The Effect of Non-tightness on Bayesian Estimation of PCFGs

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Probabilistic context-free grammars (PCFGs)

<table>
<thead>
<tr>
<th>Probability</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>S → NP VP</td>
</tr>
<tr>
<td>1.0</td>
<td>NP → Det N</td>
</tr>
<tr>
<td>1.0</td>
<td>VP → V NP</td>
</tr>
<tr>
<td>0.7</td>
<td>Det → the</td>
</tr>
<tr>
<td>0.3</td>
<td>Det → a</td>
</tr>
<tr>
<td>0.4</td>
<td>N → cat</td>
</tr>
<tr>
<td>0.6</td>
<td>N → dog</td>
</tr>
<tr>
<td>0.2</td>
<td>V → chased</td>
</tr>
<tr>
<td>0.8</td>
<td>V → liked</td>
</tr>
</tbody>
</table>

Parse tree

Tree probability = \(1.0 \times 1.0 \times 0.7 \times 0.4 \times 1.0 \times 0.2 \times 1.0 \times 0.7 \times 0.6 = 0.02352\)
PCFGs and tightness

- $p \in [0, 1]^{|\mathcal{R}|}$ is a vector of rule probabilities indexed by rules $\mathcal{R}$
- A PCFG associates each tree $t$ with a measure $m_p(t)$:

$$m_p(t) = \prod_{A \rightarrow \alpha \in \mathcal{R}} p^n_{A \rightarrow \alpha}(t),$$

where:

- $n_{A \rightarrow \alpha}(t)$ is the number of times rule $A \rightarrow \alpha$ is used in the derivation of $t$
- The partition function $Z$ of a PCFG is:

$$Z_p = \sum_{t \in T} m_p(t)$$

- PCFGs require the rule probabilities expanding a non-terminal to be normalised, but this does not guarantee that $Z_p = 1$
- When $Z_p < 1$, we say the PCFG is “non-tight.”
Catalan grammar: an example of a non-tight PCFG

- PCFG has two rules: $S \rightarrow SS$ and $S \rightarrow x$
- It generates strings of $x$ of arbitrary length
- It generates all possible finite binary trees
  - or equivalently, all possible well-formed brackettings
  - called the *Catalan grammar* because the number of parses of $x^n$ is Catalan number $C_{n-1}$
- The PCFG is non-tight when $p_{S \rightarrow SS} > 0.5$

![Diagram of Catalan grammar and probability distribution]

\[
\begin{align*}
Z_p &= 0.5 \\
p_{S \rightarrow SS} &= 0.5
\end{align*}
\]
Why can the Catalan grammar be non-tight?

- Every binary tree over $n$ terminals has $n - 1$ non-terminals
  - $\Rightarrow$ probability of a tree decreases exponentially with length
- The number of different binary trees with $n$ terminals is $C_{n-1}$
  - $\Rightarrow$ number of trees grammar grows exponentially with length
- When $p_{S \rightarrow SS} \geq 0.5$, the PCFG puts non-zero mass on non-terminating derivations
  - this grammar defines a branching processes
  - At each step, $p_{S \rightarrow SS}$ is probability of reproducing, $p_{S \rightarrow x}$ is probability of dying
    - $p_{S \rightarrow SS} < 0.5 \Rightarrow$ population dies out (subcritical)
    - $p_{S \rightarrow SS} > 0.5 \Rightarrow$ population grows unboundedly (supercritical)
- Mini-theorem: every linear PCFG is tight (except on cases of measure zero under continuous priors)
  - CFG is linear $\iff$ RHS of every rule contains at most one non-terminal
  - HMMs are linear PCFGs $\Rightarrow$ always tight
Bayesian inference of PCFGs

- Bayesian inference uses Bayes rule to compute a posterior over rule probability vectors \( p \)

\[
P(p \mid D) \propto P(D \mid p) P(p)
\]

where \( D = (D_1, \ldots, D_n) \) is the training data (trees or strings)

- Bayesians prefer the full posterior distribution \( P(p \mid D) \) to a point estimate \( \hat{p} \)

- *If the prior assigns non-zero mass to non-tight grammars, in general the posterior will too*

- As the number of independent observations \( n \) in the training data grows, the posterior concentrates around the MLE
  - MLE is always a tight PCFG (Chi and Geman 1998)
  - *As \( n \to \infty \) the posterior concentrates on tight PCFGs*
3 approaches to non-tightness in the Bayesian setting

- If the grammar is linear, then all continuous priors lead to tight PCFGs
- Three different approaches to Bayesian inference with non-tight grammars:
  1. “Sink element”: assign mass of “infinite trees” to a sink element, implicitly assumed by Johnson et al (2007)
  2. “Only tight”: redefine prior so it only places mass onto tight grammars
  3. “Renormalisation”: divide by partition function to ensure normalisation

Assume for now that trees and strings are observed in $D$ (supervised learning)
“Only tight” approach

Let $I(p)$ be 1 if $p$ is tight and 0 otherwise.

Given a “non-tight prior” $P(p)$, define a new prior $P'$ as:

$$P'(p) \propto P(p) I(p)$$

If $P(p)$ is conjugate family of priors with respect to PCFG likelihood, then $P'(p)$ is also conjugate.

We can draw samples from $P'(p \mid D)$ using rejection sampling:

- Draw PCFG parameters $p$ from $P(p \mid D)$ until $p$ is tight
  - $P(p \mid D)$ is a product of Dirichlets
  - can use textbook algorithms for sampling from Dirichlets
Renormalisation approach

Renormalise the measure $\mu_p(t)$ over finite trees (Chi, 1999)

If $P(p \mid \alpha)$ is a product of Dirichlets, posterior is:

$$P(p \mid D) = \prod_{i=1}^{n} \frac{\mu_p(t_i)}{Z_p} P(p \mid \alpha) \propto \frac{1}{Z_p^n} P(p \mid \alpha + n(D)).$$

where $n(D)$ is the count vector over all rules for the data $D$

- Use a *Metropolis-Hastings sampler* to sample from $P(p \mid D)$
  - proposal distribution is product of Dirichlets

*Samplers for each approach can be used within a component-wise Gibbs sampler for the unsupervised case where only strings are observed.*
**Toy example**

Consider the grammar $S \rightarrow S \ S \ S | S \ S | a$

Let $w = a \ a \ a$

\[
\begin{align*}
  t_1 &= S \\
   &\quad S \ S \ S \\
   &\quad \quad a \ a \ a
  \\
  t_2 &= S \\
   &\quad S \ S \\
   &\quad \quad a \ S \ S \\
   &\quad \quad \quad a \ a \\
  t_3 &= S \\
   &\quad S \ S \\
   &\quad \quad S \ S \\
   &\quad \quad \quad a \ a
\end{align*}
\]

- **Uniform prior** ($\alpha = 1$)
- **Sink-element approach**: $P(t_1 \mid w) = \frac{7}{11} \approx 0.636364$.
- **Only-tight approach**: $P(t_1 \mid w) = \frac{11179}{17221} \approx 0.649149$.
- **Renormalisation approach**: $P(t_1 \mid w) \approx 0.619893$.

$\Rightarrow$ All three approaches induce different posteriors from uniform prior.
Experiments on WSJ10

- 100 runs of each sampler for 1,000 MCMC sweeps
- Computed average $F_1$ score on every 10th sweep for last 100 sweeps
- Kolmogorov-Smirnov tests did not show a statistically significant difference
Conclusion

- *Linear* CFGs are tight regardless of the prior
- For non-linear CFGs, three approaches are suggested for handling non-tightness
- The three approaches are not mathematically equivalent, but experiments on WSJ Penn treebank showed that they behave similarly empirically

Open problem: are the approaches reducible in the following sense?

*Given a prior $P$ for one of the approaches, is there a prior $P'$ for another approach such that for all data $D$, the posteriors under both approaches are the same.*