \mathcal{L}_\square(e_2)$. 