BIPRODUCTS WITHOUT ZERO MORPHISMS

MARTTI KARVONEN

Abstract. We show how to define biproducts up to isomorphism in an arbitrary category without assuming any enrichment. This generalizes the usual definitions, and we characterize when a category has all binary biproducts in terms of an ambidextrous adjunction. Finally, we give some new examples of biproducts that our definition recognizes.

1. Introduction

Given two objects A and B living in some category $\mathbf{C}$, their biproduct - according to a standard definition [3] - consists of an object $A \oplus B$ together with maps

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} A \oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B$$

such that

$$p_A i_A = \mathrm{id}_A \qquad p_B i_B = \mathrm{id}_B \tag{1.1}$$

$$p_B i_A = 0_{A,B} \qquad p_A i_B = 0_{B,A} \tag{1.2}$$

and

$$\mathrm{id}_{A\oplus B} = i_A p_A + i_B p_B. \tag{1.3}$$

For us to be able to make sense of the equations, we must assume that $\mathbf{C}$ is enriched in commutative monoids. One can get a slightly more general definition that only requires zero morphisms but no addition – that is, enrichment in pointed sets – by replacing the last equation with the condition that $(A \oplus B, p_A, p_B)$ is a product of A and B and that $(A \oplus B, i_A, i_B)$ is their coproduct. This definition is not vastly more general, for one can show that if $\mathbf{C}$ has all binary biproducts in this latter sense, then $\mathbf{C}$ is uniquely enriched in commutative monoids and the biproducts in $\mathbf{C}$ are biproducts in the earlier sense as well.

But what if we do not assume that $\mathbf{C}$ is enriched over pointed sets? One alternative would be to postulate, just for the objects A and B in question, maps $0_{A,B} \colon A \to B$ and $0_{B,A} \colon B \to A$ that act like zero maps in that it does not matter what you pre- and postcompose them with, but this is not very satisfactory – it is as if one tried to generalize the first definition by assuming that just the particular homset $\hom(A \oplus B, A \oplus B)$ happens to be a commutative monoid.

In this paper we show that one can get a well-behaved notion of a biproduct in any category $\mathbf{C}$, with no assumptions about enrichment, by replacing the equations referring to zero with the single equation

$$i_A p_A i_B p_B = i_B p_B i_A p_A, \tag{1.4}$$

which states that the two canonical idempotents on $A \oplus B$ commute with one another. We first prove that biproducts thus defined behave as one would expect, e.g. that they are defined up to unique isomorphism compatible with the biproduct structure, and that the notion agrees with the other definitions whenever $\mathbf{C}$ is appropriately enriched. We also show how to characterize them in terms of ambidextrous adjunctions.

2. Main results

We start with the new, enrichment-free definition of a biproduct.

Definition 2.1. *A biproduct of A and B in $\mathbf{C}$ is a tuple $(A \oplus B, p_A, p_B, i_A, i_B)$ such that $(A \oplus B, p_A, p_B)$ is a product of A and B, $(A \oplus B, i_A, i_B)$ is their coproduct, and the following equations hold:*

$$p_A i_A = \mathrm{id}_A$$
$$p_B i_B = \mathrm{id}_B$$
$$i_A p_A i_B p_B = i_B p_B i_A p_A$$

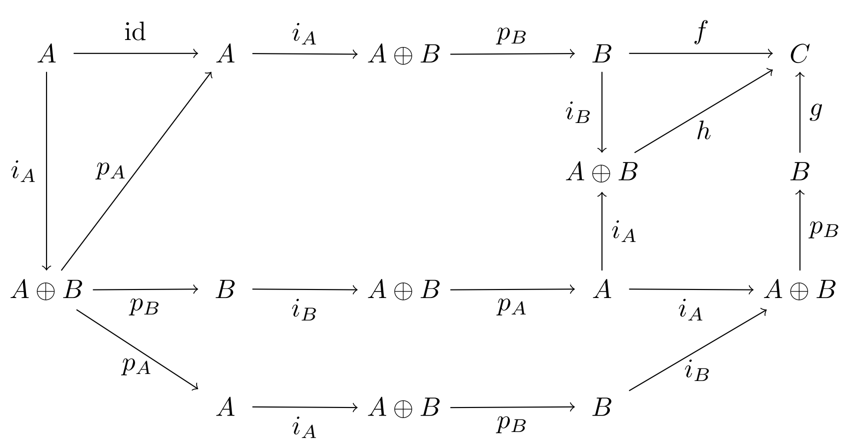
Even though the definition above does not refer to zero morphisms, the map $p_B i_A \colon A \to A \oplus B \to B$ behaves a lot like one.

Definition 2.2. *A morphism $a : A \to B$ is constant if $af = ag$ for all $f, g \colon C \to A$. Coconstant morphisms are defined dually and a morphism is called a zero morphism if it is both constant and coconstant. A category has zero morphisms if for every pair of objects B and C there is a morphism $0_{B,C}$ such that for every $g \colon C \to D$ and $f \colon A \to B$ we have $g 0_{A,B} f = 0_{A,B}$.*

Remark 2.3. *If there are zero morphisms $A \to B$ and $B \to A$, then they are unique. Furthermore, a category has zero morphisms iff for any A and B there is a zero morphism $A \to B$, and this is equivalent to the category being enriched in pointed sets. The collection of all zero morphisms forms a partial zero structure in the sense of* [1].

Lemma 2.4. *Let $(A \oplus B, p_A, p_B, i_A, i_B)$ be the biproduct of A and B. Then $p_B i_A$ is a zero morphism.*

Proof. As the definition of biproducts is self-dual, it suffices to prove that $p_B i_A$ is coconstant. This follows from the fact that the diagram



commutes, where h is the cotuple $[gp_Bi_A, f]$. □

Corollary 2.5. *If* $\mathbf{C}$ *has all binary biproducts, then it has zero morphisms.*

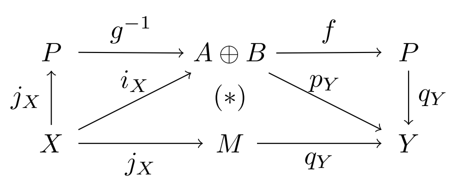
Proof. Combine Lemma 2.4 and Remark 2.3. □

Given Lemma 2.4, it is easy to check that whenever $\mathbf{C}$ has zero morphisms: equation (1.4) is equivalent to equation (1.2). Given a biproduct in the sense of Definition 1.2, it is a biproduct in the sense of Definition 2.1, since $i_Ap_Ai_Bp_B = 0 = i_Bp_Bi_Ap_A$. Conversely, let $(A \oplus B, p_A, p_B, i_A, i_B)$ be a biproduct in the sense of Definition 2.1 in a category with zero morphisms. Now by Lemma 2.4 and Remark 2.3 we have $p_Bi_A = 0_{A,B}$, as desired, and similarly $p_Ai_B = 0_{B,A}$. If $\mathbf{C}$ is enriched in commutative monoids, then our definition is equivalent to (1.3) just because the other definition in terms of (1.2) and universal properties is.

Besides being equivalent to the usual definitions when $\mathbf{C}$ is appropriately enriched, it behaves well even when $\mathbf{C}$ is not assumed to be enriched.

Proposition 2.6. *The biproduct of A and B, if it exists, is unique up to unique isomorphism compatible with the biproduct structure.*

Proof. Let $(A \oplus B, p_A, p_B, i_A, i_B)$ and (P, q_A, q_B, j_A, j_B) be biproducts of A and B. Then there exists a unique isomorphism $f\colon A \oplus B \to P$ compatible with the product structures, and a unique isomorphism $g\colon A \oplus B \to P$ compatible with the coproduct structures, and we wish to show that they coincide. We show this by proving that $fg^{-1} = \mathrm{id}_P$. By the two universal mapping properties of P, it suffices to show that the diagram



commutes for all $X, Y \in \{A, B\}$. Clearly we need to only consider the region marked by $(*)$. By definition, it commutes whenever $X = Y$. The case $X \neq Y$ follows from Lemma 2.4 and Remark 2.3. □

Using Lemma 2.4, one can then proceed to check that biproducts in our sense work just like one would expect. For example, Definition 2.1 and other results of this section generalize from the binary case to the biproduct of an arbitrary-sized collection of objects, and one can easily show that if $(A \oplus B)$ and $(A \oplus B) \oplus C$ exist, then $(A \oplus B) \oplus C$ satisfies the axioms for the ternary biproduct of A, B, C. Similarly, using Lemma 2.4 one can show that for $f\colon A \to C$ and $g\colon B \to D$ we have $f + g = f \times g$ whenever the biproducts $A \oplus B$ and $C \oplus D$ exist.

Recall that an ambiadjoint to a functor $F\colon \mathbf{C} \to \mathbf{D}$ is a functor $D\colon \mathbf{D} \to \mathbf{C}$ that is simultaneously both left and right adjoint to F.

Theorem 2.7. $\mathbf{C}$ *has biproducts iff the diagonal* $\Delta\colon \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ *has an ambiadjoint* $(-) \oplus (-)$ *such that the unit* $(i_A, i_B)\colon (A, B)\colon (A \oplus B, A \oplus B)$ *of the adjunction* $(-) \oplus (-) \dashv \Delta$*, is a section of the counit* $(p_A, p_B)\colon (A \oplus B, A \oplus B) \to (A, B)$ *of the adjunction* $\Delta \dashv (-) \oplus (-)$*, i.e.* $(p_A \circ i_A, p_B \circ i_B) = (\mathrm{id}_A, \mathrm{id}_B)$ *for* $A, B \in \mathbf{C}$.

Proof. The implication from left to right is routine. For the other direction, a right adjoint to the diagonal is well-known to fix binary products, and dually, a left adjoint fixes binary coproducts. Thus it remains to check that the required equations governing p_A, p_B, i_A and i_B are satisfied. By naturality, the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\quad i_A \quad} & A \oplus B & \xrightarrow{\quad p_B \quad} & B \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \oplus f} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\quad i_A \quad]{} & A \oplus C & \xrightarrow[\quad p_C \quad]{} & C \end{array}$$

commutes for any f. Thus $i_A p_B$ is coconstant and by duality it is constant, so it is zero. Hence $\mathbf{C}$ has zero morphisms and $i_A p_B = 0$. Thus $i_A p_A i_B p_B = 0 = i_B p_B i_A p_A$. As $(p_A \circ i_A, p_B \circ i_B) = (\mathrm{id}_A, \mathrm{id}_B)$ by assumption, this concludes the proof. □

3. Examples

Given the results of the previous section, genuinely new examples must be in categories that have neither all binary biproducts nor zero morphisms.

- In **Set** the biproduct $\emptyset \oplus \emptyset$ exists and is the empty set.
- Let $\mathbf{C}$ be any category with biproducts, and let $\mathbf{D}$ be any non-empty category. Then in the coproduct category $\mathbf{C} \sqcup \mathbf{D}$, the biproduct $A \oplus B$ exists whenever $A, B \in \mathbf{C}$. More concretely, in $\mathbf{Ab} \sqcup \mathbf{Set}$ the binary biproduct of any two abelian groups exists and is computed just as in **Ab**, even though $\mathbf{Ab} \sqcup \mathbf{Set}$ lacks zero morphisms.
- In any preorder $A \oplus B$ exists if and only if $A \cong B$.
- Commutative inverse semigroups are sets equipped with a binary operation that is commutative and associative and in which for every element x there is an unique y such that $xyx = x$ and $yxy = x$ [2]. The obvious choice of morphism is a function that preserves the binary operation. Not every such semigroup has a neutral element, so we call a homomorphism $f\colon S \to T$ unital if it preserves neutral elements. Let $\mathbf{C}$ be the category of commutative inverse semigroups and unital homomorphisms. Then $S \oplus T$ exists if and only if S and T both have a neutral element, in which case $S \oplus T$ is constructed just as in **Ab**.
- A function $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is non-expansive if $d_X(x, y) \geq d_Y(f(x), f(y))$ for all $x, y \in X$. It is contractive if there is some $c \in [0, 1)$ such that $c d_X(x, y) \geq d_Y(f(x), f(y))$ for all $x, y \in X$. Let **Met** be the category of metric spaces and non-expansive maps, and let **Con** be the category of contractions. More specifically, let $\mathbb{N}$ denote the monoid of natural numbers. Then **Con** is the full subcategory of $[\mathbb{N}, \mathbf{Met}]$ with objects given by contractive endomorphisms. In **Con**, the terminal object is $!\colon \{*\} \to \{*\}$, and for any s in **Con**, the biproduct $s \oplus !$ exists if and only if s has a (necessarily unique) fixed point.

References

[1] Amartya Goswami and Zurab Janelidze. On the structure of zero morphisms in a quasi-pointed category. *Applied Categorical Structures*, 25(6):1037–1043, 2017.

[2] Mark V. Lawson. *Inverse semigroups: the theory of partial symmetries*. World Scientific, 1998.
[3] Saunders Mac Lane. *Categories for the working mathematician*. Springer, second edition, 1998.