

Dependent Types and Fibred Computational Effects

Danel Ahman¹

(joint work with Neil Ghani² and Gordon Plotkin¹)

¹LFCS, University of Edinburgh

²MSP Group, University of Strathclyde

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Outline

Language design principles for combining

- dependent types $(\Pi, \Sigma, \text{Id}_A(V, W), \dots)$
- computational effects (state, I/O, probability, recursion, ...)

Our goal

- have a mathematically natural story
- use established math. techniques
- cover a wide range of computational effects

This work was guided by two problems

- effectful programs in types
- assigning types to effectful programs

Effectful programs in types

(type-dependency in the presence of effects)

Effectful programs in types

Let's assume that we have a dependent type $A(x)$, e.g.:

$x:\text{Nat} \vdash A(x) \stackrel{\text{def}}{=} \text{if } (x \bmod 2 == 0) \text{ then String else Char}$

Q: Should we allow $A[M/x]$ if M is an effectful program?

- e.g., if M is `receive(y.N)`

A1: In this work we say no

- types should only depend on static information
- e.g., how would one compute $A[\text{receive}(y.M)/x]$ statically?
- we recover dependency on effectful computations via thunks

A2: In a separate line of work, we are also looking at yes

- type-dependency ($z:\underline{C} \vdash A(z)$) becomes “homomorphic”
- lifting effect operations from terms to types, e.g., $\langle \text{receive} \rangle(y.A)$
- similarities with refinement types and op. modalities [A.,P.'15]

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Effectful programs in types ctd.

Aim: Types should only depend on static info about effects

Solution: CBPV/EEC style distinction between vals. and comps.

- value types $\Gamma \vdash A$ (MLTT + thunks + ...)
- computation types $\Gamma \vdash \underline{C}$ (dep. version of CBPV/EEC)
- where Γ contains **only** value variables $x_1:A_1, \dots, x_n:A_n$

Note: Some of the other options are λ_{ML} and FGCBV

- but basing the work on CBPV/EEC gives a more general story
- especially for treating of sequential composition
- also for systematically integrating dependent- and effect-typing (ongoing work)

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Assigning types to effectful programs

(i.e., typing sequential composition)

Assigning types to effectful programs

The problem: The standard typing rule for seq. composition

$$\frac{\Gamma \vdash M : F A \quad \Gamma, x:A \vdash N : \underline{C}}{\Gamma \vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

is not correct any more because x can appear free in the type

C

in the conclusion

Assigning types to effectful programs ctd.

Aim: To fix the typing rule of sequential composition

Option 1: We could restrict the free variables in \underline{C} , i.e.,

$$\frac{\Gamma \Vdash M : FA \quad \Gamma \vdash \underline{C} \quad \Gamma, x:A \Vdash N : \underline{C}}{\Gamma \Vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

But sometimes it is necessary for \underline{C} to depend on x !

- e.g., even to write effectful programs modularly
- take monadic parsing of well-typed syntax
and consider writing a parser for function application
- it is natural to modularly decompose the code into

$\cdot \Vdash \text{parseFun} : F (\Sigma y_1 : \text{LangType}. \Sigma y_2 : \text{LangType}. \text{LangSyntax}(\text{fun } y_1 y_2))$

$x : \Sigma y_1. \Sigma y_2. \text{LangSyntax}(\text{fun } y_1 y_2) \Vdash \text{parseFunArg} : F (\text{LangSyntax}(\text{fst } x))$

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Option 2: One could lift sequential composition to type level

$$\Gamma \Vdash M \text{ to } x:A \text{ in } N : M \text{ to } x:A \text{ in } \underline{C}$$

But then all comp. types would be singleton-like

- comp. types would contain exactly the terms we want to type!

Option 3: In the monadic metalanguage λ_{ML} , one could also try

$$\frac{\Gamma \vdash M : T A \quad \Gamma, x:A \vdash N : T B}{\Gamma \vdash M \text{ to } x:A \text{ in } N : T (\Sigma x : A. B)}$$

But what makes this a principled solution? Why is it correct?

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Option 4: We draw inspiration from algebraic effects

- and combine it with Option 1, i.e., restricting \underline{C} in seq. comp.

E.g., consider the stateful program (for some $x:\text{Nat} \Vdash N : \underline{C}$)

$$M \stackrel{\text{def}}{=} \text{lookup}(\text{return } 2, \text{return } 3) \text{ to } x:\text{Nat} \text{ in } N$$

After looking up the bit, this program evaluates as either

$$N[2/x] \text{ at type } \underline{C}[2/x] \quad \text{or} \quad N[3/x] \text{ at type } \underline{C}[3/x]$$

Idea: M denotes an element of the coproduct of algebras

$$\underline{C}[2/x] + \underline{C}[3/x] \stackrel{\text{def}}{=} F\left(U(\underline{C}[2/x]) + U(\underline{C}[3/x])\right) / \equiv$$

- actually, we use a Nat-indexed coproduct (i.e., $\Sigma x:\text{Nat}. \underline{C}$)

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Putting these ideas together

(a core dependently-typed calculus with comp. effects)

A computational dep.-typed language

Recall: We aim to define a *dependently-typed language* with

- general computational effects
- a clear distinction between
 - values
 - computations
- with a principled treatment of sequential composition
 - restricting free variables in seq. composition
 - based on coproducts of algebras
- with a natural denotational semantics, using standard techniques
 - dep. types — comprehension categories
 - comp. effects — adjunction models

A computational dep.-typed language

Value types: MLTT's types + **thunks** + ...

$A, B ::= \text{Nat} \mid 1 \mid \prod x:A. B \mid \sum x:A. B \mid \text{Id}_A(V, W) \mid \underline{U} \underline{C} \mid \dots$

- $\underline{U} \underline{C}$ is the type of **thunked** (i.e., suspended) **computations**

Computation types: dep.-typed version of EEC's comp. types

$\underline{C}, \underline{D} ::= F A \mid \prod x:A. \underline{C} \mid \sum x:A. \underline{C}$

- $\prod x:A. \underline{C}$ is the type of dependent effectful functions
 - it generalises CBPV's and EEC's
computational function type $A \rightarrow \underline{C}$ and product type $\underline{C} \times \underline{D}$
- $\sum x:A. \underline{C}$ is the generalisation of coproducts of algebras
 - it generalises EEC's
computational tensor type $A \otimes \underline{C}$ and sum type $\underline{C} + \underline{D}$

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A computational dep.-typed language

Value terms: MLTT's terms + *thunks* + ...

$V, W ::= x \mid \text{zero} \mid \text{succ } V \mid \dots \mid \text{thunk } M \mid \dots$

- equational theory based on MLTT with intensional id.-types
- value terms are typed using a judgment $\Gamma \vdash V : A$

Computation terms: dep.-typed version of CBPV/EEC c. terms

$M, N ::=$

	$\text{force } V$	
	$\text{return } V$	
	$M \text{ to } x:A \text{ in } N$	
	$\lambda x:A. M$	
	MV	
	$\langle V, M \rangle$	(comp. Σ intro.)
	$M \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K$	(comp. Σ elim.)

But: These val. and comp. terms alone do not suffice, as in EEC!

A computational dep.-typed language

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Note: We need to define K in such a way that the intended evaluation order is preserved, e.g., as in

$$\Gamma \Vdash \langle V, M \rangle \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K = K[V/x, M/z] : \underline{D}$$

Homomorphism terms: dep.-typed version of EEC's linear terms

$$\begin{array}{l} K, L ::= z \quad \text{(linear comp. vars.)} \\ \quad | K \text{ to } x:A \text{ in } M \\ \quad | \lambda x:A. K \\ \quad | KV \\ \quad | \langle V, K \rangle \quad \text{(comp-}\Sigma \text{ intro.)} \\ \quad | K \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } L \quad \text{(comp-}\Sigma \text{ elim.)} \end{array}$$

Computation and homomorphism terms are typed using judgments

- $\Gamma \Vdash M : \underline{C}$
- $\Gamma \mid z:\underline{C} \Vdash K : \underline{D}$ (linear in z ; comp. bound to z happens first)

Note: Formal presentation has more type-annotations on terms

A computational dep.-typed language

Note: We need to define K in such a way that the intended evaluation order is preserved, e.g., as in

$$\Gamma \Vdash \langle V, M \rangle \text{ to } \langle x:A, z:\underline{C} \rangle \text{ in } K = K[V/x, M/z] : \underline{D}$$

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A computational dep.-typed language

Typing rules: Dep.-typed versions of CBPV and EEC, e.g.:

$$\frac{\Gamma \Vdash V : A}{\Gamma \Vdash \text{return } V : F A} \qquad \frac{\Gamma \Vdash M : F A \quad \Gamma \vdash \underline{C} \quad \Gamma, x:A \Vdash N : \underline{C}}{\Gamma \Vdash M \text{ to } x:A \text{ in } N : \underline{C}}$$

...

$$\frac{\Gamma \vdash \underline{C}}{\Gamma \mid z:\underline{C} \Vdash z : \underline{C}}$$

...

$$\frac{\Gamma \Vdash V : A \quad \Gamma \mid z:\underline{C} \Vdash K : \underline{D}[V/x]}{\Gamma \mid z:\underline{C} \Vdash \langle V, K \rangle : \Sigma x:A. \underline{D}}$$

$$\frac{\Gamma \mid z_1:\underline{C} \Vdash K : \Sigma x:A. \underline{D}_1 \quad \Gamma \vdash \underline{D}_2 \quad \Gamma, x:A \mid z_2:\underline{D}_1 \Vdash L : \underline{D}_2}{\Gamma \mid z_1:\underline{C} \Vdash K \text{ to } \langle x:A, z_2:\underline{D}_1 \rangle \text{ in } L : \underline{D}_2}$$

A computational dep.-typed language

We can then account for **type-dependency in seq. comp.** by

$$\frac{\Gamma \Vdash M : F A \quad \frac{\Gamma, x:A \Vdash N : \underline{C}(x)}{\Gamma, x:A \Vdash \langle x, N \rangle : \Sigma y:A. \underline{C}(y)}}{\Gamma \Vdash M \text{ to } x:A \text{ in } \langle x, N \rangle : \Sigma y:A. \underline{C}(y)}$$

The **seq. comp. rule for λ_{ML}** is justified by the type isomorphism

$$\Gamma \vdash \Sigma x:A. F(B) \cong F(\Sigma x:A. B)$$

Operations and equations

(primitives for programming with side-effects)

Algebraic operations and equations

Effect theories:

- we consider signatures of **typed operation symbols**

$$\frac{\cdot \vdash I \quad x_i : I \vdash O \quad I \text{ and } O \text{ are both pure value types}}{\text{op} : (x_i : I) \longrightarrow O}$$

- equipped with equations on derivable effect terms
- type-dependency in operation symbols mostly a convenience

Algebraic operations:

$$\frac{\Gamma \Vdash V : I \quad \Gamma \vdash \underline{C} \quad \Gamma, x : O[V/x_i] \Vdash M : \underline{C}}{\Gamma \Vdash \text{op}_V^{\underline{C}}(x.M) : \underline{C}}$$

Generic effects:

$$\frac{\Gamma \Vdash V : I}{\Gamma \Vdash \text{genop}_V : F(O[V/x_i])}$$

Example: Global store with two locations (modeled as booleans)

$\text{lookup} : (x_i : \text{Bool}) \longrightarrow (\text{if } x_i \text{ then String else Nat})$

$\text{update} : (x_i : \Sigma x : \text{Bool}.(\text{if } x \text{ then String else Nat})) \longrightarrow 1$

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What about handlers?

We ensure that K 's behave like homomorphisms via the rule

$$\Gamma \mid z : \underline{C} \Vdash K : \underline{D} \implies \Gamma \Vdash K[\text{op}_{\underline{V}}^{\underline{C}}(x.M)/z] = \text{op}_{\underline{V}}^{\underline{D}}(x.K[M/z]) : \underline{D}$$

Recall: Plotkin-Pretnar presentation of handlers is given by:

$\Gamma \Vdash M$ handled with $\{\text{op}_x(y) \mapsto M_{\text{op}}\}_{\text{op}}$ to $x:A$ in $M_{\text{ret}} : \underline{C}$

- semantically, $\{\text{op}_x(y) \mapsto M_{\text{op}}\}_{\text{op}}$ defines an algebra on $U[\underline{C}]$
- and M handled ... is the unique homomorphism out of $F[A]$

Note: We have homomorphisms in the language, namely, the K 's

Q: So, could we simply add?

$$\Gamma \mid z : \underline{C} \Vdash K \text{ handled with } \{\text{op}_x(y) \mapsto M_{\text{op}}\}_{\text{op}} \text{ to } x:A \text{ in } M_{\text{ret}} : \underline{D}$$

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One way forward with handlers

User-defined algebra types:

(definitional equational proof obligations about V_{op} 's omitted)

$$\frac{\Gamma \vdash A \quad \{\Gamma, x_1 : I, x_2 : O[x_1/x_i] \rightarrow A \Vdash V_{\text{op}} : A\}_{\text{op}:(x_j:I) \rightarrow O}}{\Gamma \vdash \langle A, \{(x_1, x_2).V_{\text{op}}\}_{\text{op}:(x_j:I) \rightarrow O} \rangle}$$

Introduction: $\text{force} \langle A, \{(x_1, x_2).V_{\text{op}}\}_{\text{op}} \rangle V$ (where $V : A$)

Elimination: (comp. term version)

(definitional equational proof obligations about N omitted)

$$\frac{\Gamma \Vdash M : \langle A, \{(x_1, x_2).V_{\text{op}}\}_{\text{op}} \rangle \quad \Gamma, x : A \Vdash N : \underline{C}}{\Gamma \Vdash \text{run } M \text{ as } x \text{ in } N : \underline{C}}$$

Equations:

- $U \langle A, \{(x, y).V_{\text{op}}\}_{\text{op}} \rangle = A$
- $\text{op}_V^{\langle A, \{(x_1, x_2).V_{\text{op}}\}_{\text{op}} \rangle} (x.M) = \text{force} (V_{\text{op}}[V/x_1, \lambda x.\text{thunk } M/x_2])$
- (η - and β -equations for intro.-elim. interaction)

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Encoding Plotkin-Pretnar handlers:

M handled with $\{\text{op}_x(y) \mapsto M_{\text{op}}\}_{\text{op}}$ to $x:A$ in $M_{\text{ret}} : \underline{C}$

$$\text{force}_{\underline{C}} \left(\text{thunk} \left(M \text{ to } x:A \text{ in } \text{force}_{\langle \underline{C}, \dots, \text{thunk}(M_{\text{op}}) \dots \rangle} \left(\underbrace{\text{thunk } M_{\text{ret}}}_{:\underline{C}} \right) \right) \right)$$

$\underbrace{\hspace{15em}}_{:\langle \underline{C}, \dots, \text{thunk}(M_{\text{op}}) \dots \rangle}$

$\underbrace{\hspace{25em}}_{:\underline{C}}$

Categorical semantics

(fibrations and adjunctions)

Categorical semantics

Using fibred cat. theory, we define **fibred adjunction models**

- a sound and complete class of models

given by: i) a split closed comprehension category \mathcal{P}



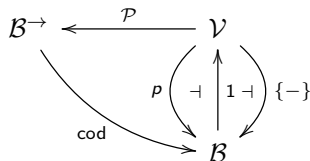
- following Streicher and Hoffmann, we have a partial interpretation function $\llbracket - \rrbracket$ on raw syntax, that maps (if defined):
- a context Γ to an object $\llbracket \Gamma \rrbracket$ in \mathcal{B}
- a context Γ and a value type A to an object $\llbracket \Gamma; A \rrbracket$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a context Γ and a value term V to $\llbracket \Gamma; V \rrbracket : 1_{\llbracket \Gamma \rrbracket}} \rightarrow X$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$

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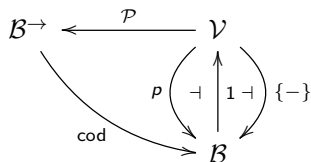
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- the display maps $\pi_A = \mathcal{P}(A) : \{A\} \rightarrow p(A)$ in \mathcal{B}
- induce the weakening functors $\pi_A^* : \mathcal{V}_{p(A)} \rightarrow \mathcal{V}_{\{A\}}$
- and the **value** Σ - and Π -types are interpreted as **adjoints**

$$\Sigma_A \dashv \pi_A^* \dashv \Pi_A$$

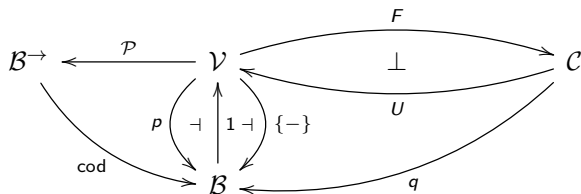
(Σ_A is also required to be strong, i.e., support dep. elimination)

Categorical semantics

Using fibred cat. theory, we define **fibred adjunction models**

- a sound and complete class of models

given by: ii) a **split fibration** q and a **split fib. adj.** $F \dashv U$



- we extend $\llbracket - \rrbracket$ so that it maps (if defined):
- a ctx. Γ and a **comp. type** \underline{C} to an object $\llbracket \Gamma; \underline{C} \rrbracket$ in $\mathcal{C}_{\llbracket \Gamma \rrbracket}$
- a ctx. Γ and a **comp. term** M to $\llbracket \Gamma; M \rrbracket : 1_{\llbracket \Gamma \rrbracket} \rightarrow U(Z)$ in $\mathcal{V}_{\llbracket \Gamma \rrbracket}$
- a ctx. Γ , a comp. type \underline{C} and a **hom. term** K to

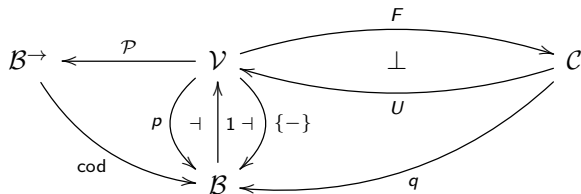
$$\llbracket \Gamma; \underline{C}; K \rrbracket : \llbracket \Gamma; \underline{C} \rrbracket \rightarrow Z \text{ in } \mathcal{C}_{\llbracket \Gamma \rrbracket}$$

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$$\Sigma_A \dashv \pi_A^* \dashv \Pi_A$$

Examples of fibred adjunction models

- for a split closed comprehension cat. $\mathcal{P} : \mathcal{V} \rightarrow \mathcal{B}^{\rightarrow}$, we have

$$\text{Id}_{\mathcal{V}} \dashv \text{Id}_{\mathcal{V}} : \mathcal{V} \rightarrow \mathcal{V}$$

- for a model of EEC (\mathcal{V} is CCC, \mathcal{C} is \mathcal{V} -enriched, \mathcal{V} -enr. adj., etc.)

$$F_{\text{EEC}} \dashv U_{\text{EEC}} : \mathbf{s}(\mathcal{V}, \mathcal{C}) \rightarrow \mathbf{s}(\mathcal{V})$$

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- for a model of EEC (\mathcal{V} is CCC, \mathcal{C} is \mathcal{V} -enriched, \mathcal{V} -enr. adj., etc.)

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- for $\mathcal{P}_{\text{fam}} : \text{Fam}(\text{Set}) \rightarrow \text{Set}^{\rightarrow}$ and $F \dashv U : \mathcal{C} \rightarrow \text{Set}$, when \mathcal{C} has **set-indexed products** and **set-indexed coproducts**, we have

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- for the **continuations monad** $R^{R^{(-)}} : \text{Set} \rightarrow \text{Set}$, we have

$$\widehat{R^{(-)}} \dashv \widehat{R^{(-)}} : \text{Fam}(\text{Set}^{\text{op}}) \rightarrow \text{Fam}(\text{Set})$$

and analogously for the **state monad** $(S \times (-))^S$

Examples of fibred adjunction models

Another example:

- for a \mathcal{CPO} -enriched monad $T : \mathcal{CPO} \rightarrow \mathcal{CPO}$ with a least algebraic operation $\Omega : 0$ and reflexive coequalizers in \mathcal{CPO}^T

$$\widehat{F}^T \dashv \widehat{U}^T : \text{CFam}(\mathcal{CPO}^T) \rightarrow \text{CFam}(\mathcal{CPO})$$

where $\text{CFam}(\mathcal{CPO})$ is the cat. of continuous families

$$\left((X, \sqsubseteq_X), A : (X, \sqsubseteq_X) \rightarrow \mathcal{CPO}^{\text{EP}} \right)$$

- this allows us to treat general recursion as a comp. effect by

$$\frac{\Gamma, x : \underline{UC} \Vdash M : \underline{C}}{\Gamma \Vdash \mu x : \underline{UC}. M : \underline{C}}$$

- but have to restrict A in $\text{Id}_A(V, W)$ to be discrete to define

$$\text{Id}_{(X,A)} \stackrel{\text{def}}{=} (\{\pi_{(X,A)}^*(X, A)\}, \langle x, a, a' \rangle \mapsto \prod_{\{\star \mid a=a'\}} 1)$$

Conclusions

A dependently-typed computational language with

- clear distinction between values and computations
- systematic treatment of seq. composition (comp. Σ -types)
- algebraic effects and handlers
- natural denotational semantics, using standard math. tools

Ongoing work

- integrating **dependent-** and **effect-typing**
 - e.g., fibred parametrised adjunctions for a principled account of resource-dependent effects in Idris

$$\text{EffM } \varepsilon_1 \left((x:A). \varepsilon_2(x) \right) = U_{\varepsilon_1} \left(\Sigma x:A. F_{\varepsilon_2(x)}(1) \right)$$

- **homomorphic type-dependency** on effectful computations

Thank you for listening!

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